Solutions for the Final

#1. Both $(\tan x)$ and (x) are differentiable on $(0,+\infty)$ and $\frac{d}{dx}(x) = 1 \neq 0$ so we can use L'Hospital's Rule.

Thus $\lim_{x\to 0^+} \frac{\tan x}{x} = \lim_{x\to 0^+} \frac{\tan' x}{1} = \lim_{x\to 0^+} \frac{1}{\cos^2 x} = 1$.

#2 (a) We know that $x \mapsto x$ is continuous on $\mathbb{R}$ and that $x \mapsto |x-2|$ is also continuous on $\mathbb{R}$ (composition of 2 cont. functions)

Thus by the product rule, $g(x)$ is continuous on $\mathbb{R}$.

(b) On $(-\infty, 2)$, $g(x) = -x.(x-2)$ (a polynomial fc.) so it is differentiable on $(-\infty, 2)$.

On $(2,+\infty)$, $g(x) = x.(x-2)$ " " " " " .

At the point $x=2$: From the right: $\lim_{x\to 2^+} \frac{g(x)-g(2)}{x-2} = \lim_{x\to 2^+} \frac{+x.(x-2)-0}{x-2} = +2$.

From the left: $\lim_{x\to 2^-} \frac{g(x)-g(2)}{x-2} = \lim_{x\to 2^-} \frac{-x.(x-2)-0}{x-2} = -2$

Since $-2 \neq +2$, $\lim_{x\to 2} \frac{g(x)-g(2)}{x-2}$ doesn't exist, so $g'(2)$ doesn't exist.

#3 One has $\left|f(x) - \frac{1}{2}\right| = \left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| = \left|\frac{2x^2-2x+2-x-1}{2(x+1)}\right| = \left|\frac{2x^2-3x+1}{2(x+1)}\right| = |x-1| \cdot \frac{|2x-1|}{2|x+1|}$.

Let's prove that $\frac{|2x-1|}{2|x+1|}$ is bounded on $(0,2)$ (which is a neighborhood of 1):

Indeed $0<x<2 \Rightarrow -1 < 2x-1 < 3$ so $|2x-1| \leq 3$
and $0<x<2 \Rightarrow 2 < 2(x+1) < 6$ so $\frac{1}{2|x+1|} < \frac{1}{2}$
Thus on $(0,2)$ one has $\left|f(x) - \frac{1}{2}\right| \leq \frac{3}{2} \cdot |x-1|$.

Therefore, for a given $\varepsilon > 0$, pick $\delta = \min\left\{1, \frac{2}{3}\varepsilon\right\}$: for any $x \in (1-\delta, 1+\delta)$ one has $\left|f(x) - \frac{1}{2}\right| \leq \frac{3}{2} \cdot \frac{2}{3}\varepsilon = \varepsilon$.

#4 By the density of $\mathbb{Q}$ in $\mathbb{R}$, any $x \in \mathbb{R}$ is the limit of a sequence (r_n) where each $r_n \in \mathbb{Q}$.

By continuity of f, one has: $f(x) = \lim f(r_n) = \lim 2r_n = 2x$, and we are done.

If you take now g defined by $\begin{cases} g(r) = 2r & \text{if } r \in \mathbb{Q} \\ g(x) = 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$ then clearly $g(x) \neq 2x$ everywhere (this g is not continuous).

#5. Each function $x \mapsto x$ and $x \mapsto \sin x$ is continuous on $\mathbb{R}$, and the values agree at 0 ($0 = \sin 0$) so g is cont. on $\mathbb{R}$.

~~Each fc $x \mapsto x$~~ $\begin{cases} \text{Since } x \mapsto x \text{ is differentiable on } (-\infty, 0), \text{ so is } g. \\ \quad " \quad x \mapsto \sin x \quad " \quad " \quad " \quad (0,+\infty) \quad " \quad " \end{cases}$

At the point $x=0$: $\lim_{x\to 0^-} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^-} \frac{x}{x} = 1$ and $\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$ (L'Hospital's).

Thus $\lim_{x\to 0} \frac{g(x)-g(0)}{x}$ exists and g is differentiable on $\mathbb{R}$.