1 Lecture 10 - Simpson's rule and examples

1.1 S_n, Simpson's rule with n intervals

Simpson's rule uses small pieces of parabolas (that is, graphs of the kind $y = ax^2 + bx + c$) to approximate the graphs. One can easily figure out the equation of a parabola passing through three points, say $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_{i+1}, f(x_{i+1}))$. and one can easily figure out a general formula for the area under a small piece of a parabola. In the end, we get

$$\begin{aligned} \triangle x &= \frac{b-a}{n} \\ x_i &= a + i\triangle x \end{aligned}$$

and the total approximate area under the curve is

$$\begin{aligned} &\text{approximate area under the graph} \\ &\quad = \frac{1}{3}\triangle x\,\left(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right). \end{aligned}$$

1.2 Examples of uses of approximation techniques

Example 1 Approximate the integral $\int_1^3 x^2\,dx$ using L_4, R_4, M_4, T_4, and S_4.

Solution
First off, in all cases we have $n = 4$, so we can use the formulas to find

$$\begin{aligned} \triangle x &= \frac{3-1}{4} = \frac{1}{2} \\ x_0 &= 1 \quad x_1 = \frac{3}{2} \quad x_2 = 2 \quad x_3 = \frac{5}{2} \quad x_4 = 3. \end{aligned}$$

Then

$$
\begin{aligned}
L_4 &= \sum_{i=1}^{4} f(x_{i-1}) \triangle x \\
&= f(x_0)\triangle x \;+\; f(x_1)\triangle x \;+\; f(x_2)\triangle x \;+\; f(x_3)\triangle x \\
&= (1)^2 \cdot \frac{1}{2} \;+\; \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} \;+\; (2)^2 \cdot \frac{1}{2} \;+\; \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} \\
&= \frac{27}{4} \;=\; 6.75.
\end{aligned}
$$

$$
\begin{aligned}
R_4 &= \sum_{i=1}^{4} f(x_i) \triangle x \\
&= f(x_1)\triangle x \;+\; f(x_2)\triangle x \;+\; f(x_3)\triangle x \;+\; f(x_4)\triangle x \\
&= \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} \;+\; (2)^2 \cdot \frac{1}{2} \;+\; \left(\frac{5}{2}\right)^2 \cdot \frac{1}{2} \;+\; (3)^2 \cdot \frac{1}{2} \\
&= \frac{43}{4} \;=\; 10.75.
\end{aligned}
$$

$$
\begin{aligned}
M_4 &= \sum_{i=1}^{4} f\left(\frac{x_{i-1}+x_i}{2}\right) \triangle x \\
&= f\left(\frac{x_0+x_1}{2}\right)\triangle x \;+\; f\left(\frac{x_1+x_2}{2}\right)\triangle x \;+\; f\left(\frac{x_2+x_3}{2}\right)\triangle x \;+\; f\left(\frac{x_3+x_4}{2}\right)\triangle x \\
&= f\left(\frac{5}{4}\right)\triangle x \;+\; f\left(\frac{7}{4}\right)\triangle x \;+\; f\left(\frac{9}{4}\right)\triangle x \;+\; f\left(\frac{11}{4}\right)\triangle x \\
&= \left(\frac{5}{4}\right)^2 \cdot \frac{1}{2} \;+\; \left(\frac{7}{4}\right)^2 \cdot \frac{1}{2} \;+\; \left(\frac{9}{4}\right)^2 \cdot \frac{1}{2} \;+\; \left(\frac{11}{4}\right)^2 \cdot \frac{1}{2} \\
&= \frac{69}{8} \;=\; 8.675.
\end{aligned}
$$

$$
\begin{aligned}
T_4 &= \frac{\triangle x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right) \\
&= \frac{1}{4}\left(f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + f(3)\right) \\
&= \frac{1}{4}\left((1)^2 + 2\cdot\left(\frac{3}{2}\right)^2 + 2\cdot(2)^2 + 2\cdot\left(\frac{5}{2}\right)^2 + (3)^2\right) \\
&= \frac{35}{4} = 8.75.
\end{aligned}
$$

$$
\begin{aligned}
S_4 &= \frac{\triangle x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right) \\
&= \frac{1}{6}\left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + f(3)\right) \\
&= \frac{1}{6}\left((1)^2 + 4\left(\frac{3}{2}\right)^2 + 2(2)^2 + 4\left(\frac{5}{4}\right)^2 + (3)^2\right) \\
&= \frac{26}{3} = 8.\bar{6}.
\end{aligned}
$$

True value of the integral:

$$
\int_1^3 x^2 dx = \frac{1}{3}x^3\Big|_1^3 = 9 - \frac{1}{3} = \frac{26}{3} = 8.\bar{6}.
$$

Note that Simpson's rule is not only the best approximation, in this case it is dead on.