

1 Lecture 18 - Applications to Work

The principle behind the applications here is that if you can solve the infinitesimal problem, then the large-scale problem can be solved by integrating.

If a particle moves in a straight path for a distance of s under the influence of the constant force F directed either with or directly against the motion, the work done by that force is

$$W = F s.$$

Of course force is often nonconstant and motion is rarely in a straight line, so this formula has limited applicability. But even if the force is nonconstant, it will still be approximately constant on small enough length scales, so that the formula still holds on the infinitesimal scale:

$$dW = F ds.$$

Example 1 A particle moves along the x-axis from $x = 0$ to $x = 1$ under the influence of the force $F = (1 + x^2)^{-1}$. Find the work done by this force.

Solution The displacement variable is x (not s) so we use the formula $dW = F dx = \frac{dx}{1+x^2}$. Then

$$\begin{aligned} W &= \int dW \\ &= \int_0^1 \frac{dx}{1+x^2} \\ &= \int_0^{\pi/4} \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta \\ &= \int_0^{\pi/4} d\theta = \frac{\pi}{4}. \end{aligned}$$

Example 2 (problem 7 from section 6.5) Suppose 2J of work is needed to stretch a spring from its natural length of 30cm to a length of 42cm. Then (a) How much work is needed to stretch it from 30cm to 40cm and (b) how far beyond its natural length will a force of 30 N keep the spring stretched?

Solution The one thing we know about springs is Hooke's law: $F = kx$ where x is the distance the spring is stretched from its natural length, and k is the (usually unknown) spring constant. The spring's natural length is 30 cm, so it is stretched 12 cm from the natural length, we can use

$$\begin{aligned}
 W &= \int dW = \int F dx \\
 2 J &= \int_0^{.12m} k x dx \\
 &= \left. \frac{k}{2} x^2 \right|_0^{.12m} \\
 &= \frac{k}{2} .0144 m^2 \\
 k &= \frac{4Nm}{.0144m^2} = \frac{100}{9} N/m
 \end{aligned}$$

(recall $1 J = 1 N \cdot m$). There is one and only one piece of information that characterizes an ideal spring: its spring constant. Knowing the spring constant allows us to find out any other information we need. The solution to part (a) is

$$\begin{aligned}
 W &= \int dW = \int_0^{.1} F dx \\
 &= \int_0^{.1} \frac{100}{9} x dx \\
 &= \left. \frac{50}{9} x^2 \right|_0^{.1} \\
 &= \frac{1}{18}
 \end{aligned}$$

The solution to part (b) is even simpler: given that the force is 30 N, we get

$$\begin{aligned}
 F &= kx \\
 30 &= \frac{100}{9} x \\
 x &= \frac{27}{10}
 \end{aligned}$$

Example 3 (problem 11 from section 6.5) A cable weighing 2 lbs/ft is used to lift 800 lbs of coal up a mineshaft 500 ft deep. find out how much work is done.

Solution We divide into two parts: the work needed to lift the coal and the work needed to lift the cable. The work lifting the coal is easy: it weighs 800 lbs and is lifted 500 ft, so

$$\begin{aligned} W &= F s \\ &= 800 \cdot 500 \\ &= 400000. \end{aligned}$$

Finding the work done lifting the cable is a little harder. Let x be the height from the shaft's bottom. A piece of cable, at initial position x and of length dx , must be lifted $500 - x$ feet. The force on the piece of cable due to gravity is $dF = 2 \cdot dx$, so the work done to lift it is $dW = 2dx \cdot (500 - x)$. Thus

$$\begin{aligned} W &= \int dW \\ &= \int_0^{500} 2(500 - x) dx \\ &= 2\left(500x - \frac{1}{2}x^2\right)\Big|_0^{500} \\ &= 250000. \end{aligned}$$

Thus the total work done is

$$\begin{aligned} W_{tot} &= W_{coal} + W_{cable} \\ &= 650000. \end{aligned}$$