

1 Lecture 19 - Applications to work and finance

1.1 Pumping Water out of tanks

Knowing the geometry of a tank and the amount of fluid it holds, one can determine how much work is required to pump the fluid out. One considers thin ‘slabs’ of water of constant gravitational potential, and of thickness dx . To find the work required to lift that ‘slab’ out of the tank, you must calculate its weight (force due to gravity) and the distance it is lifted against the gravitational pull.

Example 1 A tank has semicircular cross section of radius $4m$, and length of $8m$. If the tank is filled with water up to a depth of $4m$, determine the work needed to pump the water out.

Solution Let x measure the height above the tank’s bottom. Consider a ‘slab’ of water, at height x , of thickness dx . Using basic geometry, you can calculate the width of the slab to be $2\sqrt{8x - x^2}$. Its length is obviously 8. Thus the slab’s volume is

$$dV = 16\sqrt{8x - x^2} dx.$$

The density of water is $1000kg/m^3$, and weight (force due to gravity) obeys $F = mg$ with $g = 9.8m/s^2$, so we get

$$\text{Infinitesimal weight} = 156800\sqrt{8x - x^2} dx.$$

The slab must be lifted $4 - x$ meters, so the work done to the slab is

$$\begin{aligned} dW &= F \cdot s \\ &= 156800\sqrt{8x - x^2} dx \cdot (4 - x). \end{aligned}$$

Thus the total work is

$$\begin{aligned} W &= \int dW \\ &= 156800 \int_0^4 (4-x)\sqrt{8x-x^2} dx \\ &= 88400 \int_0^{16} u^{\frac{1}{2}} du \\ &= 88400 \left. \frac{2}{3} u^{3/2} \right|_0^{16} \\ &= \frac{10135200}{3}. \end{aligned}$$

1.2 Applications to Finance

If you sell exactly X many items, you must charge P dollars. If you charge any more, you will sell fewer items. If you charge less, you will sell more. We call $P = P(X)$ the demand function. In addition we have the cost function $C = C(X)$ giving the total cost of producing X units, the revenue function $R = R(X)$ giving the total revenue obtained by selling X units (obviously $R(X) = X \cdot P(X)$), and the profit function $T(X) = R(X) - C(X)$.

The *marginal* cost is the derivative of the cost function $C'(X)$, and likewise for the other quantities: marginal revenue is $R'(x)$, marginal profit is $T'(X)$.

The *consumer surplus* is the total amount of value your customers receive from doing business with you. Namely, if you charge 1 dollar for something that a consumer was willing to pay 2.50 for, the consumer receives a value of 1.50.

Example 2 The demand function for your product is $P = \frac{8}{1+x^2}$. Suppose you sell 10 units. What is the consumer surplus?

Solution To sell 10 units the price you charge is $P = \frac{8}{101}$, just under 8 cents. The consumer surplus is the total amount of value received by the consumer: the integrate of the price consumers would have paid (ie, what it is worth to

them), minus the price they actually paid.

$$\begin{aligned}\text{consumer surplus} &= \int_0^{10} \left(\frac{8}{1+x^2} - \frac{8}{101} \right) dx \\ &= 8 \int_0^{10} \frac{1}{1+x^2} dx - \frac{800}{101} \\ &= 8 \tan^{-1}(x) \Big|_0^{10} - \frac{800}{101} \\ &\approx 3.58\end{aligned}$$

So consumers netted a grand total of \$3.58 of value.