

1 Lecture 1 - FTC II

Calculus is the mathematics of change. It is divided into two branches, differential calculus and integral calculus, which interact strongly with each other.

1.1 Derivatives

Derivatives measure instantaneous rates of change.

Given a function $y = f(x)$, we can measure the discrete change in the y -value, denoted Δy , when a discrete change, Δx , occurs in the x -value:

Average rate of change of $f(x)$ when x changes by the amount $\Delta x = \frac{\Delta y}{\Delta x}$.

If we make the discrete change Δx smaller and smaller, thereby measuring the change of the function $f(x)$ over smaller and smaller intervals, in the limit we get the instantaneous rate of change

$$\text{Instantaneous rate of change} = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} .$$

Here “ dx ” and “ dy ” indicate *infinitesimal*, as opposed to discrete, changes in the variables x and y ¹. If $y = f(x)$ is a function, the symbols

$$f'(x) \qquad \frac{df}{dx} \qquad \frac{d}{dx}(f(x)) \qquad \frac{dy}{dx}$$

all mean precisely the same thing: the derivative of f with respect to x . In class, on tests, and in homeworks, all of these notations will be used.

You will be required to know the following basic differentiation rules:

¹the term “infinitesimal” is not mathematically precise, but we shall not deal with this in this here.

Power Rule:

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

Exponential Rule:

$$\frac{d}{dx}(e^{kx}) = k e^{kx}$$

Logarithm Rule:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Trig rules:

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \cos(x) & \frac{d}{dx}(\cos(x)) &= -\sin(x) \\ \frac{d}{dx}(\tan(x)) &= \sec^2(x) & \frac{d}{dx}(\cot(x)) &= -\csc^2(x) \\ \frac{d}{dx}(\sec(x)) &= \sec(x)\tan(x) & \frac{d}{dx}(\csc(x)) &= -\csc(x)\cot(x) \end{aligned}$$

Constant multiple rule:

$$\frac{d}{dx}(a f(x)) = a \frac{d}{dx}(f(x))$$

Sum/difference rule:

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

Product rule:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \frac{d}{dx}(g(x))}{(g(x))^2}$$

Chain Rule:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

1.2 Integrals

Integrals measure total accumulated change.

In terms of graphs of functions, this equates to the (signed) area under a curve.

Given a function $y = f(x)$, one can *approximate* the area under its graph, say between $x = a$ and $x = b$, by breaking the graph into uniformly-spaced rectangles of width Δx . If you use n many rectangles and x_i is a point in the i^{th} rectangle, then the height of the rectangle should be $f(x_i)$, and

$$\text{Area of the } i^{\text{th}} \text{ rectangle} = \text{height} \times \text{length} = f(x_i) \cdot \Delta x$$

$$\text{Sum of the rectangles' areas} = \sum_{i=1}^n f(x_i) \cdot \Delta x$$

But using rectangles with discrete length Δx will give just an approximation. To make the approximation better, you make Δx smaller and smaller: in the limit, the discrete width Δx become the infinitesimal width dx , and the discrete sum $\sum_{i=1}^n$ becomes the continuous sum \int_a^b .

$$\sum_{i=1}^n f(x_i) \Delta x \quad \text{becomes} \quad \int_a^b f(x) dx.$$

The symbol $\int_a^b f(x) dx$ literally means “the (signed) area under the graph of $y = f(x)$ between $x = a$ and $x = b$ ”. Theoretically, it is obtained by summing up the infinitesimal areas of the infinitely many rectangles that live under the graph.²

²Again, this is mathematically imprecise. For the purposes of this class however, we consider this to be a minor issue.

1.3 The Fundamental Theorem of Calculus

If $y = f(x)$ is a function, we usually denote the derivative by

$$f'(x) \qquad \frac{df}{dx} \qquad \text{etc.}$$

and we denote an *antiderivative* of $f(x)$ by using capitals:

$$F(x).$$

Remarkably, derivatives (rates of change) and integrals (areas under graphs, or total accumulated change) are actually related to each other. This is the **Fundamental Theorem of Calculus**, version II of which we state here:

Theorem 1.1 (Fundamental Theorem of Calculus, version II) *If the antiderivative of $f(x)$ is $F(x)$, then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

□

Because of the FTC, it is as important to know the rules for antiderivatives as it is for derivatives:

$$\text{Power Rule for } n \neq 1: \quad f(x) = x^n \quad \implies \quad F(x) = \frac{1}{n+1} x^{n+1}$$

$$\text{Power Rule for } n = 1: \quad f(x) = x^{-1} \quad \implies \quad F(x) = \ln(x)$$

$$\text{Exponential Rule: } \quad f(x) = e^{kx} \quad \implies \quad F(x) = \frac{1}{k} e^{kx}$$

$$\text{Trig rules: } \quad f(x) = \sin(x) \quad \implies \quad F(x) = -\cos(x)$$

$$f(x) = \cos(x) \quad \implies \quad F(x) = \sin(x)$$

There is no product rule or quotient rule for integrals!