I would like to finish the proof of the following (we already proved the "if" statement):

Theorem. *A number $\alpha \in \mathbb{R} - \mathbb{Q}$ is quadratic irrational if and only if its continued fraction expansion is eventually periodic.*

Proof of the theorem, taken from Hardy-Wright.

Lemma. *If $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_{n-1} + \alpha_n}}}$, then one has $\alpha = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$.*

This is not too difficult to prove (a proof by induction works: see the book if you need help).

Assume now that α is a root of an irreducible polynomial $P(X) = aX^2 + bX + c$ with integer coefficients. After substituting the value for α, one gets that

$$A_n \alpha_n^2 + B_n \alpha_n + C_n = 0$$

with explicit formulas $A_n = ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2$, $B_n = 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + 2cq_{n-1}q_{n-2}$, and also $C_n = ap_{n-2}^2 + bp_{n-2}q_{n-2} + cq_{n-2}^2$.

Notice that $A_n \neq 0$ (otherwise p_{n-2}/q_{n-2} would be a root of $P(X)$). Also notice the following:

$$B_n^2 - 4A_nC_n = (b^2 - 4ac).(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}) = \pm(b^2 - 4ac).$$

Now you need to remember that we had proven in class that $\alpha = \frac{p_{n-1}}{q_{n-1}} + \frac{x_{n-1}}{q_{n-1}}$, with $|x_{n-1}| < 1/q_{n-1}$ and thus $p_{n-1} = q_{n-1}\alpha + x_{n-1}$.

Therefore,

$$A_n = a(q_{n-1}\alpha + x_{n-1})^2 + bq_{n-1}(q_{n-1}\alpha + x_{n-1}) + cq_{n-1}^2,$$

but this is also

$$A_n = (a\alpha^2 + b\alpha + c)q_{n-1}^2 + 2a\alpha x_{n-1}.q_{n-1} + ax_{n-1}^2 + bx_{n-1}.q_{n-1}$$

, and since α is a root of $P(X)$, one gets

$$A_n = 2a\alpha x_{n-1}.q_{n-1} + ax_{n-1}^2 + bx_{n-1}.q_{n-1},$$

which implies $|A_n| < 2|a\alpha| + |a| + |b|$. Now $C_n = A_{n-1}$, so the same estimate holds. Using the relation on the discriminant, one gets also $B_n^2 \leq 4|A_nC_n| + |b^2 - 4ac| \leq 4(2|a\alpha| + |a| + |b|)^2 + |b^2 - 4ac|$.

All these upper bounds do not depend on n, therefore there is only a finite possible number of values for A_n, B_n, C_n.Thus one can find a triple (A, B, C) of values that is taken three times, for $\alpha_{n_1}, \alpha_{n_2}, \alpha_{n_3}$. Thus one has three roots of the same quadratic polynomial $AX^2 + BX + C$, therefore two of them must be equal, say $\alpha_{n_1} = \alpha_{n_2}$. Thus the integers in the expansion must satisfy $a_{n_1} = a_{n_2}, a_{n_1+1} = a_{n_2+1}, \dots$ □