Problem 1. *Expand in continued fractions the following rational numbers:* $\frac{67}{41}, \frac{111}{19}$.

Answer. You will find $67/41 = \langle 1,1,1,1,2,1,3\rangle$ and $111/19 = \langle 5,1,5,3\rangle$.

Problem 2. *We write a continued fraction* $a_0 + \frac{1}{a_1+\frac{1}{a_2+\dots}}$ *as* $\langle a_0, a_1, \dots\rangle$*. You can truncate the continued fraction in order to get* $\langle a_0, a_1, \dots, a_n\rangle$ *and reduce the result to a fraction* $r_n = \frac{p_n}{q_n}$*. For* $n \geq 1$*, prove that* $\frac{q_n}{q_{n-1}} = \langle a_n, a_{n-1}, \dots, a_2, a_1\rangle$*. Find and prove a similar continued fraction expansion for* $\frac{p_n}{p_{n-1}}$*, assuming* $a_0 \geq 0$*.*

Answer. Prove it by induction:

1. For $n = 1$, one has $q_0 = 1, q_1 = a_1$ so one gets $q_1/q_0 = a_1$;
2. Assume the result is true for n: then one has $\langle a_{n+1}, a_n, \dots, a_2, a_1\rangle = a_{n+1} + \dfrac{1}{\langle a_n, \dots, a_2, a_1\rangle} = a_{n+1} + q_{n-1}/q_n = \frac{a_{n+1}q_n+q_{n-1}}{q_n} = \frac{q_{n+1}}{q_n}$ (remember how we get the expansion using Euclid's algorithm). A similar proof will show that $p_n/p_{n-1} = \langle a_n, \dots, a_1, a_0\rangle$.

Problem 3. *Let* u_0/u_1 *be a rational number in its lowest terms, and write* $u_0/u_1 = \langle a_0, a_1, \dots, a_n\rangle$*. Show that if* $0 \leq i < n$*, then* $|r_i - u_0/u_1| \leq 1/(q_i q_{i+1})$*, with equality if and only if* $i = n-1$*.(Here* $r_i = p_i/q_i$ *is the truncated fraction equal to* $\langle a_0, a_1, \dots, a_i\rangle$*).*

Answer. The inequality has been proved in class. Now if $i = n-1$, one has $|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}| = \frac{1}{q_n q_{n-1}}$ because we know that $p_n q_{n-1} - p_{n-1} q_n = \pm 1$.

Assume now that one has equality: since we know that the even terms p_{2k}/q_{2k} are strictly increasing towards the limit p_n/q_n, that the odd terms strictly decrease, and that the difference between consecutive terms is $\pm 1/q_i q_{i+1}$, we deduce that the equality is possible only when the initial fraction is exactly one of the approximants (and this happens only with the last one).

Problem 4 (Geometric interpretation of the denominators q_n)**.** *For an irrational number* ζ *(this greek letter is called "zeta"), consider the point on the unit circle* $\lambda = e^{2\pi i \zeta}$ *(this greek letter is called "lambda").We study the orbit* $1 \mapsto \lambda \mapsto \lambda^2 \mapsto \dots$ *under the rotation* $z \mapsto \lambda z$ *of the circle. We say that a point* λ^q *on this orbit is a* **closest return** *to* 1 *if*

$$|\lambda^q - 1| < |\lambda^m - 1|$$

for every m *with* $0 < m < q$*, so that* λ^q *is closer to* 1 *than any preceding point on the orbit.*

Show that the point $\lambda^q = e^{2\pi i \zeta q}$ *is a closest return to* 1 *along the orbit*

$$1 \mapsto \lambda \mapsto \lambda^2 \mapsto \dots$$

if and only if q *is one of the denominators* $1 = q_1 \le q_2 < q_3, \dots$ *in the continued fraction approximations to* ζ*. Furthermore, if* $q = q_n$ *with* $n \ge 2$ *then the order of magnitude of the distance* $|\lambda^q - 1|$ *is given by*

$$\frac{2}{q_{n+1}} < |\lambda^{q_n} - 1| < \frac{2\pi}{q_{n+1}}$$

Answer. As in the class, let's prove that a best approximation to ζ is necessarily of the form p_n/q_n, and that for $n \ge 1$, q_n is the smallest integer $q > q_{n-1}$ such that $\|q\zeta\| < \|q_{n-1}\zeta\|$. Let's consider a/b a best approximation to ζ.

First, suppose $a/b < p_0/q_0 = a_0/1$, then $|\zeta - a_0| < |\zeta - a/b| \le |b\zeta - a|$ (contradiction with the fact that a/b is a best approximation). Second, suppose $a/b > p_1/q_1$, then $|a/b - \zeta| > |a/b - p_1/q_1| \ge \frac{1}{bq_1}$ and therefore one would have $|b\zeta - a| > \frac{1}{q_1} = \frac{1}{a_1} \ge |\zeta - a_0|$. (contradiction). Finally assume that a/b is strictly between $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_{n+1}}{q_{n+1}}$, then

$$\frac{1}{bq_{n-1}} \le |\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}| < |\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}| = \frac{1}{q_n q_{n-1}},$$

from which we deduce that $q_n < b$. On the other hand we know that

$$\frac{1}{bq_{n+1}} \le |\frac{a}{b} - \frac{p_{n+1}}{q_{n+1}}| \le |\zeta - \frac{a}{b}|,$$

which implies

$$|q_n\zeta - p_n| < \frac{1}{q_{n+1}} \le |b\zeta - a|,$$

and together with $q_n < b$, this is a contradiction to the fact that a/b is a best approximation.

Now let's prove by induction on n the second part of the theorem (that q_n is the smallest integer $q > q_{n-1}$ such that $\|q\zeta\| < \|q_{n-1}\zeta\|$):

For $n = 0$, there is nothing to prove (because $q_0 = 1$), assume the property is true for $n \ge 0$. Let q be the smallest integer $> q_n$ such that $\|q\zeta\| < \|q_n\zeta\|$ and let p be such that $\|q\zeta\| = |q\zeta - p|$. Then by induction p_n/q_n is a best approximation, so p/q is also a best approximation., therefore it must be of the form $p_{n'}/q_{n'}$, but q is chosen as the smallest such that $\|q\zeta\| < \|q_n\zeta\|$, so $q = q_{n+1}$, and then automatically $p = p_{n+1}$ and we are done.