Problem 1. *Expand in continued fraction* $\sqrt{2}$ *and* $\sqrt{15}$.

Problem 2. Diophantine conditions

Given some fixed real number $k \geq 2$, *let us say that an irrational number* ζ *satisfies a Diophantine condition of order* k *if there is some* $\epsilon > 0$ *(depending on* ζ*) so that*

$$|\zeta - \frac{p}{q}| > \frac{\epsilon}{q^k},$$

for every rational number $\frac{p}{q}$.*We write* D_k *the set of all irrational numbers* ζ *which satisfy such a condition.*

Now let f *be a polynomial of degree* d *with integer coefficients, and suppose that* $f(\alpha) = 0$ *where* α *is irrational. If every other root of this equation has distance at least* ϵ *from* α, *and if* $|f'(x)| < K$ *in the open interval* $(\alpha - \epsilon, \alpha + \epsilon)$, *show that*

$$K.|\alpha - p/q| \geq |f(p/q)| \geq 1/q^d$$

for every rational number p/q *in* $(\alpha - \epsilon, \alpha + \epsilon)$. *Conclude that* $\alpha \in D_d$, *and hence that all irrational numbers in the complement of the union of all the* D_d *are transcendental (this means that they cannot be roots of a polynomial with integer coeffients).*

Problem 3. *Example of transcendental numbers:(due to Liouville). Show that the number*

$$\alpha = \sum_{n=0}^{\infty} \frac{1}{10^{n!}}$$

is transcendental.

Hint: *Look at the partial sum* $\frac{p_k}{q_k} = \sum_{n=0}^{\infty} \frac{1}{10^{n!}}$, *with* $q_k = 10^{k!}$. *Then try to find a constant* S *such that* $|\alpha - \frac{p_k}{q_k}| \leq \frac{S}{q_k^{k+1}}$. *Conclude with the previous problem.*

Problem 4. *Find two rational numbers* a/b *such that*

$$|\sqrt{2} - \frac{a}{b}| < \frac{1}{\sqrt{5}b^2}.$$