Math 319 Some remarks and hints for HW9

Since we didn't practice too much with that material (uniform continuity, continuity on intervals), I would like to give you some remarks, hints, and treat some similar examples...

Problem 1. *Show that every polynomial of odd degree with real coefficients has at least one real root.*

Answer. We did that in class already, but here is a complete solution. Consider the dominant term in your polynomial $P(x) : a_n x^n$. If you factor by it, you will get $P(x) = a_n x^n(1 + \frac{a_{n-1}}{x} + \ldots + \frac{a_0}{x^n})$. Clearly $\lim_{x\to+\infty} 1 + \frac{a_{n-1}}{x} + \ldots + \frac{a_0}{x^n} = 1$, therefore by the comparison theorem, $\lim P(x) = \lim a_n x^n$ which is $\pm\infty$ depending on the sign of a_n. Assume now that $a_n > 0$: then the limit of $P(x)$ at $+\infty$ is $+\infty$, and the limit at $+\infty$ is $-\infty$, therefore there exists an α such that for any $x > \alpha$ we have $f(x) > 1$ (for example), and there is a β such that for any $x < \beta$ we have $f(x) < -1$. Thus we found two real numbers such that $f(\alpha) \geq 1 > 0$ and $f(\beta) \leq -1 < 0$. Apply now the intermediate value theorem on $[\beta, \alpha]$, and get a zero of the function. The case where the coefficient $a_n < 0$ can be treated similarly.

Remark. The same idea can be used for problem 13 in section 5.3: far away your function is bounded (for example by 1), and in the middle you have a continuous function on a closed interval.

On problem 3, section 5.3. Try to build a sequence of points x_n that is converging and such that the sequence $f(x_n)$ converges to zero. At some point you might have to use Bolzano-Weierstrass...

On problem 11. This is actually how I proved the theorems in class. The key fact is to know that if a continuous function is strictly positive at a point c then there is a small δ-neighborhood of c on which the function is still strictly positive.

Problem 2. *Show that* $f(x) = \frac{1}{x}$ *is uniformly continuous on* $[1, +\infty)$

Answer. The key is to understand that a continuous function on a closed interval is uniformly continuous, and then to notice that since the function goes to zero when x becomes large, it will be uniformly continuous "at the infinity".

More precisely: given $\epsilon > 0$: there exists an $\alpha > 1$ such that for any $x > \alpha$ we have $|f(x)| < \frac{\epsilon}{2}$. Now consider the closed interval $[1, \alpha + 2]$: since f is continuous, it is uniformly continuous on that closed interval, therefore there exists a $\delta > 0$ such that for any $x, y \in [1, \alpha + 2]$, we have $|f(x) - f(y)| < \epsilon$. By possibly taking $\delta' = min(\delta, 1)$ we can even assume that $\delta < 1$. Now pick any two real numbers u, v in $[1, \infty)$ satisfying $|u - v| < \delta$, and assume that for example $u < v$:

1. if $v \leq \alpha + 2$: then both u, v are in $[1, \alpha + 2]$, and therefore they satisfy $|f(u) - f(v)| < \epsilon$;

2. if $v \geq \alpha + 2$, then necessarily $\alpha < \alpha + 1 < u < v$, and therefore by the triangle inequality we have $|f(u) - f(v)| \leq |f(u)| + |f(v)| < \epsilon/2 + \epsilon/2 = \epsilon$.

Thus we proved the uniform continuity of the function. Notice that basically we only used the fact that the function is converging to zero at infinity. Another possible approach would be to use the particular form of the function: $|f(x) - f(y)| = |\frac{x-y}{xy} \leq |x - y|$ if x, y are larger than 1 (and then for a given ϵ just pick $\delta = \epsilon$: the same δ will now work for any pair x, y.