1 Lecture 5 - Examples of Integration by parts and substitution

Many techniques for integration exist, some harder than others.

When figuring out which techniques to use against a particular problem, you usually try the simplest technique first, and try harder and harder techniques if they seem necessary.

For example, substitution is easier that integration by parts, so you should usually try substitution first. If substitution doesn't work, consider using integration by parts.

Example 1 Find $\int v\sqrt{1+v^2}\,dv$.

Solution
There is a product inside the integral, so integration-by-parts is tempting. However, you should try a substitution first, just because it is easier:
Use $u = 1+v^2$, $du = 2vdv$ to get

$$\begin{aligned}
\int v\sqrt{1+v^2}\,dv &= \int v\left(1+v^2\right)^{\frac{1}{2}}\,dv \\
&= \frac{1}{2}\int u^{\frac{1}{2}}\,du \\
&= \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}}\cdot u^{-\frac{1}{2}} + C \\
&= -\left(1+v^2\right)^{-\frac{1}{2}} + C
\end{aligned}$$

Example 2 Find $\int y^5\,\ln(y)\,dy$.

Solution
Substitution will not work. For integration by parts, let u be something that gets *simpler* when you differentiation it. Thus we choose

$$\begin{aligned}
u &= \ln(y) \\
dv &= y^5\,dy
\end{aligned}$$

and we calculate

$$\begin{aligned} du &= y^{-1}\,dy \\ v &= \frac{1}{6}y^6 \end{aligned}$$

To get

$$\begin{aligned} \int u\,dv &= u\,v - \int v\,du \\ \int \ln(y)\,y^5\,dy &= \frac{1}{6}\,y^6\,\ln(y) - \frac{1}{6}\int y^6\,y^{-1}\,dy \\ &= \frac{1}{6}\,y^6\,\ln(y) - \frac{1}{6}\int y^5\,dy \\ &= \frac{1}{6}\,y^6\,\ln(y) - \frac{1}{36}y^6 + C. \end{aligned}$$

Example 3 Find $\int x^2 e^{-x}\, dx$.

Solution

This is another integration by parts problem. The function e^{-x} does not get simpler when you differentiate, however, the function x^2 *does*, so we should let

$$\begin{aligned} u &= x^2 \\ dv &= e^{-x}\, dx \end{aligned}$$

and compute

$$\begin{aligned} du &= 2x\, dx \\ v &= -e^{-x}. \end{aligned}$$

Thus

$$\begin{aligned} \int u\, dv &= u\, v - \int v\, du \\ \int x^2 e^{-x}\, dx &= -x^2 e^{-x} + 2 \int x\, e^{-x}\, dx \end{aligned}$$

Despite using integration by parts, we *still* have a difficult integral to evaluate. We have to use integration by parts again: choose

$$\begin{aligned} u &= x \\ dv &= e^{-x}\, dx \end{aligned}$$

and compute

$$\begin{aligned} du &= dx \\ v &= -e^{-x}. \end{aligned}$$

Thus

$$\begin{aligned} \int x^2 e^{-x}\, dx &= -x^2 e^{-x} + 2 \int x\, e^{-x}\, dx \\ &= -x^2 e^{-x} + 2 \left(-x\, e^{-x} + \int e^{-x}\, dx \right) \\ &= -x^2 e^{-x} - 2x\, e^{-x} - 2e^{-x} + C \end{aligned}$$

Example 4 Find $\int r^2\sqrt{1 + r^3}\,dr$.

Solution This is a substitution:

$$\begin{aligned} u &= 1+r^3 \qquad\qquad du = 3r^2dr \\ \int r^2\left(1+r^3\right)^{\frac{1}{2}}\,dr &= \frac{1}{3}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{3}\cdot\frac{1}{-\frac{1}{2}}\cdot u^{-\frac{1}{2}} \\ &= -\frac{2}{3}\cdot\left(1+r^3\right)^{-\frac{1}{2}}. \end{aligned}$$

Example 4 Find $\int r^5\sqrt{1\,+\,r^3}\,dr$.

Solution We use integration by parts: select

$$\begin{aligned} u &= r^3 \\ dv &= r^2\sqrt{1+r^3}. \end{aligned}$$

and compute

$$\begin{aligned} du &= 4r^2 \\ v &= -\frac{2}{3}\cdot\left(1+r^3\right)^{-\frac{1}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ \int r^5\left(1+r^3\right)^{\frac{1}{2}}\,dr &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{2}{3}\int r^2\left(1+r^3\right)^{-\frac{1}{2}}\,dr. \end{aligned}$$

The last integral on the right is *still* very difficult! But we can use the substitution

$$u = 1+r^3 \qquad\qquad du = 3r^2\,dr$$

to get

$$\begin{aligned} \int r^5\left(1+r^3\right)^{\frac{1}{2}}\,dr &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{2}{3}\left(\frac{1}{3}\int u^{-\frac{1}{2}}\,du\right) \\ &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{4}{27}u^{-\frac{3}{2}} \\ &= -\frac{2}{3}r^3\left(1+r^3\right)^{-\frac{1}{2}} + \frac{4}{27}\left(1+r^3\right)^{-\frac{3}{2}} + C \end{aligned}$$