HW 8

Exercise 1. Consider a sphere in $\mathbb{R}^3$ with radius 1, and centered at the point $(0, 0, 1)$. Find a formula for the projection from the North pole sending a point from that sphere to the horizontal plane ($z = 0$). Find a formula for the inverse also.

What is the image by this projection of a circle (on the sphere) going through the North pole?

Exercise 2. Two models of the hyperbolic plane.

In this problem we want to construct directly an isomorphism between Δ_K (the interior of the unit disk, where lines are chords: this is called Klein model) and Δ_P (the interior of the unit disk, where lines are circle arcs perpendicular to the boundary).

1. Show that the following map expressed in polar coordinates $f\colon (r, \theta) \mapsto (\frac{r}{1+\sqrt{1-r^2}}, \theta)$ is a bijection from Δ_K to Δ_P.
2. Find the expression of f in cartesian coordinates (x, y).
3. In Δ_K consider the two points P, Q with respective coordinates $(s, \sqrt{1-s^2}), (s, -\sqrt{1-s^2})$ where $0 < s < 1$. Find the center C and radius R of the circle γ_{PQ} going through P,Q and orthogonal to the unit circle.
4. Show that any point on the chord joigning P to Q will be mapped by f to a point on the circle with center C and radius R. (Hint: any such point has coordinates (s, y) where $-\sqrt{1-s^2} < y < \sqrt{1-s^2}$ and then use the expression of f in cartesian coordinates).
5. Conclusion: explain why chords in Δ_K are mapped to orthogonal circular arcs in Δ_P.