1 Lecture 11 - Improper Integrals

An integral $\int_{x_0}^{x_1} f(x)\,dx$ is considered 'improper' if

1) $f(x)$ is 'singular,' meaning there is an infinite discontinuity, somewhere in the closed interval $[x_0, x_1]$
2) Either $x_0 = -\infty$ or $x_1 = \infty$ or both.

1.1 First type: $f(x)$ is singular somewhere

You CANNOT integrate right up to, or across, an infinite discontinuity. You must use limits to APPROACH any discontinuities.

If $f(s)$ is singular, we have to set up the integral as

$$\int_{x_0}^{x_1} f(x)\,dx = \lim_{a\to s^+} \int_a^t f(x)\,dx.$$

If $f(t)$ is singular, we have to set up the integral as

$$\int_{x_0}^{x_1} f(x)\,dx = \lim_{a\to t^-} \int_s^a f(x)\,dx.$$

If $f(m)$ is singular for some m between x_0 and x_1, we have to set up the integral as

$$\begin{aligned}\int_{x_0}^{x_1} f(x)\,dx &= \int_{x_0}^{m} f(x)\,dx + \int_{m}^{x_1} f(x)\,dx \\ &= \lim_{a\to m^-} \int_{x_0}^{a} f(x)\,dx + \lim_{b\to m^+} \int_{b}^{x_1} f(x)\,dx.\end{aligned}$$

1.2 Second type: and infinity in the limits of integration

If $x_1 = \infty$, set up the integral

$$\int_{x_0}^{\infty} f(x)\,dx = \lim_{a\to\infty} \int_{x_0}^{a} f(x)\,dx$$

If $x_0 = -\infty$, set up the integral

$$\int_{\infty}^{x_1} f(x)\,dx \;=\; \lim_{a \to -\infty} \int_a^{x_1} f(x)\,dx$$

If both $x_0 = -\infty$ and $x_1 = \infty$, set up the integral

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx \;+\; \int_0^{\infty} f(x)\,dx \\ &= \lim_{a \to -\infty} \int_a^0 f(x)\,dx \;+\; \lim_{b \to \infty} \int_0^b f(x)\,dx.\end{aligned}$$

1.3 The comparison test

Sometimes it will not be possible to evaluate an integral directly, for example

$$\int_1^\infty \frac{1}{\sqrt{1+x^3}}\, dx.$$

Nevertheless, it is often possible to determine of the integral is finite or not, using the comparison test.

Consider an improper integral

$$\int_a^b f(x)\, dx.$$

(ie, possibly $a = -\infty$, $b = \infty$, or $f(x)$ has a singularity somewhere). The first step is to choose a comparison function $g(x)$.

To prove that the integral $\int_a^b f(x)dx$ *converges*, you must

- Prove that the integral of the comparison function converges, namely that $\int_a^b g(x)dx$ is finite
- Prove that the function is *absolutely smaller* than the comparison function, namely that $|f(x)| \leq g(x)$.

To prove that the integral $\int_a^b f(x)dx$ *diverges*, you must

- Prove that the integral of the comparison function diverges, namely that $\int_a^b g(x)dx$ is infinite
- Prove that the function is *bigger* than the comparison function, namely that $f(x) \geq g(x)$.

Choosing an appropriate comparison function is largely a matter of intuition. After you've done a few examples, you can usually tell pretty well what the comparison function should be. Basically the idea is to extract the key features of the original function.

Example 1 Is $\int_1^\infty \frac{1}{\sqrt{1+x^3}}\,dx$ finite or infinite?

Solution It is not possible to evaluate the integral directly, but we can use the comparison test.

Our function is $f(x) = \frac{1}{\sqrt{1+x^3}}$. First we try to pick a comparison function. Since we are integrating up to infinity, and since $1 + x^3 \approx x^3$ as x gets big, we should let

$$g(x) = \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$$

be our comparison function. Since $\int_1^\infty g(x)\,dx$ is FINITE (by the p-test), we guess that the original integral is also finite. Our chain of inequalities is

$$f(x) = \frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}} = g(x).$$

Thus $f(x) < g(x)$, so it follows that $\int_1^\infty f(x)dx < \int_1^\infty g(x)dx < \infty$.

Example 2 Is $\int_3^\infty \frac{1}{\sqrt[4]{x^4-1}}\,dx$ finite or infinite?

Solution Again, we cannot evaluate directly. Since we are integrating up to ∞, and since $x^4 - 1 \approx x^4$ when x is very big, we should let

$$g(x) = \frac{1}{\sqrt[4]{x^4}} = \frac{1}{x}$$

be our comparison function. We know, by the p-test, that $\int_3^\infty g(x)dx = \infty$. Now we try to compare $f(x)$ to $g(x)$ by using the chain of inequalities

$$f(x) = \frac{1}{\sqrt[4]{x^4-1}} > \frac{1}{\sqrt[4]{x^4}} = \frac{1}{x} = g(x).$$

Therefore $\int_3^\infty f(x)\,dx > \int_3^\infty g(x)\,dx = \infty$, so therefore the original integral diverges.