1 Lecture 4 - Integration by parts

1.1 Odds and ends: the "d" operator

The notation

$$\triangle x$$

indicates a small but finite change in the variable x. The notation

$$dx$$

indicates an infinitesimal[1] change in the variable x. If $f(x)$ is a function, then the derivative of f w.r.t x, written

$$\frac{df}{dx}$$

literally indicates the infinitesimal change in f compared to the infinitesimal change in x.

But the operator "d" actually behaves like a derivative:

$$\begin{aligned} \text{given} \quad f(x) &= x^2 \\ \text{then} \quad df(x) &= 2x\,dx. \end{aligned}$$

If we divide both sides by dx, we get

$$\frac{df}{dx} = 2x,$$

which is the correct derivative. The d-operator (the infinitesimal change operator) is used very frequently in calculus.

[1] Mathematically speaking, this is imprecise. After all, what, exactly, is the mathematical meaning of "infinitesimal"? Maybe a number that is really really really really really small?? Maybe some kind of number smaller than all positive numbers but bigger that 0?? Or is and 'infinitesimal' really a number at all?? We shall sweep such questions under the rug.

1.2 Integration by parts for Indefinite integrals

The theory: Integration by parts is, roughly speaking, the product rule for integrals.

To develop integration by parts, we start with the product rule for derivatives: given arbitrary functions $v(x)$ and $u(x)$:

$$u(x), \qquad v(x)$$

Take "d" of the product $u(x)v(x)$:

$$d\ (u(x) \cdot v(x)) \ = \ du(x) \ \cdot \ v(x) \ + \ u(x) \cdot dv(x) \qquad \text{(product rule)}$$

Now take the integral, $\int$, of both sides:

$$\int d\,(u(x)\,v(x)) \ = \ \int du(x) \ \cdot \ v(x) \ + \ \int u(x) \cdot dv(x)$$

$$\text{rearrange}: \ \int u(x)\,dv(x) \ = \ \int d\,(u(x)\,v(x)) \ - \ \int v(x)\,du(x).$$

The **Fundamental Theorem of Calculus** states that the *integral of the derivative* gives the function back again: we can interpret this to mean $\int d(f(x)) = f(x)$, or in our case, that $\int d(u\,v) = u\,v$.

This gives us the **Formula for Integration by Parts:**

$$\int u\,dv \ = \ u \cdot v - \int v\,du.$$

Example 1 Find the antiderivative of $f(x) = x \sin(x)$.

Solution

We must evaluate

$$\int x \, \sin(x) \, dx.$$

This appears to be a product of two unlike functions, so we should use Integration by Parts:

Choose

$$\begin{aligned} u &= x \\ dv &= \sin(x). \end{aligned}$$

Then calculate

$$\begin{aligned} du &= dx \\ v &= -\cos(x). \end{aligned}$$

By the formula for integration by parts we get

$$\begin{aligned} \int u \, dv &= u \cdot v - \int v \, du \\ \int x \, \sin(x) \, dx &= -x \cos(x) + \int \cos(x) \, dx \\ &= -x \cos(x) + \sin(x). \end{aligned}$$

1.3 Integration by Parts for Definite integrals

The formula is nearly the same:

$$\int_a^b u\,dv \;=\; u\cdot v\Big|_a^b - \int_a^b v\,du.$$

<u>Example 2</u> Evaluate $\int_1^2 x\,e^x\,dx$.

<u>Solution</u>
This is a product of unlike functions, so we should use integration by parts. Choose

$$\begin{aligned} u &= x \\ dv &= e^x\,dx. \end{aligned}$$

The calculate

$$\begin{aligned} du &= dx \\ v &= e^x \end{aligned}$$

Thus we get

$$\begin{aligned} \int u\,dv &= u\cdot v - \int v\,du \\ \int_1^2 x\,e^x\,dx &= x\,e^x\Big|_1^2 - \int_1^2 e^x\,dx \\ &= x\,e^x\Big|_1^2 - \int_1^2 e^x\,dx \\ &= x\,e^x\Big|_1^2 - e^x\Big|_1^2 \\ &= 2\cdot e^2 - e^1 - e^2 + e^1 \\ &= e^2. \end{aligned}$$