Problem 1. *Expand in continued fractions the following rational numbers:* $\frac{67}{41}$, $\frac{111}{19}$.

Problem 2. *We write a continued fraction* $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$ *as* $\langle a_0, a_1, \dots \rangle$. *You can truncate the continued fraction in order to get* $\langle a_0, a_1, \dots, a_n \rangle$ *and reduce the result to a fraction* $r_n = \frac{p_n}{q_n}$. *For* $n \geq 1$, *prove that* $\frac{q_{n+1}}{q_n} = \langle a_n, a_{n-1}, \dots, a_2, a_1 \rangle$. *Find and prove a similar continued fraction expansion for* $\frac{p_n}{p_{n-1}}$, *assuming* $a_0 \geq 0$.

Problem 3. *Let* u_0/u_1 *be a rational number in its lowest terms, and write* $u_0/u_1 = \langle a_0, a_1, \dots, a_n \rangle$. *Show that if* $0 \leq i < n$, *then* $|r_i - u_0/u_1| \leq 1/(q_i q_{i+1})$, *with equality if and only if* $i = n-1$.*(Here* $r_i = p_i/q_i$ *is the truncated fraction equal to* $\langle a_0, a_1, \dots, a_i \rangle$*).*

Problem 4 (Geometric interpretation of the denominators q_n)**.** *For an irrational number* ζ *(this greek letter is called "zeta"), consider the point on the unit circle* $\lambda = e^{2\pi i \zeta}$ *(this greek letter is called "lambda").We study the orbit* $1 \mapsto \lambda \mapsto \lambda^2 \mapsto \dots$ *under the rotation* $z \mapsto \lambda z$ *of the circle. We say that a point* λ^q *on this orbit is a* **closest return** *to* 1 *if*

$$|\lambda^q - 1| < |\lambda^m - 1|$$

for every m *with* $0 < m < q$, *so that* λ^q *is closer to* 1 *than any preceding point on the orbit.*

Show that the point $\lambda^q = e^{2\pi i \zeta q}$ *is a closest return to* 1 *along the orbit*

$$1 \mapsto \lambda \mapsto \lambda^2 \mapsto \dots$$

if and only if q *is one of the denominators* $1 = q_1 \leq q_2 < q_3, \dots$ *in the continued fraction approximations to* ζ. *Furthermore, if* $q = q_n$ *with* $n \geq 2$ *then the order of magnitude of the distance* $|\lambda^q - 1|$ *is given by*

$$\frac{2}{q_{n+1}} < |\lambda^{q_n} - 1| < \frac{2\pi}{q_{n+1}}$$

Some hints for that: *prove that* $|\lambda^m - 1| = 2\sin(\pi << m\zeta >>)$, *where* $<< x >>$ *represents* $min|x+n|, n \in \mathbb{Z}$ *(the distance from the point* x *to the closest integer).Then use the fact that* $4 < 2\sin(\pi t)/t < 2\pi$ *for* $t \in (0, 1/2)$. *Also notice that* $q_n \zeta \equiv x_n$ *mod* $\mathbb{Z}$, *and remember that we proved that* $|x_n| < 1/2$ *for* n *larger than 2.*