

1 Lecture 13 - Volumes of rotation, slicing, and arclength

1.1 Volumes of Rotation - Disks and Washers

Volume of a shells:

$$dV = 2\pi R h dx$$

where R is the radius, h is the height, and dx is the thickness of the shell.

Volume of a disks:

$$dV = \pi R^2 dx$$

where R is the disk's radius and dx is its thickness.

Volume of a washer:

$$dV = \pi (R^2 - r^2) dx$$

where R is the outer radius, r is the inner radius, and dx is the thickness.

Example 1 Find the volume of the solid formed by rotating the region between the graph of $y = 1 - x^2$ and $y = 0$ about the x-axis.

Solution If you were finding the area of bounded by these graphs, a test rectangle would have height $f(x) = 1 - x^2$ and width dx . Rotating a test rectangle gives a disk of radius $1 - x^2$ and thickness dx . Thus a given disk has volume $dV = \pi (1 - x^2)^2 dx$. Adding up all the volumes gives

$$\begin{aligned} V &= \int dV \\ &= \int_{-1}^1 \pi (1 - x^2)^2 dx \\ &= \pi \int_{-1}^1 (1 - 2x^2 + x^4) dx \\ &= \pi \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_{-1}^1 \\ &= \pi \frac{16}{15}. \end{aligned}$$

Example 2 Find the volume of the solid formed by rotating the region between the graphs of $f(x) = 4 - x^2$ and $g(x) = 6 - 3x$ around the x-axis.

Solution In the bounded region, $f(x)$ is the top graph and $g(x)$ is the bottom graph. The points of intersection are found by setting $f(x) = g(x)$ and solving:

$$\begin{aligned}4 - x^2 &= 6 - 3x \\0 &= 2 - 3x + x^2 \\0 &= (2 - x)(1 - x)\end{aligned}$$

so the points of intersection are $x = 1$ and $x = 2$.

If you were to find area between the graphs, your test rectangle would have height $f(x) - g(x)$ and width dx . Rotating about the x-axis, you get a washer with outer radius $f(x)$, inner radius $g(x)$, and thickness dx . Thus the volume of a given washer is

$$\begin{aligned}dV &= \pi (f(x)^2 - g(x)^2) dx \\&= \pi ((4 - x^2)^2 - (6 - 3x)^2) dx \\&= \pi (-20 + 36x - 17x^2 + x^4) dx\end{aligned}$$

Thus the total volume is

$$\begin{aligned}V &= \int dV \\&= \int_1^2 \pi (-20 + 36x - 17x^2 + x^4) dx \\&= \pi \left(-20x + 18x^2 - \frac{17}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_1^2 \\&= \pi \frac{812}{35}.\end{aligned}$$

Example 3 Find the volume of the solid formed by rotating the region between the graphs of $f(x) = x^2$ and $g(x) = -x + 2$ around the line $y = 4$.

Solution This time $g(x)$ is the top graph and $f(x)$ is the bottom graph. The points of intersection are found by setting $f(x) = g(x)$ and solving:

$$\begin{aligned}x^2 &= -x + 2 \\x^2 + x - 2 &= 0 \\(x + 2)(x - 1) &= 0\end{aligned}$$

so the points of intersection are $x = -2$ and $x = 1$.

If you were to find area between the graphs, your test rectangle would have height $g(x) - f(x)$ and width dx . Rotating about the line $y = 4$, you get a washer with outer radius $4 - f(x)$, inner radius $4 - g(x)$, and thickness dx . Thus the volume of a given washer is

$$\begin{aligned}dV &= \pi ((4 - f(x))^2 - (4 - g(x))^2) dx \\&= \pi ((4 - x^2)^2 - (2 + x)^2) dx \\&= \pi (12 + 4x - 7x^2 + x^4) dx\end{aligned}$$

Thus the total volume is

$$\begin{aligned}V &= \int dV \\&= \int_{-2}^1 \pi (12 + 4x - 7x^2 + x^4) dx \\&= \pi \left(12x + 2x^2 - \frac{7}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_{-2}^1 \\&= \pi \frac{78}{5}.\end{aligned}$$

1.2 Slicing

Not every 3-dimensional shape is a solid of revolution.

Example 4 Consider the region between the graph of $y = \sqrt{1-x^2}$ and the x-axis. A solid is formed by letting each vertical segment be the base of an equilateral triangle. Find the volume of the solid.

Solution A test rectangle would have height $f(x) = \sqrt{1-x^2}$ and width dx . But each test rectangle forms the base of a very thin triangular prism. The formula of the area of an equilateral triangle of sidelength s is $A = \frac{\sqrt{3}}{4}s^2$. The formula for the volume of a prism is $V = (\text{Face Area}) \cdot (\text{thickness})$. Thus each thin prism has side-length $\sqrt{1-x^2}$, thickness dx , and volume

$$\begin{aligned}dV &= (\text{Face Area}) \cdot (\text{thickness}) \\&= \frac{\sqrt{3}}{4} \left(\sqrt{1-x^2}\right)^2 dx \\&= \frac{\sqrt{3}}{4} (1 - 2x^2 + x^4) dx\end{aligned}$$

Thus the volume of the solid is

$$\begin{aligned}V &= \int dV \\&= \int_{-1}^1 \frac{\sqrt{3}}{4} (1 - 2x^2 + x^4) dx \\&= \frac{\sqrt{3}}{4} \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^4 \right) \Big|_{-1}^1 \\&= \frac{4\sqrt{3}}{15}.\end{aligned}$$

1.3 Arclength

The problem is to determine the *length* of a curve, not the area under the curve. As always, we look at the problem on the infinitesimal level first, where the Pythagorean Theorem indicates proves that the infinitesimal length of arc dl (called the *arclength element*) is related to the infinitesimal changes in the x- and y-coordinate values, dx and dy , by

$$dl = \sqrt{(dx)^2 + (dy)^2}.$$

If the curve in question is given by the graph of a function, $y = f(x)$, then we can manipulate this

$$\begin{aligned} dl &= \sqrt{\left(1 + \frac{(dy)^2}{(dx)^2}\right) (dx)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \sqrt{(dx)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \end{aligned}$$

Example 5 Find the length of the curve given by the graph of $y = \frac{1}{2}x^2$ for $-1 < x < 1$.

Solution We compute the arclength element

$$\begin{aligned} dl &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \sqrt{1 + x^2} dx, \end{aligned}$$

then integrate

$$\begin{aligned} \text{Arclength} &= \int dl \\ &= \int_{-1}^1 \sqrt{1 + x^2} dx. \end{aligned}$$

A trigonometric substitution is require: use $x = \tan \theta$, $dx = \sec^2 \theta d\theta$ to get

$$\begin{aligned} \text{Arclength} &= \int_{-1}^1 \sqrt{1+x^2} dx \\ &= \int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 \theta} \sec^2 \theta d\theta \\ &= \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 \theta} \sec^2 \theta d\theta \\ &= \int_{-\pi/4}^{\pi/4} \sec^3 \theta d\theta. \end{aligned}$$

Now we use the reduction formula for the sec function to get

$$\begin{aligned} \text{Arclength} &= \int_{-\pi/4}^{\pi/4} \sec^3 \theta d\theta \\ &= \frac{1}{2} \tan \theta \sec \theta \Big|_{-\pi/4}^{\pi/4} + \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec \theta d\theta \\ &= \frac{1}{2} \tan \theta \sec \theta \Big|_{-\pi/4}^{\pi/4} + \frac{1}{2} \ln (\sec \theta + \tan \theta) \Big|_{-\pi/4}^{\pi/4} \\ &= \sqrt{2} + \frac{1}{2} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right). \end{aligned}$$