1 Lecture 2 - FTC I

1.1 The Chain Rule

We begin with a review of old material. Everyone should be familiar with the chain rule:

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

It is equally important to understand the chain rule in differential notation:

$$\text{substitution} : u = g(x)$$

$$\begin{aligned}\frac{d}{dx}(f(g(x))) &= \frac{d}{dx}(f(u)) \\ &= \frac{du}{dx}\frac{d}{du}(f(u)).\end{aligned}$$

Often, this version of the chain rule is written in abbreviated form $\frac{d}{dx} = \frac{du}{dx}\frac{d}{du}$.

Example 1: Evaluate $\frac{d}{dx}\sqrt{x^2+1}$.
Solution: Use the substitution $u = x^2 + 1$ to write

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}\sqrt{u}$$

But there is a problem! The function is $\sqrt{u}$, with variable u, but the derivative $\frac{d}{dx}$ is with respect to x, not u!
To rectify this, use the **chain rule** $\frac{d}{dx} = \frac{du}{dx}\frac{d}{du}$, to get

$$\begin{aligned}\frac{d}{dx}\sqrt{x^2+1} &= \frac{d}{dx}\sqrt{u} \\ &= \frac{du}{dx} \cdot \frac{d}{du}\sqrt{u} \\ &= 2x \cdot \frac{1}{2}u^{-1/2} \\ &= x \cdot \left(x^2+1\right)^{-1/2}.\end{aligned}$$

Example 2: Evaluate $\frac{d}{dx}\sin\left(x^4+x\right)$.
Solution: use $u = x^4 + x$ and the chain rule $\frac{d}{dx} = \frac{du}{dx}\frac{d}{dx}$ to get

$$\begin{aligned}
\frac{d}{dx}\sin\left(x^4+x\right) &= \frac{d}{dx}\sin(u) \\
&= \frac{du}{dx}\cdot\frac{d}{du}\left(\sin(u)\right) \\
&= \frac{d}{dx}\left(x^4+x\right)\cdot\frac{d}{du}\left(\sin(u)\right) \\
&= \left(3x^3+1\right)\cdot\cos(u) \\
&= \left(3x^2+1\right)\cdot\cos\left(x^4+x\right).
\end{aligned}$$

1.2 FTC I

The first version of the fundamental theorem of calculus states explicitly that the derivative is the inverse of the integral

Theorem 1.1 (Fundamental Theorem of Calculus, version I)

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$

□

Example 3. Evaluate

$$\frac{d}{dx}\int_{-1}^x u^4\,du$$

in two ways: a) by directly evaluating and b) by using the fundamental theorem.
a) Direct evaluation:

$$\begin{aligned}
\frac{d}{dx}\left(\int_{-1}^x u^4\,du\right) &= \frac{d}{dx}\left(\frac{1}{5}u^5\Big|_{-1}^x\right)\\
&= \frac{d}{dx}\left(\frac{1}{5}x^5 - \frac{1}{5}(-1)^5\right)\\
&= \frac{d}{dx}\left(\frac{1}{5}x^5\right)\\
&= x^4
\end{aligned}$$

b) Fundamental theorem: there is no work involved! This problem fits the pattern of the fundamental theorem exactly:

$$\frac{d}{dx}\int_{-1}^x u^4\,du = x^4.$$

Example 4. Use the fundamental theorem to evaluate

$$\frac{d}{dx}\int_2^{x^2}(u^2+1)\,du$$

Solution: This problem DOES NOT directly fit the pattern of the fundamental theorem. We have to use a substitution

$$v = x^2$$
$$\frac{d}{dx}\int_2^v (u^2+1)\,du.$$

Now our variable is v, but the derivative is with respect to x. Thus we use the chain rule: $\frac{d}{dx} = \frac{dv}{dx}\frac{d}{dv}$ to get

$$\begin{aligned} \frac{d}{dx}\int_2^v (u^2+1)\,du &= \frac{dv}{dx}\cdot\frac{d}{dv}\int_2^v (u^2+1)\,du \\ &= 2x\cdot\big(v^2+1\big) \\ &= 2x\cdot\big(x^4+1\big) = 2x^5 + 2x. \end{aligned}$$