1 Lecture 7 - More Integration Techniques: Trigonometric substitution

1.1 Odds & Ends: Derivation of the reduction formulae

The two reduction formulae for *indefinite* integrals are

$$\begin{aligned}\int \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\,\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx\\ \int \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\,\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.\end{aligned}$$

The two reduction formulae for *definite* integrals are

$$\begin{aligned}\int_a^b \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\,\cos(x)\Big|_a^b + \frac{n-1}{n}\int_a^b \sin^{n-2}(x)\,dx\\ \int_a^b \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\,\sin(x)\Big|_a^b + \frac{n-1}{n}\int_a^b \cos^{n-2}(x)\,dx.\end{aligned}$$

We show how to derive these. This is not knowledge you need for the test, but it is good to see how it is done.

1.1.1 Derivation of the sin reduction formula

We use a clever integration by parts argument:

$$\begin{aligned}\int \sin^n(x)\,dx &= \int \sin^{n-1}(x)\,\sin(x)\,dx\\ \text{use}u &= \sin^{n-1}(x) \qquad du = (n-1)\sin^{n-2}(x)\,\cos(x)\,dx\\ dv &= \sin(x)\,dx \qquad v = -\cos(x)\,dx\\ \int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,\cos^2(x)\,dx.\end{aligned}$$

Now integration by parts is supposed to make things simpler, but the expression on the right does not look simpler. However, we can use some algebra to manipulate the right side of the equation. Use $\cos^2(x) = 1 - \sin^2(x)$ to get rid of the cos on the right:

$$\begin{aligned}
\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\left(1 - \sin^2(x)\right)\,dx \\
\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx.
\end{aligned}$$

On both the left side and the right side, we have a "$\int \sin^n(x)\,dx$" expression, so we can move both of them to the left side, then simplify:

$$\begin{aligned}
(n-1)\int \sin^n(x)\,dx - \int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx \\
n\int \sin^n(x)\,dx &= -\sin^{n-1}(x)\,\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx.
\end{aligned}$$

Now divide both sides by n to get

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\,\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.$$

Thus, after our algebra tricks, the integral on the right is indeed simpler than the integral on the left, which is the integral we started with. We have arrived at the reduction formula for sin.

1.1.2 Derivation of the cos reduction formula

We use the same procedure we used for sin. First use integration by parts:

$$\begin{aligned}
\int \cos^n(x)\,dx &= \int \cos^{n-1}(x)\,\cos(x)\,dx \\
\text{use}u &= \cos^{n-1}(x) \qquad du = -(n-1)\cos^{n-2}(x)\,\sin(x)\,dx \\
dv &= \cos(x)\,dx \qquad v = \sin(x)\,dx \\
\int \sin^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x)\,\sin^2(x)\,dx.
\end{aligned}$$

Then get rid of the "$\sin^2(x)$" on the right by using $\sin^2(x) = 1 - \cos^2(x)$:

$$\begin{aligned}
\int \cos^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x)\left(1 - \cos^2(x)\right)\,dx \\
\int \cos^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x) - (n-1)\int \cos^n(x)\,dx.
\end{aligned}$$

Then add $(n-1)\int \cos^n(x)\,dx$ to both sides to get

$$\begin{aligned} n\int \cos^n(x)\,dx &= \cos^{n-1}(x)\,\sin(x) + (n-1)\int \cos^{n-2}(x) \\ \int \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\,\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x). \end{aligned}$$

Done!

1.2 Trigonometric substitution

Guidelines for trigonometric substitution:

If you see	Consider substituting
$a^2 - x^2$	$x = a\cos(\theta)$ or $x = a\sin(\theta)$
$a^2 + x^2$	$x = a\tan(\theta)$ or $x = a\cot(\theta)$
$x^2 - a^2$	$x = a\sec(\theta)$ or $x = a\csc(\theta)$

Let's do some examples.

Example 1 Evaluate $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Solution Use $x = 2\cos(\theta)$, $dx = -2\sin(\theta)\,d\theta$ to get

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx &= \int_{\pi/3}^{\pi/6} \frac{-2\sin(\theta)}{\sqrt{4-4\cos^2(\theta)}}\,d\theta \\
&= \int_{\pi/3}^{\pi/6} \frac{-2\sin(\theta)}{\sqrt{4\sin^2(\theta)}}\,d\theta \\
&= \int_{\pi/3}^{\pi/6} \frac{-2\sin(\theta)}{2\sin(\theta)}\,d\theta \\
&= -\int_{\pi/3}^{\pi/6} d\theta \\
&= -\theta\Big|_{\pi/3}^{\pi/6} \\
&= -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}.
\end{aligned}$$

Example 2 Evaluate $\int_3^{2\sqrt{3}} \frac{1}{x\sqrt{x^2-9}}\,dx$

Solution Use $x = 3\sec(\theta)$, $dx = 3\sec(\theta)\tan(\theta)\,d\theta$ to get

$$\begin{aligned}
\int_3^{2\sqrt{3}} \frac{1}{x\sqrt{x^2\,-\,9}}\,dx &= \int_0^{\frac{\pi}{6}} \frac{3\sec(\theta)\tan(\theta)}{3\sec(\theta)\sqrt{9\sec^2(\theta)\,-\,9}}\,d\theta \\
&= \int_0^{\frac{\pi}{6}} \frac{3\sec(\theta)\tan(\theta)}{3\sec(\theta)\sqrt{9\tan^2(\theta)}}\,d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{6}} d\theta \\
&= \frac{1}{3}\theta\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{18}.
\end{aligned}$$

Example 3(a) Evaluate $\int_4^{4\sqrt{3}} \frac{x}{x^2+16}\,dx$

Solution Use $x = 4\tan(\theta)$, $dx = 4\sec^2(\theta)\,d\theta$ to get

$$\begin{aligned}
\int_4^{4\sqrt{3}} \frac{x}{\sqrt{x^2+16}}\,dx &= \int_{\pi/4}^{\pi/3} \frac{16\tan(\theta)\,\sec^2(\theta)}{16\tan^2(\theta)\,+\,16}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \frac{16\tan(\theta)\,\sec^2(\theta)}{16\sec^2(\theta)}\,d\theta \\
&= \int_{\pi/4}^{\pi/3} \tan(\theta)\,d\theta \\
&= -\ln\big|\cos(\theta)\big|\Big|_{\pi/4}^{\pi/3} \\
&= -\ln\left(\frac{1}{2}\right) + \ln\left(\frac{1}{\sqrt{2}}\right) \\
&= -\ln\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

Example 3(b) Evaluate $\int_4^{4\sqrt{3}} \frac{x}{x^2+16}\,dx$

Solution This problem is identical to the problem from the previous example. But this time, we will use the substitution $u = x^2 + 16$, $du = 2x\,dx$ to get

$$\begin{aligned}
\int_4^{4\sqrt{3}} \frac{x}{x^2+16}\,dx &= \frac{1}{2}\int_{32}^{64} \frac{1}{u}\,du \\
&= \frac{1}{2}\ln|u|\Big|_{32}^{64} \\
&= \frac{1}{2}\ln(64) - \frac{1}{2}\ln(32) \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

This is a good time to recall the **rules of logarithms**:

$$\begin{aligned}
\ln(x) + \ln(y) &= \ln(xy) \\
\ln(x) - \ln(y) &= \ln\left(\frac{x}{y}\right) \\
a\ln(x) &= \ln(x^a)
\end{aligned}$$