1 Lecture 3 - Substitution in Integrals

1.1 Odds and ends

Given a function $f(x)$, its antiderivative can be denoted as either

$$F(x) \qquad \text{or} \qquad \int f(x)\,dx.$$

The antiderivative is also known as the *indefinite integral*.

An expression like

$$\int_1^2 x^3\,dx$$

involves no uncertainty whatsoever. It has a definite value: it comes to 15/16 if you work it out. However there *appears* to be a variable, namely x. But actual variables may freely take on any value, while x is constrained to move between 1 and 2. The x in this case is called an *apparent variable*; it's only there to serve as a placeholder until the integral has been evaluated.

Example of an FTC I problem Given $h(x) = \int_{e^x}^0 \sin^3(t)\,dt$, find $h'(x)$. Solution: use the substitution $u = e^x$ to get

$$\begin{aligned}
\frac{dh}{dx} &= -\frac{d}{dx}\int_0^{e^x} \sin^3(t)\,dt \\
&= -\frac{du}{dx}\frac{d}{du}\int_0^{u} \sin^3(t)\,dt \\
&= -e^x \sin^3(u) \\
&= -e^x \sin^3\left(e^x\right).
\end{aligned}$$

1.2 Substitution in Integrals

- The rule of thumb is to pick the substitution u to be either 1) a function inside a function, or 2) the denominator.
- You must convert all x's and dx's to u's and du's
- For indefinite integrals, switch back to x's at the end
- For definite integrals, you don't need to switch back to x's, but you do need to convert the limits.

Example 1 (Indefinite Integral) Find the antiderivative of $f(x) = \sqrt{2x+1}$.

Solution

$$\begin{aligned}
F(x) &= \int \sqrt{2x+1}\, dx \\
& \quad \text{substitute} \quad u = 2x+1 \\
& \qquad\qquad \frac{du}{dx} = 2 \qquad \frac{1}{2}du = dx \\
F(x) &= \int \sqrt{u}\, \frac{1}{2}\, du \\
&= \frac{1}{2} \int u^{\frac{1}{2}}\, du \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} + C \\
&= 3\,(2x+1)^{\frac{3}{2}} + C
\end{aligned}$$

Example 2 (Indefinite Integral) Find the antiderivative of $g(x) = x^2\sqrt[3]{x^3+2}$.

Solution

$$
\begin{aligned}
G(x) &= \int x^2\,(x^3+2)^{\frac{1}{3}}\,dx \\
&\quad \text{substitute} \quad u = x^3+2 \\
&\qquad\qquad \frac{du}{dx} = 3x^2 \qquad \frac{1}{3}du = x^2\,dx \\
G(x) &= \int u^{\frac{1}{3}}\,\frac{1}{3}\,du \\
&= \frac{1}{3}\cdot\frac{3}{4}\cdot u^{\frac{4}{3}} + C \\
&= \frac{1}{4}\left(x^3 + 2\right)^{\frac{4}{3}} + C.
\end{aligned}
$$

Example 3 (Definite Integral) Find $\int_0^1 \frac{e^x}{e^x+1}\,dx$.

Solution Use $u = e^x + 1$, so that $du = e^x\,dx$.

$$
\begin{aligned}
\int_1^2 \frac{e^x}{e^x+1}\,dx &= \int_{e+1}^{e^2+1} \frac{1}{u}\,du \\
&= \ln|u|\Big|_{e+1}^{e^2+1} = \ln(e^2+1) - \ln(e+1) \\
&= \ln\left(\frac{e^2+1}{e+1}\right).
\end{aligned}
$$

Example 4 (Definite Integral) Find $\int_0^{\frac{\pi}{4}} \frac{\sin(x)\cos(x)}{1+\cos^2(x)} dx$.

Solution Use $u = 1 + \cos^2(x)$, so that $du = -2\cos(x)\sin(x)$.

We will also have to change the limits:

- Lower limit: $x = 0$ implies $u = 2$
- Upper limit: $x = \frac{\pi}{4}$ implies $u = \frac{3}{2}$

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin(x)\cos(x)}{1+\cos^2(x)} dx &= -\frac{1}{2}\int_2^{\frac{3}{2}} \frac{du}{u} \\
&= -\frac{1}{2}\int_2^{\frac{3}{2}} u^{-1}\, du \\
&= -\frac{1}{2}\ln|u|\Big|_2^{\frac{3}{2}} \\
&= \left(-\frac{1}{2}\ln\left(\frac{3}{2}\right)\right) - \left(-\frac{1}{2}\ln(2)\right) \\
&= -\frac{1}{2}\ln\left(\frac{4}{3}\right).
\end{aligned}$$