Wild singularities of translation surfaces

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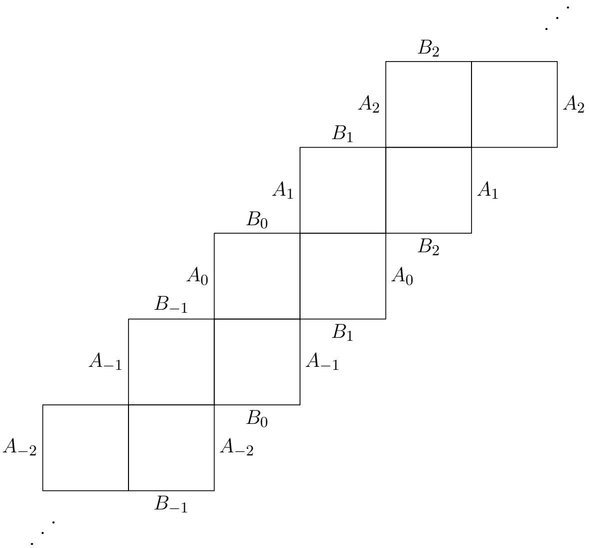
Background

joint with F. Valdez (Instituto de Matemáticas UNAM)

Definition. A *translation surface* X is a (connected) topological surface together with an atlas whose transition maps are all translations.

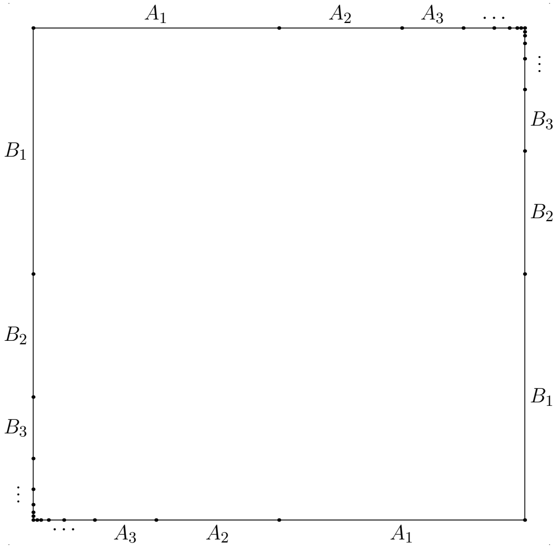
Notation. $\Gamma(X) =$ Veech group of X
$= \{Df \in \mathrm{GL}_2^+(\mathbb{R}) \mid$
$f : X \to X$ affine, orientation-preserving homeomorphism$\}$

Examples



Infinite staircase X_s (Hooper–Hubert–Weiss)

Examples



"Baker's map" surface X_b (Chamanara–Gardiner–Lakic)

Examples

Veech groups of these:

- $\Gamma(X_s)$ has index 3 in $SL_2(\mathbb{Z})$ (Hubert–Weiss)
- $\Gamma(X_b) \cong F_2$ (Chamanara)

These examples are topologically identical: infinite genus with no punctures and one end (Loch Ness monsters).

They are distinguished from each other by their Veech groups and infinite vs. finite area.

Are there local characteristics that distinguish them?

In particular, is there an object to play the role of "cone angle"?

Linear approaches

Definition. Let X be a translation surface.
For each $\varepsilon > 0$, denote

$$\mathcal{L}^\varepsilon(X) := \{\text{unit speed geodesic embeddings } \gamma : (0,\varepsilon) \to X\}.$$

For each $\varepsilon' < \varepsilon$, the restriction map $\gamma \mapsto \gamma|_{(0,\varepsilon')}$ defines a natural injection $\mathcal{L}^\varepsilon(X) \to \mathcal{L}^{\varepsilon'}(X)$.
Define

$$\mathcal{L}(X) := \lim_{\varepsilon \to 0} \mathcal{L}^\varepsilon(X) = \{[\gamma] \mid \gamma : (0,\varepsilon) \to X\},$$

where $[\gamma]$ is the equivalence class of γ under the relation

$$\gamma_1 \sim \gamma_2 \quad \text{if} \quad \gamma_1 = \gamma_2 \quad \text{on} \quad (0, \min\{\varepsilon_1, \varepsilon_2\}).$$

$\mathcal{L}$ is used to mean the set of "linear approaches" to points of X.

Motivations

Why study $\mathcal{L}(X)$?

- Provides analogues of "cone points" in non-compact case.
- Critical graphs for geodesic flow on X embedded in $\mathcal{L}(X)$.
- Produces local invariants that are preserved by the affine group / Veech group.
- First step towards constructing "strata" of non-compact translation surfaces.

Topology

Each $\mathcal{L}^{\varepsilon}(X)$ has the topology of uniform convergence.

We give $\mathcal{L}(X)$ the direct limit topology.

Universal property of $\mathcal{L}(X)$: If $\mathfrak{T}$ is a topological space and $f_{\varepsilon} : \mathcal{L}^{\varepsilon}(X) \to \mathfrak{T}$ are continuous maps that commute with restriction, then there is a unique continuous map $f : \mathcal{L}(X) \to \mathfrak{T}$ that restricts to f_{ε} on each $\mathcal{L}^{\varepsilon}(X)$.

Let $\overline{X}$ denote the metric completion of X.

Recall that translation structure trivializes the unit tangent bundle of X: $T^1 X \cong X \times S^1$.

Define:

- *basepoint map* $\mathcal{L}(X) \to \overline{X}$, $\mathrm{bp}([\gamma]) = \lim_{t \to 0} \gamma(t)$
- *direction map* $\mathcal{L}(X) \to S^1$, $\mathrm{dir}([\gamma]) = \dot{\gamma}(t)$ for any $t \in (0, \varepsilon)$

Functoriality

Theorem

$\mathcal{L}(\,\cdot\,)$ is a functor from the category of translation surfaces to the category of Hausdorff spaces.

Why only "Hausdorff"?
$\mathcal{L}(X)$ is not in general metrizable, or even regular.

Each open affine map $f : X \to Y$ of translation surfaces induces a "push-forward" map $f_* : \mathcal{L}(X) \to \mathcal{L}(Y)$:

$$\begin{array}{ccc} \mathcal{L}(X) & \xrightarrow{f_*} & \mathcal{L}(Y) \\ \text{bp}\downarrow & & \downarrow\text{bp} \\ \overline{X} & \xrightarrow[\bar{f}]{} & \overline{Y} \end{array} \qquad \begin{array}{ccc} \mathcal{L}(X) & \xrightarrow{f_*} & \mathcal{L}(Y) \\ \text{dir}\downarrow & & \downarrow\text{dir} \\ S^1 & \xrightarrow[Df]{} & S^1 \end{array}$$

$\mathcal{L}(X)$ "extends" the unit tangent bundle of X to points of $\overline{X} \setminus X$.

Tame vs. wild

Assume hereafter that $\operatorname{sing}(X) = \overline{X} \setminus X$ is discrete.

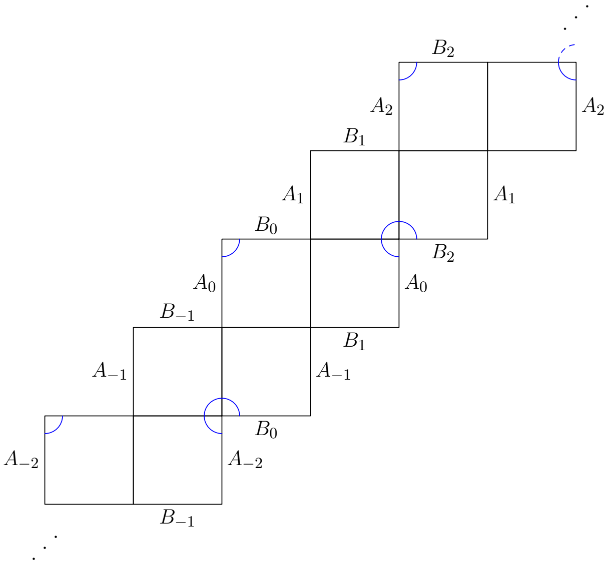
To study points of $\operatorname{sing}(X)$, set $\mathcal{L}(x) = \operatorname{bp}^{-1}(x)$.

Definition. $x \in \overline{X}$ is *tame* if $\mathcal{L}(x)$ is contained in some $\mathcal{L}^{\varepsilon}(X)$.
(Short saddle connections do not accumulate on x.)
Otherwise, x is a *wild singularity*.

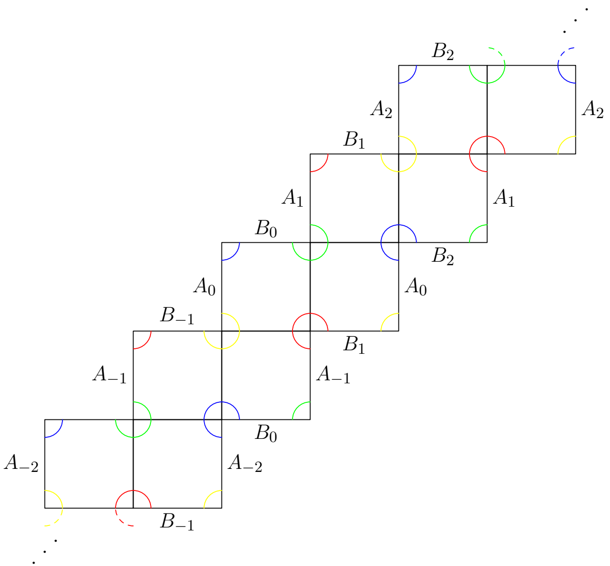
Three kinds of tame points:

- points of X
- cone points of finite angle $2k\pi$
- "infinite cone angles" having neighborhood isometric to the universal cover of some punctured disk

Tame singularities of a non-compact surface



Tame singularities of a non-compact surface



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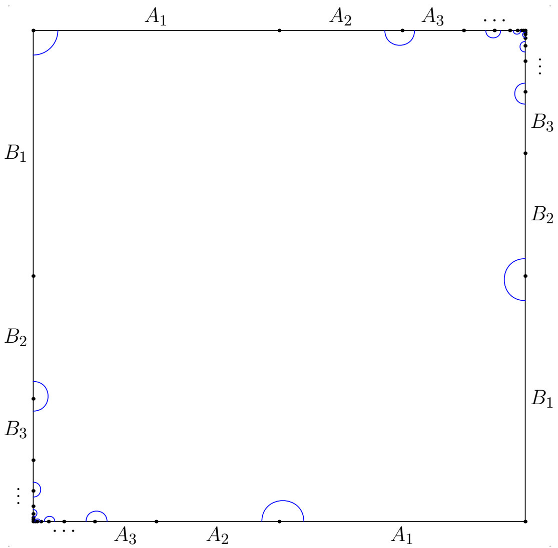
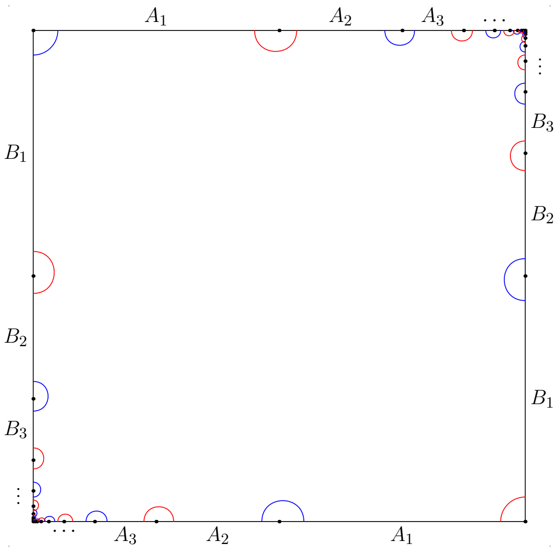


Wild singularity of a non-compact surface



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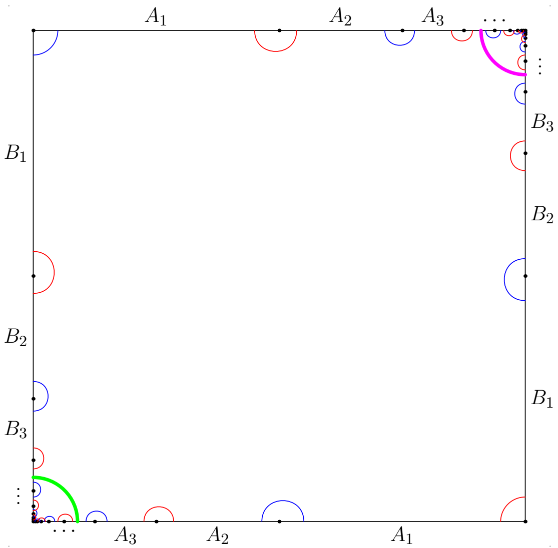


Components of $\mathcal{L}(x)$

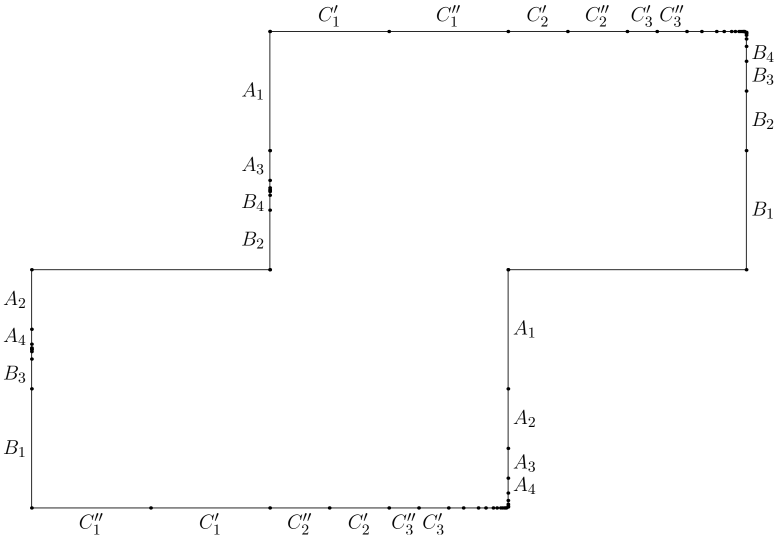
Theorem
$\mathcal{L}(x)$ is the closure of a union of connected 1-manifolds (possibly with boundary), each of which carries a canonical (angular) metric and is injectively immersed in $\mathcal{L}(x)$.

Each 1-manifold is a *rotational component*, obtained from some $[\gamma_0] \in \mathcal{L}(x)$ by "rotating around the basepoint".

Definition. A rotational component of $\mathcal{L}(x)$ is:

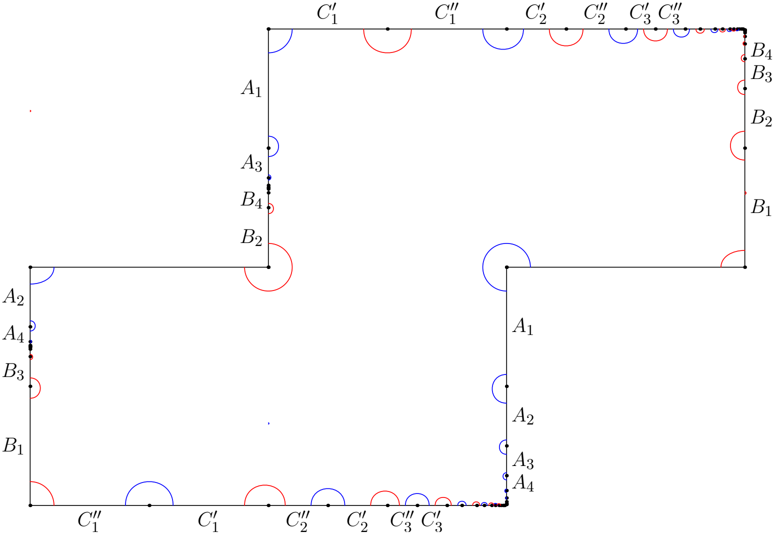
- a *spire* if it is unbounded with respect to its angular metric;
- a *double spire* if it is unbounded in both directions;
- an *arc* if it is bounded and not homeomorphic to S^1.

Wild singularity of a non-compact surface



Infinite genus Arnoux–Yoccoz surface X_{AY}

Wild singularity of a non-compact surface



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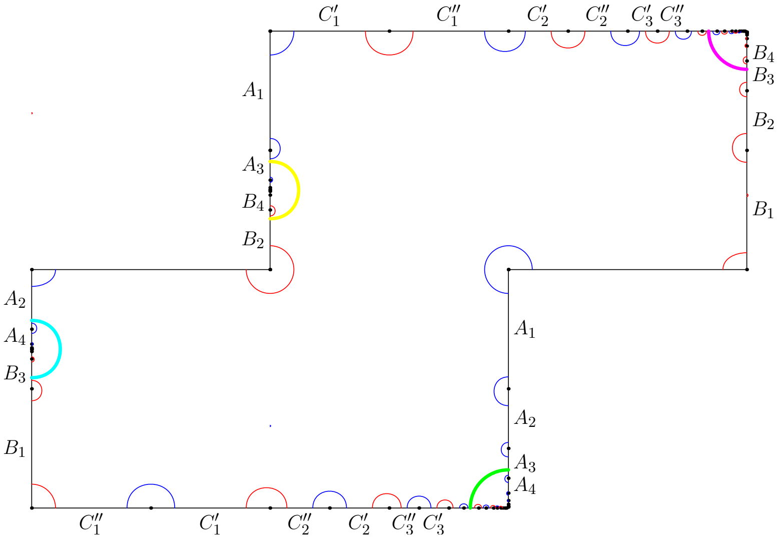

$\Gamma(X_{AY}) \cong \mathbb{Z}$ (B.)

Conditions for local isometry

Two cone points are locally isometric
$\iff$ they have the same angle.

How does this translate to wild singularities?

Theorem

Let $x \in \overline{X}$, $y \in \overline{Y}$. Then there exist neighborhoods $N(x)$ and $N(y)$ and an isometry $N(x) \to N(y)$ if and only if there exists a homeomorphism $F : \mathcal{L}(x) \to \mathcal{L}(y)$ such that

- *F preserves the angular metric;*
- *for some $\varepsilon > 0$, F sends trajectories of length at most ε to trajectories of the same length; and*
- *F preserves the “pairing” of short saddle connections.*

Comparison with other work

Gromov–Hausdorff limit as opposed to direct limit:

- not well-defined; depends on rate of rescaling
- translation surfaces do not always have necessary volume-doubling properties

Alexandrov tangent cone to a point in a metric space:

- always metrizable
- loses some information about “nearness” of trajectories
- can be recovered from $\mathcal{L}(X)$

de Carvalho–Hall study singularities *extrinsically* in terms of the *scar* on a surface left by identifying sides of a polygon.

Questions for future work

- Can the topological spaces that arise as $\mathcal{L}(x)$ be characterized?
  Certainly Hausdorff, topological dimension 1.
- How does $\mathcal{L}(x)$ change under bi-Lipschitz or quasiconformal maps of translation surfaces?
- Does the same description of local structure still work if we lift the restriction that $\text{sing}(X)$ be discrete (e.g., what if $\text{sing}(X)$ contains a Cantor set)?
- Can this construction be useful in the study of other Riemannian / affine manifolds?

Muchas gracias a todos

Thanks for coming!