MAT 319/320

SOLUTIONS FOR HW9

Exercise 1. Section 5.3, #3.

Proof. Start with any point, $x_0 = 1/2$ for example. Then we know that there exists x_1 such that $\left|f(x_1)\right| \leqslant \frac{1}{2}\left|f(x_0)\right|$. If $\left|f(x_1)\right| = 0$, then we are done; if not then necessarily $x_1 \neq x_0$, and there is then another point x_2, such that $\left|f(x_2)\right| \leqslant \frac{1}{2}\left|f(x_1)\right|$. Continue this process and build a sequence (x_n) satisfying $\left|f(x_{n+1})\right| \leqslant \frac{1}{2}\left|f(x_n)\right|$. Since all these points are in a bounded interval, the sequence is bounded and by Bolzano-Weierstrass we can extract a sequence (x_{k_n}) converging to some $c \in [a, b]$. If we write $k_{n+1} = k_n + \alpha(n)$, we get $\left|f(x_{k_{n+1}})\right| \leqslant \left(\frac{1}{2}\right)^{\alpha(n)}\left|f(x_{k_n})\right| \leqslant \frac{1}{2}\left|f(x_{k_n})\right|$.

(This just means that the extracted sequence satisfies the same inequality).

Since $(x_{k_n}) \to c$, and f is continuous, then we deduce that $f(x_{k_n}) \to f(c)$, but then the previous inequality says that: $\left|f(c)\right| \leqslant \frac{1}{2}\left|f(c)\right|$ so $f(c) = 0$ and we are done.

□

Exercise 2. Section 5.3, #6.

Proof. The hint says it all: $g(0) = f(0) - f(1/2)$ and $g(1/2) = f(1/2) - f(1) = -g(0)$. Now if $g(0) = 0$ we are done. If not, then apply the intermediate value theorem to g: you will get a zero for g between 0 and 1/2. But $g(x) = 0 \Rightarrow f(x) = f(x + \frac{1}{2})$.

□

Exercise 3. Section 5.3, #11.

Proof. This has been proved in class.Let me quickly give the argument:

if $f(w) < 0$, then necessarily $w < b$ (because $f(b) > 0$). By continuity of f there will be a small neighborhood of w on which the function is strictly negative: this contradicts the definition of w.

If $f(w) > 0$, then by continuity of f there is a small $\delta -$ neighborhood where the function is strictly positive: this is a contradiction, because between $w - \delta$ and w there should be a point in W. □

Exercise 4. Section 5.3, #13.

Proof. Take $\varepsilon = 1$: then there exists an $\alpha > 0$ such that for any $x > \alpha$ we have $|f(x)| < 1$, and there is a $\beta < 0$ such that for any $x < \beta$ we have $|f(x)| < 1$. But now f is continuous on $[\beta, \alpha]$ so it is bounded, say by $M > 0$. Putting everything together, we get that f is bounded on the entire line by $\max(1, M)$.

Since it is bounded, we can consider $L = \sup_{x\in\mathbb{R}} (f(x))$ and $l = \inf_{x\in\mathbb{R}} (f(x))$. If $L = l$ then the function is constant. If $L \neq l$, then one of them is nonzero. Assume it is L: pick $\varepsilon = \frac{1}{2}\Big|L\Big|$. Then there is an $\alpha > 0$ such that for any $x > \alpha$ we have $\Big|f(x)\Big| < \varepsilon$, and there is a $\beta < 0$ such that for any $x < \beta$ we have $\Big|f(x)\Big| < \varepsilon$. Now since again f is continuous on $[\beta, \alpha]$, it reaches a maximum at a point X in that closed interval. I claim that $f(X) = L$. Indeed if we have $f(X) < L$ then this would contradict the definition of L (because we know we can find points $y \in \mathbb{R}$, such that $f(y)$ is arbitrarily close to L, and such points are necessarily in $[\beta, \alpha]$ because outside of this closed interval everybody has an image less than $\frac{1}{2}.L$).

If $\inf f < 0$, then similarly we obtain a global minimum for the function.

Now it can happen that one of the two values $\inf f, \sup f$ is zero, in which case it is possible that the extremum is not reached: take for example $f(x) = \frac{1}{x^2+1}$ (max is reached at zero, but infimum is zero, not reached).

□

Exercise 5. Section 5.4, #2.

Proof. Just notice that $\Big|\frac{1}{x^2} - \frac{1}{y^2}\Big| = \Big|x - y\Big|.\Big|\frac{x+y}{x^2.y^2}\Big|$.

But observe now that for $x \geqslant 1, y \geqslant 1$ we have $\Big|\frac{x+y}{x^2.y^2}\Big| \leqslant \frac{|x|+|y|}{x^2.y^2} \leqslant \frac{x^2+y^2}{x^2y^2} \leqslant \frac{1}{y^2} + \frac{1}{x^2} \leqslant 2$.

So now given $\varepsilon > 0$, just pick $\delta = \frac{\varepsilon}{2}$: then for any x, y such that $\Big|x - y\Big| < \delta$ we get: $\Big|\frac{1}{x^2} - \frac{1}{y^2}\Big| < 2\Big|x - y\Big| < 2\frac{\varepsilon}{2} = \varepsilon$.

The key thing is that the first inequality is true for anybody in $[1, +\infty)$ (it is "uniformly true"!).

It is not uniformly continuous on $(0, \infty)$: pick the sequence $(x_n) = \frac{1}{2^n}$, then notice that $(x_{n+1} - x_n) \to 0$ but that $\Big|f(x_n) - f(x_{n+1})\Big| = 2^{2(n+1)} - 2^{2n} = 2^{2n}.(3) > 3$.

□

Exercise 6. Section 5.4, #4.

Proof. Same stuff: $\Big|\frac{1}{1+x^2} - \frac{1}{1+y^2}\Big| = \Big|x - y\Big|.\Big|\frac{x+y}{(1+x^2).(1+y^2)}\Big| \leqslant \Big|x - y\Big|.\frac{|x|}{1+x^2}.\frac{|y|}{1+y^2} \leqslant \Big|x - y\Big|$.

The last inequality comes from the fact that for any $x \in \mathbb{R}$ we have $\Big|x\Big| \leqslant 1 + |x|^2$. Indeed either $\Big|x\Big| \leq 1$ and then $\Big|x\Big| \leqslant 1 + |x|^2$, or $\Big|x\Big| > 1$ but then $\Big|x\Big| < x^2 \leqslant 1 + x^2$.

Now for a given $\varepsilon > 0$, just pick $\delta = \varepsilon$: then for any pair x, y satisfying $|x - y| < \delta$ we get that $\Big|\frac{1}{1+x^2} - \frac{1}{1+y^2}\Big| \leqslant \Big|x - y\Big| < \delta = \varepsilon$.

□

Exercise 7. Section 5.4, #9.

Proof. Just notice $\Big|\frac{1}{f(x)} - \frac{1}{f(y)}\Big| = \Big|f(x) - f(y)\Big|.\Big|\frac{1}{f(x).f(y)}\Big| \leqslant \frac{1}{k^2}\Big|f(x) - f(y)\Big|$.

Now : f is uniformly continuous so for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any x, y satisfying $\Big|x - y\Big| < \delta$ we have $\Big|f(x) - f(y)\Big| < k^2.\varepsilon$.

This will imply that for that given $\varepsilon > 0$, if we have $|x - y| < \delta$ we deduce $\Big|\frac{1}{f(x)} - \frac{1}{f(y)}\Big| < \frac{1}{k^2}k^2.\varepsilon = \varepsilon$.

□

Exercise 8. Section 5.6, #10.

Proof. Let c be the interior point where f attains a max. We have $a < c < b$. If either $f(a) = f(c)$ or $f(b) = f(c)$ then we are done (f will not be injective). If not, then pick any k between

$\max(f(a), f(b))$ and $f(c)$. Then by the intermediate value theorem, you know that f will reach that value k once in $[a, c]$, and once in $[c, b]$ (observe that k is not reached at c, so we really get two different points having the same value), and this proves that f is not injective. $\square$