

SOLUTIONS FOR HW7

Exercise 1. Section 4.1, #2.

Proof. Notice that $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2} \leq \frac{1}{2}|x-4|$, therefore if you take $|x-4| < 1$ you will get that $|\sqrt{x} - 2| < 1/2$ and for the second inequality it is sufficient to take $|x-4| < 2 \cdot 10^{-2}$. \square

Exercise 2. Section 4.1, #9d.

Proof. One has $\left| \frac{x^2-x+1}{x+1} - \frac{1}{2} \right| = \left| \frac{2x^2-x+1-x-1}{2x+2} \right| = \left| \frac{2x \cdot (x-1)}{2x+2} \right| = \left| \frac{2x}{2x+2} \right| \cdot |x-1|$.

Now, on the neighborhood $[0, +\infty)$ of the point 1, we have that $\left| \frac{2x}{2x+2} \right| \leq 1$. Therefore for a given $\varepsilon > 0$, for any $x \in [0, +\infty)$ satisfying $|x-1| < \varepsilon$ one has that $\left| \frac{x^2-x+1}{x+1} - \frac{1}{2} \right| < \varepsilon$. \square

Exercise 3. Section 4.1, #10b.

Proof. One has $\left| \frac{x+5}{2x+3} - 4 \right| = \left| \frac{x+5-8x-12}{2x+3} \right| = \frac{7}{|2x+3|} \cdot |x+1|$

Now it will be sufficient to prove that $\frac{7}{|2x+3|}$ is bounded in a neighborhood of -1 . (Notice that it is NOT bounded everywhere! More precisely that function is large when you are too close to $-3/2$). Let's consider the following neighborhood of -1 given by $V = (-5/4, 0)$.

Since $(x > -5/4) \Rightarrow 2x+3 > \frac{-5}{2} + 3 = \frac{1}{2}$ we get the following inequality:

on V , $0 < \frac{7}{|2x+3|} < 2.7 = 14$. Therefore, for a given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{14}$. Then for any $x \in V$ satisfying $|x+1| < \frac{\varepsilon}{14}$, one has $\left| \frac{x+5}{2x+3} - 4 \right| < \varepsilon$. Thus $\lim_{x \rightarrow -1} f(x) = 4$. \square

Exercise 4. Section 4.1, #11c.

Proof. The functions $x \mapsto x + \operatorname{sgn}(x)$ has a left-hand limit at zero equal to -1 and a right-hand limit equal to $+1$, therefore it has no limit at zero. \square

Exercise 5. Section 4.1, #13.

Proof. When $x \rightarrow c$, to say that $|f(x)|^2 \rightarrow 0$ implies that $|f(x)| \rightarrow 0$ (square root rule) which is equivalent to the following phrase: $\lim_{x \rightarrow c} f(x) = 0$.

Now take $c = 0$, and the function $f(x) = \operatorname{sign}(x)$, and declare by convention that $f(0) = 1$. The square of this function is a constant function equal to 1, but the function itself has no limit at 0. \square

Exercise 6. Section 4.2, #1d.

Proof. The sum rule implies that the numerator has a limit equal to 1, and by the sum and product rules the denominator has a limit equal to 2. Then the Quotient rule implies that the limit is $1/2$. (Notice that the denominator is never zero!)

□

Exercise 7. Section 4.2, #4.

Proof. We proved in class the non-existence of $\lim_{x \rightarrow 0} \cos(1/x)$ (take the two sequences converging to zero given by $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{(2n+1)\pi}$ the function takes constant values equal to 1 on the first one, and constant values equal to -1 on the other).

Now the function $x \cdot \cos(1/x)$ has a limit equal to zero at zero because of the squeeze theorem applied to the following inequality:

$$0 \leq |x \cdot \cos(1/x)| \leq |x| \rightarrow 0, \text{ when } x \rightarrow 0.$$

□

Exercise 8. Section 4.2, #5.

Proof. On a neighborhood V of c one has an inequality of the form $0 \leq |f(x) \cdot g(x)| \leq M \cdot |g(x)|$, where M is an upper bound for f on V . Now apply the squeeze theorem to that inequality to get the result.

□