1 Lecture 12 - Areas between graphs, and volumes of rotation

1.1 Areas between graphs

The area under the graph of $f(x)$, between $x = a$ and $x = b$ is

$$\int_a^b f(x)\,dx.$$

Now consider two functions $f(x)$ and $g(x)$ which intersect at $x = x_0$ and $x = x_1$. The area under $f(x)$ between x_0 and x_1 is $\int_{x_0}^{x_1} f(x)\,dx$, and the area under $g(x)$ between x_0 and x_1 is $\int_{x_0}^{x_1} g(x)\,dx$. To get the area between the graphs, you subtract:

$$\int_{x_0}^{x_1} (f(x) - g(x))\,dx \qquad \text{or} \qquad \int_{x_0}^{x_1} (g(x) - f(x))\,dx,$$

depending on which one has the upper graph and which one has the lower graph.

Example 1 Find the area bounded between the graphs of $f(x) = 3x + 4$ and $g(x) = x^2$.

Solution First we find the points of intersection by setting the functions equal to each other:

$$\begin{aligned} f(x) &= g(x) \\ 3x + 4 &= x^2 \\ 0 &= x^2 - 3x - 4 \\ 0 &= (x-4)(x+1). \end{aligned}$$

Thus the points of intersection are $x = -1$ and $x = 4$. It can easily be seen from the graphs that $f(x) = 3x - 4$ is the upper function. Thus the bounded area is

$$\begin{aligned}
\inf_{-1}^{4} \left(3x + 4 - x^2\right) dx &= \frac{3}{2}x^2 + 4x - \frac{1}{3}x^3\Big|_{-1}^{4} \\
&= \left(\frac{3}{2}\cdot 16 + 4\cdot 4 - \frac{1}{3}\cdot 64\right) - \left(\frac{3}{2} - 4 + \frac{1}{3}\right) \\
&= 44 - \frac{3}{2} - \frac{65}{3} = \frac{125}{6}.
\end{aligned}$$

1.2 Solids of rotation: Shells

The volume of a shell of radius R height h and thickness dx is

$$dV = 2\pi\, R\, h\, dx.$$

Example 2 Find the area of the solid obtained by rotating about the y-axis the region bounded by $f(x) = -x^2 + 5x - 6$ and the x-axis.

Solution The graph of $f(x) = -x^2 + 5x - 6 = (3 - x)(x - 2)$ is a downward opening parabola that intersects the x-axis at $x = 2$ and $x = 3$. At each value of x lies a test-rectangle of height $f(x)$ and width dx. This rectangle, when rotated about the y-axis, produces a shell of radius x, height $f(x)$, and width dx. Thus each shell has volume

$$dV = 2\pi\, f(x)\, x\, dx.$$

Summing up the infinitesimal volumes, we get

$$\begin{aligned}
V &= \int_2^3 dV \\
&= \int_2^3 2\pi \left(-x^2 + 5x - 6\right) x\, dx \\
&= 2\pi \int_2^3 \left(-x^3 + 5x^2 - 6x\right) dx \\
&= 2\pi \left(-\frac{1}{4}x^4 + \frac{5}{3}x^3 - 3x^2\right) \Big|_2^3 \\
&= \frac{\pi}{12}.
\end{aligned}$$