1 Lecture 20 - Basic concepts of differential equations

A differential equation is an equation relating a function to one or more of its derivatives.

The first main example is Newton's Law of Cooling. If a body of temperature T is immersed in surroundings of constant temperature T_{AMB}, Newton says the rate of change in the body's temperature is proportional to the difference between the body's temperature and the ambient temperature. That is

$$\frac{dT}{dt} = -k\,(T - T_{AMB}),$$

which is a first-order differential equation.

The second main example is Hooke's Law, which gives the force exerted by an ideal spring as $F = -kx$, where k is the spring constant and x is the displacement from the natural length. Combine this with Newton's Second Law $F = m\frac{d^2x}{dt^2}$, and we have the following equation for a mass-spring problem:

$$m\frac{d^2x}{dt^2} = -k\,x$$

which is a second-order differential equation.

<u>Example 1</u> Show that $x(t) = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$ solves the differential equation $\frac{d^2x}{dt^2} = -2x$.

<u>Solution</u> We compute

$$\begin{aligned}
\frac{d}{dt}\frac{d}{dt}x(t) &= \frac{d}{dt}\left(-\sqrt{2}A\sin(\sqrt{2}t) + \sqrt{2}B\cos(\sqrt{2}t)\right) \\
&= -2A\cos(\sqrt{2}t) - 2B\sin(\sqrt{2}t) \\
&= -2x.
\end{aligned}$$

Example 2 Show that $y(t) = \frac{1}{\sqrt{-2t+C}}$ solves the differential equation $\frac{dy}{dt} = y^3$.

Solution We compute

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(-2t + C)^{-\frac{1}{2}} \\ &= -\frac{1}{2}(-2t + C)^{-\frac{3}{2}} \cdot (-2) \\ &= (-2t + C)^{-\frac{3}{2}} \\ &= y^3. \end{aligned}$$

Example 3 Find the solution to $\frac{dy}{dt} = y^3$ given that $y(0) = \frac{1}{2}$.

Solution We know that $y(t) = (-2t + C)^{-\frac{1}{2}}$ solves the differential equation, but we still have the unknown constant C to deal with. To find the value of C we use the initial condition

$$\begin{aligned} y(0) &= (-2(0) + C)^{-\frac{1}{2}} \\ \frac{1}{2} &= \frac{1}{\sqrt{C}} \\ C &= 4. \end{aligned}$$

Thus the solution is

$$y(t) = \frac{1}{\sqrt{4 - 2t}}.$$