

SOLUTIONS OF PRACTICE MIDTERM II

Name:

Student I.D:

Problem 1. (25 points)

1. Prove that in any Hilbert space every angle has a unique bisector. (You can use the existence and unicity of the midpoint of a segment).
2. Consider a triangle $\triangle ABC$. Using the above result, bisect the angle $\sphericalangle A$. Explain why the bisector meets segment BC at some point D . Use this point D to reprove the triangle inequality (namely $\overline{AB} + \overline{AC} > \overline{BC}$).

Proof.

1. Suppose you want to bisect an angle $\sphericalangle BAC$. You can assume, to simplify, that AB is congruent to AC (so that the triangle is isosceles). Consider D the midpoint of segment BC . By S-A-S, the two triangles ACD and ABD are congruent, therefore angle $\sphericalangle CAD$ is congruent to angle $\sphericalangle DAB$. Now D is between B and C , therefore it is inside the angle $\sphericalangle A$, and by angle addition we deduce that the ray \overrightarrow{AD} bisects the angle $\sphericalangle A$. The bisector is unique by the axiom "transporting the angle".

2. Now we don't necessarily assume that the triangle is isosceles. But the bisector must intersect segment BC at a point D because of the crossbar theorem. Notice that now D is not necessarily the midpoint of segment BC . Now apply the Exterior Angle theorem twice to deduce that

$$\begin{aligned} (\sphericalangle ADB)^\circ &> (\sphericalangle CAD)^\circ = (\sphericalangle BAD)^\circ \text{ and} \\ (\sphericalangle ADC)^\circ &> (\sphericalangle BAD)^\circ = (\sphericalangle CAD)^\circ \end{aligned}$$

Apply now proposition P4.5 from the axioms list ("the greater side lies opposite the greater angle") and therefore deduce that:

$\overline{AC} > \overline{CD}$ and $\overline{AB} > \overline{BD}$, but this implies the result by addition, using the fact that D lies between B and C .

□

Problem 2. (25 points) Prove that Hilbert's Euclidean parallel postulate is equivalent to the following proposition:

if t is a transversal to l and m , l parallel to m , and $t \perp l$ then $t \perp m$.

Proof. a) Hilbert implies P:

Assume t cuts m at P without forming a right angle, then if you consider the perpendicular through P to t you would get a line distinct from m that would be another parallel to l , a contradiction.

b) P implies Hilbert:

Consider the standard configuration: l is a line, P a point not on l ; build t the perpendicular line through P to l and m the perpendicular through P to t . Assume now that there is another parallel through P to l , then it would form a different angle with respect to t , but this is a contradiction with proposition P.

□

Problem 3. (25 points) Prove that in any Hilbert plane there exists one triangle that is not isosceles.

Proof. Take a line l , pick a point P not on l , drop a perpendicular through P in order to create a right angle at $Q \in l$. On one ray of the right angle, pick any length QR . On the other ray, transport twice that length to get a segment QS . Now RQS is a right triangle such that QR is not congruent to QS .

It remains to show that RS is not congruent to any other side. But the Exterior angle theorem says that each angle at R and S is acute. Therefore by proposition P4.5 ("the greater side lies opposite the greater angle"), we know that the hypotenuse is strictly greater than any other side and we are done.

□

Problem 4. (25 points) In any Hilbert plane, prove that the interior of a circle is a convex set.

Proof. Let \mathcal{C} be the circle with center O and with radius OM .

Given any two points A, B inside \mathcal{C} , we want to show that for any point P such that $A * P * B$ we have $OP < OM$.

One of the two angles OAB , OBA is smaller than the other (or congruent to the other): assume that $\sphericalangle OAB \leq \sphericalangle OBA$ for example. Then by the Exterior angle theorem, $\sphericalangle OPB$ is larger than $\sphericalangle OAB$ which is larger than $\sphericalangle OBA$.

Therefore, again by prop P4.5 we have that side OB is larger than side OP which is what we wanted.

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