MAT 319

Solutions of the Practice FINAL

Problem 1. What is the limit of $(x_n) = \frac{n^3}{n!}$?

Proof. Observe that $\frac{x_{n+1}}{x_n} = \frac{(n+1)^3}{(n+1)!}.\frac{n!}{n^3} = \frac{(n+1)^2}{n^3} \to 0$ therefore the sequence is converging to zero by the comparison theorem.

□

Problem 2. Use the definition of the limit to prove that $\lim \frac{n^2-1}{3n^2+1} = \frac{1}{3}$.

Proof. $\left|\frac{n^2-1}{3n^2+1} - \frac{1}{3}\right| = \left|\frac{3n^2-3-3n^2-1}{(3n^2+1).3}\right| = \frac{4}{9n^2+3} \leqslant \frac{1}{n^2}$ thus for a given $\varepsilon > 0$ if we take an integer $K > \frac{1}{\sqrt{\varepsilon}}$ we have that for any $n \geqslant K$, $\left|x_n - 1/3\right| \leqslant \frac{1}{n^2} < \varepsilon$.

□

Problem 3. Prove that an increasing sequence that is bounded above is necessarily converging.

Proof. See the textbook for this one... □

Problem 4. Show that if u_n is unbounded then there is a subsequence u_{n_k} of terms that are all non zero and such that $\frac{1}{u_{n_k}} \to 0$.

Proof. Since the sequence is unbounded, for any natural number k there is a term u_{n_k} of the sequence that is strictly larger than k, therefore one has $0 < \frac{1}{u_{n_k}} < \frac{1}{k} \to 0$ and this subsequence converges to zero by the squeeze theorem.

□

Problem 5. Is the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+2}$ convergent?

Proof. As usual factor by the leading term: $x_n = \frac{1}{n^2-n+2} = \frac{1}{n^2}.\frac{1}{1-\frac{1}{n}+\frac{2}{n^2}}$

But now if you write $y_n = 1/n^2$, we know that

$x_n/y_n = \frac{1}{1-\frac{1}{n}+\frac{2}{n^2}} \to 1$, therefore by the comparison theorem for infinite series, we know that our series converges if and only if $\sum y_n$ converges, but this is the case (p-series with $p = 2 > 1$).

□

Problem 6. Evaluate the following limit, or show that it doesn't exist: $\lim_{x\to+\infty} \frac{\sqrt{x}-x^2}{\sqrt{x}+x.\sqrt{x}}$.

Proof. Factor again by the leading term:

$f(x) = \frac{\sqrt{x}-x^2}{\sqrt{x}+x.\sqrt{x}} = \frac{-x^2}{x.\sqrt{x}}.\frac{-\sqrt{x}/x^2+1}{1/x+1}$ so if we call $g(x) = \frac{-x}{\sqrt{x}} = -\sqrt{x}$, we have that $f(x)/g(x) \to 1$, so by the comparison theorem, the limit of f is the same as the limit of g which is $-\infty$.

□

Problem 7. Assume that $f: \mathbb{R} \to \mathbb{R}$ is such that: for any $x \in \mathbb{R}$ there is a $\delta > 0$ such that f is bounded on $[x-\delta, x+\delta]$. Is the function f bounded on $\mathbb{R}$? (If yes, prove it; if not give a counter-example).

Proof. Of course not! Take $f(x) = x$, it is locally bounded: on $[x-\delta, x+\delta]$, the function is bounded by $x+\delta$, but it is unbounded on the entire line.

□

Problem 8. Is the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x + |x|$ differentiable everywhere? (Prove your assertion!)

Proof. On $(-\infty, 0)$ the function is equal to $3x - x = 2x$, which is differentiable (derivative is the constant function equal to 2), and similarly on $(0, +\infty)$, the function is equal to $4x$ which is differentiable.

Now it remains to study the differentiability at zero:

But $\frac{g(x) - g(0)}{x - 0} = 2$ to the left of zero, and is equal to 4 to the right of zero, therefore the function is not differentiable at zero.

□

Problem 9. If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, then show that

$\lim (n(f(c + \frac{1}{n}) - f(c))$ exists and is equal to $f'(c)$.

Proof. For a given $\varepsilon > 0$, we know the existence of $\delta > 0$ such that for any $h \in (-\delta, \delta)$ we have

$\frac{f(c+h) - f(c)}{h} \in (f'(c) - \varepsilon, f'(c) + \varepsilon)$.

Now pick any natural number $K > 1/\delta$. Then for any $n \geqslant K$ one has that $1/n$ is less than δ and therefore

$\frac{f(c + \frac{1}{n}) - f(c)}{1/n} \in (f'(c) - \varepsilon, f'(c) + \varepsilon)$, which exactly expresses the convergence of the sequence to $f'(c)$.

□

Problem 10. Show that if $x > 0$ then we have $\sqrt[3]{1+x} \leqslant 1 + \frac{1}{3}x$

Proof. Apply Taylor's theorem at the order 2 to $f(x) = \sqrt[3]{1+x}$ between the points 0 and x.

It says that $f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(c)$ for some particular $c \in (0, x)$.

Notice now that $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$, so $f'(0) = 1/3$,

and also that $f''(x) = \frac{1}{3} . \frac{-2}{3}(1+x)^{\frac{-5}{3}}$, so that $f''(c) = \frac{-2}{9}$

Therefore the remainder is $\frac{-2}{9} . \frac{x^2}{2}$ which is less than zero, and this gives the inequality we want.

□