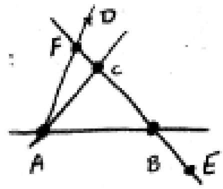
Solutions - Practice Final

#1) D is outside $\triangle ABC$ so D must be opposite to at least one vertex, say D and A are on opposite sides of $\overleftrightarrow{BC}$. Thus segment DA must cut $\overleftrightarrow{BC}$ at a pt F.

Three cases: ① $B * C * F$:



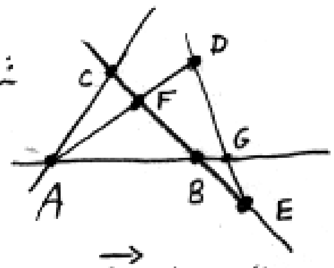
Pick any E on $\overleftrightarrow{BC}$ such that $C * B * E$. Let's show that $\overleftrightarrow{DE}$ is outside $\triangle ABC$.

Since we have $\begin{cases} D * F * A \\ B * C * F \end{cases}$ we know that D, F, C are together w.r. to $\overleftrightarrow{AB}$.

Now the entire ray $\overrightarrow{ED}$ is on the opposite side of A (w.r.t $\overleftrightarrow{BC}$) so it's outside $\triangle ABC$.

Finally the opposite ray to $\overrightarrow{ED}$ is on the opposite side of C w.r to $\overleftrightarrow{AB}$ (because D and E are on opposite sides of $\overleftrightarrow{AB}$).

② $C * F * B$:



Again pick any E on $\overleftrightarrow{BC}$ s.t $C * B * E$ and show that $\overleftrightarrow{DE}$ is outside $\triangle ABC$.

Similarly, the entire ray $\overrightarrow{ED}$ is on the opposite side of A w.r. to $\overleftrightarrow{BC}$, hence is outside $\triangle ABC$.

Now the opposite ray to $\overrightarrow{ED}$ is on the opposite side of C w.r to $\overleftrightarrow{AB}$ (to see this, notice that there is a pt $G \in$ segment DE and on $\overleftrightarrow{AB}$, and ray $\overrightarrow{GE}$ is on the opposite side of C).

③ $C * B * F$: same as ① (just permute B and C).

#2) See HW #4.

#3) It's an immediate consequence of the axioms:

Pick any line. By I2 there are at least 2 pts B, D on it.

Now by B2 there exist pts A, C, E on $\overleftrightarrow{BD}$ s.t $A * B * D$, $B * C * D$, $B * D * E$.

And we are done!

#4 Let $\mathcal{M}$ be a projective plane, therefore it satisfies: all the incidence axioms + any two lines meet + every line has at least 3 distinct pts on it.

Let's check that $\mathcal{M}'$ is a projective plane: $\left\{ \begin{array}{l} \mathcal{M}'\text{-pts. correspond to } \mathcal{M}\text{-lines} \\ \mathcal{M}'\text{-lines } \text{ " " } \text{ " } \mathcal{M}\text{-points} \end{array} \right\}$

$\boxed{I1:}$

Since any two distinct $\mathcal{M}$-lines intersect at a unique $\mathcal{M}$-pt, I1 is true for $\mathcal{M}'$!

$\boxed{I2:}$

We want to show the following: for every $\mathcal{M}$-pt there exist at least two $\mathcal{M}$-lines incident with it.

Indeed we know that in $\mathcal{M}$ there are at least 3pts such that no line is incident with all three (axiom I3 for $\mathcal{M}$). Thus given any pt P, consider the lines joining P to these pts: at least 2 of them are distinct so we are done.

$\boxed{I3}$: We want to show the following:

There exist 3 distinct $\mathcal{M}$-lines such that no $\mathcal{M}$-point is incident with all three.

But this is a consequence of I3 for $\mathcal{M}$: take the 3 non-collinear $\mathcal{M}$-pts, A, B, C : then the 3 lines $\overleftrightarrow{AB}, \overleftrightarrow{AC}, \overleftrightarrow{BC}$ cannot be collinear.

$\boxed{\text{Any 2 } \mathcal{M}'\text{-lines meet}}$: we want to show that given any 2 $\mathcal{M}$-points there is a line incident with them. But this is just I2 for $\mathcal{M}$!

$\boxed{\text{Any } \mathcal{M}'\text{-line has at least 3 distinct pts on it}}$

This can be rephrased as: given any $\mathcal{M}$-pt, there are at least 3 distinct lines incident with it.

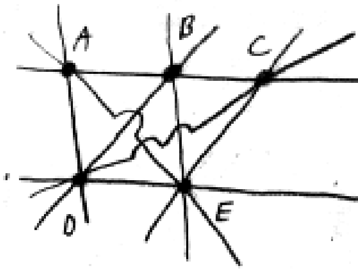
Indeed by I3 for $\mathcal{M}$, there exist a line l not going through a given P.

Now there exist at least 3 distinct pts on that line l (because $\mathcal{M}$ is a projective plane).

Join those pts to P to get the desired lines.

#5:

Just take a space with 5 pts as follows:



schematic picture of the lines

Points: A, B, C, D, E

Lines: $\{A,B,C\}$, $\{D,E\}$, $\{A,D\}$, $\{B,D\}$, $\{C,D\}$

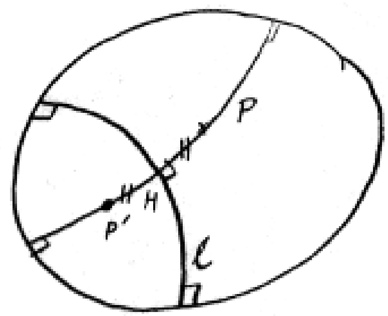
$\{A,E\}$, $\{B,E\}$, $\{C,E\}$.

Incidence: "set membership".

Then in this model, the incidence axioms are satisfied. But $\{A,B,C\}$ has a unique parallel through D, but $\{CE\}$ has 2 parallels through D, so the model doesn't have any of the 3 properties mentioned.

#6

In the Poincaré model:

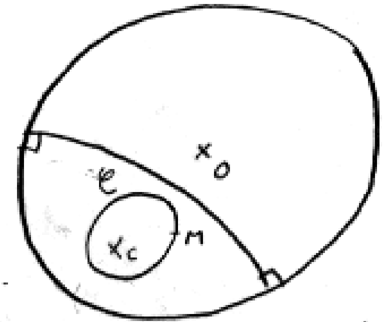


Since it is a Hilbert plane, one can drop a perpendicular through P to ℓ to get the point H and then "transport the length HP" on the other side of ℓ to get P'.

Consider now the inversion with respect to ℓ: it preserves orthogonal circles thus the "line" $\overleftrightarrow{HP}$ is globally preserved, H is preserved (because on ℓ) and the two sides of ℓ are exchanged.

We also proved that inversions preserve $\mathcal{P}$-lengths therefore $d(HP) = d(H\,i(P))$ and thus $i(P)$ coincides with P' (which is the unique pt P' on $\overleftrightarrow{HP}$ on the opposite ray to $\overrightarrow{HP}$, such that $HP \cong HP'$).

#7



Let $\mathcal{C}$ a $\mathcal{P}$-circle with $\left\{\begin{array}{l}\text{radius } CM \\ \text{center } C\end{array}\right.$ in the Poincaré-model: so $\mathcal{C}$ is the set of pts $P \in \mathcal{P}$ such that $CP \cong CM$.

We know the existence of an inversion sending C to O.

Since such an inversion preserves the $\mathcal{P}$-length, it will send $\mathcal{C}$ to a $\mathcal{P}$-circle centered at O. But we proved that $d(OP) = d(OQ)$ iff $\overline{OP} = \overline{OQ}$ (when O is the origin), therefore a $\mathcal{P}$-circle centered at the origin is the same thing as an Euclidean circle centered at O.

Now apply the inversion one more time (since $i \circ i$ = identity we return to $\mathcal{C}$): we know that inversions send circles (not going through the origin of the inversion) to circles (Euclidean circles), thus $\mathcal{C}$ is a Euclidean circle.

(Notice that the $\mathcal{P}$-center of $\mathcal{C}$ is not the same as its Euclidean center).