MAT 319/320

Solutions of Practice Midterm II

Problem 1. Define a sequence (x_n) by : $x_0=1$ and $x_{n+1}=\frac{1}{3}x_n+1$. Does (x_n) have a limit? If yes, what is this limit?

Proof. First, notice that if the sequence has a limit l, then it must satisfy the equation : $l=\frac{1}{3}.l+1$, thus the only possible limit is $l=\frac{3}{2}$.

Let's show that this sequence is monotone and bounded, and therefore it will be convergent.

1. Claim: the sequence is bounded above by $\frac{3}{2}$.

   Indeed, by induction we get that: $x_0 \leqslant 3/2$, and if $x_n \leqslant 3/2$ then we deduce that $x_{n+1} \leqslant \frac{1}{3}.\frac{3}{2}+1=3/2$, so we are done.

2. Claim: the sequence is increasing: this comes from the fact that $(x \leqslant 3/2) \Rightarrow \frac{1}{3}x+1 \geqslant x$.

Therefore the sequence is converging to 3/2.

□

Problem 2. Is the infinite series $\sum_{n=1}^{+\infty} \frac{1}{n+\sqrt{n}}$ convergent? (If yes, you do not need to find the value of the limit).

Proof. We have $a_n=\frac{1}{n+\sqrt{n}}=\frac{1}{n}.\frac{1}{1+\frac{\sqrt{n}}{n}}$. Thus $\frac{a_n}{1/n} \to 1$, which implies that $\sum a_n$ converges if and only if $\sum \frac{1}{n}$ converges, by the comparison theorem. Since $\sum \frac{1}{n}$ diverges, we deduce that $\sum a_n$ diverges. □

Problem 3. Recall the definition of the continuity of a function f at a point c.

Proof. Hehe, see the textbook! (don't write that kind of answer for the actual midterm...) □

Problem 4. What is $\lim_{x\to+\infty} \frac{x+3}{\sqrt{x+7}+1}$?

Proof. As usual we factor both numerator and denominator by the "dominant term":
$\frac{x+3}{\sqrt{x+7}+1}=\frac{x}{\sqrt{x}}.\frac{1+\frac{3}{x}}{\sqrt{1+\frac{7}{x}}+\frac{1}{\sqrt{x}}}$. Now $\lim_{x\to+\infty} 1+\frac{3}{x}=1$ and $\lim_{x\to+\infty}\sqrt{1+\frac{7}{x}}+\frac{1}{\sqrt{x}}=1$ because $\frac{1}{\sqrt{x}} \to 1$ and $\sqrt{1+\frac{7}{x}} \to 1$ by the square root rule.Therefore by the comparison theorem, we know that $\frac{x+3}{\sqrt{x+7}+1}$ has a limit equal to $+\infty$ if and only if $\lim_{x\to+\infty}\sqrt{x}=+\infty$, which is the case.

Thus, $\lim_{x\to+\infty}\frac{x+3}{\sqrt{x+7}+1}=+\infty$. □

Problem 5. Use the **definition of a limit** (I mean use "ε,δ") to prove that $\lim_{x\to-1}\frac{5x^2+2x+1}{x+3}=2$. How could you prove the same thing using an easier way?

Proof. The easy way: a rational function is continuous at any point where the denominator is not zero. Since $x+3$ is not zero at $x=-1$, we know that the limit is actually equal to $\frac{5.1+2.(-1)+1}{-1+3}=2$.

The complicated way (using the definition):
$\left|\frac{5x^2+2x+1}{x+3}-2\right|=\left|\frac{5x^2+2x+1-2x-6}{x+3}\right|=5.\left|x+1\right|.\left|\frac{x-1}{x+3}\right|$. Now let's prove that on a neighborhood of -1, the function $5.\left|\frac{x-1}{x+3}\right|$ is bounded above.

One can take for example the neighborhood $V=(-2,-\frac{1}{2})$. Then on V, one has:
$-3<x-1<\frac{-3}{2} \Rightarrow \left|x-1\right|<3$, and similarly on V one has : $1<x+3<5/2 \Rightarrow \frac{1}{|x+3|}<1$.

Therefore, on V we have that $5.\left|\frac{x-1}{x+3}\right| < 15$.

Now for a given $\varepsilon > 0$, if we take $x \in V$, we have that $\left|\frac{5x^2+2x+1}{x+3} - 2\right| < 15.\left|x+1\right|$, so it is enough to take $\delta > 0$ less than $\varepsilon/15$ and such that $(-1-\delta, -1+\delta) \subset V$. Clearly $\delta = \min(\varepsilon/15, 1/2)$ will work. ☐

Problem 6. Let $f: [0,3] \to \mathbb{R}$ be a continuous function. Assume that $f(1) > 0$, then prove the existence of a small δ-neighborhood of 1 on which the function f has no root (meaning there is no x in this neighborhood such that $f(x) = 0$).

Proof. Take $\varepsilon = \frac{f(1)}{2} > 0$. By continuity, there exists a $\delta > 0$ such that for any $x \in [0,3] \cap (1-\delta, 1+\delta)$,

one has $f(x) \in \left(\frac{f(1)}{2}, \frac{3}{2} f(1)\right)$, and thus on that neighborhood of 1, we have that $f(x) > 0$, so it has no zero. ☐