1 Lecture 6 - More Integration Techniques

In total, we will study six integration techniques:

- Substitution
- Integration by parts
- Trigonometric integration (integrating powers of sin and cos
- Trigonometric substitution (substituting a trig function into an integral that didn't ave trig functions to begin with)
- Partial fraction expansion
- Long division

We have already studied substitution and integration by parts. In this lecture we study trigonometric integration and trigonometric substitution.

Something many of you have noticed is that integration is hard, much harder than differentiation, and the techniques needed to evaluate integrals are commensurately harder.

Even still, many relatively simple integrals just cannot be evaluated, no matter what techniques are used: for instance $\int \sqrt{x^3+1}\,dx$ or $\int e^{x^2}\,dx$ can not be explicitly evaluated.

1.1 Trigonometric integration

Here we study integrals of the type

$$\int \sin^m(x)\, \cos^n(x)\, dx.$$

1.1.1 Case where either m or n (or both) is an odd number

In this case, one uses the identity

$$\sin^2(x) \;+\; \cos^2(x) \;=\; 1$$

to reduce to the case where there is either a *single* cosine or else a *single* sine. Here are some examples:

Example 1 Evaluate $\int \sin^3(x)\, dx$

Solution
Use $\sin^2(x) \;=\; 1 - \cos^2(x)$ to get

$$\begin{aligned}\int \sin^3(x)\, dx \;&=\; \int \sin^2(x)\, \sin(x)\, dx \\ &=\; \int (1 - \cos^2(x))\, \sin(x)\, dx\end{aligned}$$

Then use the substitution $u = \cos(x)$, $du = -\sin(x)\, dx$ to get

$$\begin{aligned}\int \sin^3(x)\, dx \;&=\; \int (1 - \cos^2(x))\, \sin(x)\, dx \\ &=\; -\int (1 - u^2)\, du \\ &=\; -u \;+\; \frac{1}{3}u^3 \;+\; C \\ &=\; -\cos(x) \;+\; \frac{1}{3}\cos^3(x) \;+\; C.\end{aligned}$$

Example 2 Evaluate $\int \cos^2(x)\,\sin^5(x)\,dx$.

Solution Since the power on the sin function is odd, we can reduce to the case where there is one single sin function, and then use a substitution:

$$\begin{aligned}
\int \cos^2(x)\,\sin^5(x)\,dx &= \int \cos^2(x)\,\left(\sin^2(x)\right)^2\,\sin(x)\,dx \\
&= \int \cos^2(x)\left(1-\cos^2(x)\right)^2\,\sin(x)\,dx \\
&= -\int u^2(1-u^2)^2\,du \qquad\qquad u\;=\;\cos(x)\quad du\;=\;-\sin(x) \\
&= -\int \left(u^2\;-\;2u^4\;+\;u^6\right)\;du \\
&= -\frac{1}{3}u^3\;+\;\frac{2}{5}u^5\;-\;\frac{1}{6}u^6\;+\;C \\
&= -\frac{1}{3}\cos^3(x)\;+\;\frac{2}{5}\cos^5(x)\;-\;\frac{1}{6}\cos^6(x)\;+\;C.
\end{aligned}$$

Example 3 Evaluate $\int \cos^3(x)\,\sin^5\;dx$.

Solution Here both powers are odd, so we can decide whether we want to get rid of all but a single cos or all but a single sin. Let's go with getting rid of the cosines:

$$\begin{aligned}
\int \cos^3(x)\,\sin^5(x)\,dx &= \int \left(1-\sin^2(x)\right)\,\cos(x)\,\sin^5(x)\,dx \\
&= \int (1-u^2)\,u^5\,du \qquad\qquad u\;=\;\sin(x)\quad du\;=\;\cos(x)\,dx \\
&= \int (u^5\;-\;u^7)\,du \\
&= \frac{1}{6}u^6\;-\;\frac{1}{8}u^8\;+\;C \\
&= \frac{1}{6}\sin^6(x)\;-\;\frac{1}{8}\sin^8(x)\;+\;C.
\end{aligned}$$

1.1.2 Case where both powers are even

Here we study what can be done with, for example, $\int \sin^4(x)\,dx$ or $\int \cos^2(x)\,\sin^6(x)\,dx$.

One must use one of two *reduction formulae*:

$$\begin{aligned} \int \sin^n(x)\,dx &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \\ \int \cos^n(x)\,dx &= \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx. \end{aligned}$$

Example 4 Evaluate $\int \sin^2(x)\,dx$.

Solution Use the reduction formula for sin to get

$$\begin{aligned} \int \sin^2(x)\,dx &= -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int 1\,dx \\ &= -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + C. \end{aligned}$$

This can also be solved, for example, using a half-angle formula for sin.

Example 5 Evaluate $\int \sin^4(x)\,dx$.

Solution We have to use the reduction formula *twice*:

$$\begin{aligned} \int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx \\ &= -\frac{1}{4}\sin^3\cos(x) + \frac{3}{4}\left(-\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int 1\,dx\right) \\ &= -\frac{1}{4}\sin^3\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + C \end{aligned}$$

Example 6 Evaluate $\int \cos^4(x)\sin^2(x)\,dx$.

Solution First use $\sin^2(x) \;=\; 1-\cos^2(x)$ to convert entirely to cosines:

$$\begin{aligned}\int \cos^4(x)\sin^2(x)\,dx &= \int \cos^4(x)\left(1-\cos^2(x)\right)\,dx \\ &= \int \cos^4(x)\,dx \;-\; \int \cos^6(x)\,dx.\end{aligned}$$

Now we have two problems: evaluate $\int \cos^4(x)\,dx$ and evaluate $\int \cos^6(x)\,dx$. First things first: to evaluate $\int \cos^4(x)\,dx$, use the cosine reduction formula twice:

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\cos^3(x)\sin(x) \;+\; \frac{3}{4}\int \cos^2(x)\,dx \\ &= \frac{1}{4}\cos^3(x)\sin(x) \;+\; \frac{3}{4}\left(\frac{1}{2}\cos(x)\sin(x) \;+\; \frac{1}{2}\int 1\,dx\right) \;+\; C \\ &= \frac{1}{4}\cos^3(x)\sin(x) \;+\; \frac{3}{8}\cos(x)\sin(x) \;+\; \frac{3}{8}\,x \;+\; C.\end{aligned}$$

Now we evaluate $\int \cos^6(x)\,dx$ by using using the reduction formula three times:

$$\begin{aligned}\int \cos^6(x)\,dx &= \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{5}{6}\int \cos^4(x)\,dx \\ &= \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{5}{6}\left(\frac{1}{4}\cos^3(x)\sin(x) \;+\; \frac{3}{4}\int \cos^2(x)\,dx\right) \\ &= \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{5}{24}\cos^3(x)\sin(x) \;+\; \frac{15}{24}\int \cos^2(x)\,dx \\ &= \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{5}{24}\cos^3(x)\sin(x) \;+\; \frac{15}{24}\left(\frac{1}{2}\cos(x)\sin(x) \;+\; \frac{1}{2}\int 1\,dx\right) \\ &= \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{5}{24}\cos^3(x)\sin(x) \;+\; \frac{15}{48}\cos(x)\sin(x) \;+\; \frac{15}{48}\,x \;+\; C.\end{aligned}$$

Altogether, we get

$$\begin{aligned}\int \cos^4(x)\sin^2(x)\,dx &= \int \cos^4(x)\,dx \;-\; \int \cos^6(x)\,dx \\ &= \frac{1}{4}\cos^3(x)\sin(x) \;+\; \frac{3}{8}\cos(x)\sin(x) \;+\; \frac{3}{8}\,x \\ &\quad + \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{5}{24}\cos^3(x)\sin(x) \;+\; \frac{15}{48}\cos(x)\sin(x) \;+\; \frac{15}{48}\,x \;+\; C. \\ &= \frac{1}{6}\cos^5(x)\sin(x) \;+\; \frac{11}{24}\cos^3(x)\sin(x) \;+\; \frac{33}{48}\cos(x)\sin(x) \;+\; \frac{33}{48}\,x \;+\; C.\end{aligned}$$

1.2 Trigonometric substitution

One way of writing the main trigonometric identity is

$$1 \;-\; \sin^2(\theta) \;=\; \cos^2(\theta).$$

Some problems do not have trig functions, but have some pattern resembling the main trig identity. In these problems, we can substitute a trig function into the integral, and then use the trig identity to simplify the problem.

Example 7 Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Solution The $1-x^2$ resembles the trig identity $1-\sin^2(\theta) = \cos^2(\theta)$. Therefore use the substitution $x \;=\; \sin(\theta)$, $dx \;=\; \cos(\theta)\,d\theta$ to get

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-\sin^2(\theta)}}\,\cos(\theta)\,d\theta \\
&= \int \frac{1}{\sqrt{\cos^2(\theta)}}\,\cos(\theta)\,d\theta \\
&= \int d\theta \\
&= \theta \;+\; C \\
&= \sin^{-1}(\theta) \;+\; C.
\end{aligned}$$

Example 8 Evaluate $\int \frac{x}{1+x^2}\,dx$.

Solution We have to use the trig identity $1 + \tan^2(\theta) \;=\; \sec^2(\theta)$. Substitute $x = \tan(\theta)$, $dx \;=\; \sec^2(\theta)$ to get

$$\begin{aligned}
\int \frac{x}{1+x^2}\,dx &= \int \frac{\tan(\theta)}{1+\tan^2(\theta)}\,\sec^2(\theta)\,d\theta \\
&= \int \frac{\tan(\theta)}{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta \\
&= \int \tan(\theta)\,d\theta \\
&= \ln|\cos(\theta)| \;+\; C \\
&= \ln\left|\cos\left(\tan^{-1}(x)\right)\right| \;+\; C.
\end{aligned}$$