Problem 1. *Consider the map $f : \mathbb{Z}[X] \to \mathbb{Z} \times \mathbb{Z}$, given by $P(X) \mapsto (P(1), P(2))$. Is it a surjective map? What is the kernel of it?*

Answer. The map is clearly surjective: if you pick any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, then one preimage is $P(X) = b(X-1) - a(X-2)$. The map is certainly not injective: for example $S(X) = (X-1)(X-2)$ is in the kernel. Let's find the kernel: by euclidian division, one can write: $P(X) = (X-1)(X-2).Q(X) + aX + b$ for any polynomial $P(X) \in \mathbb{Z}[X]$.But one can do more, actually $aX + b = P(2)(X-1) - P(1)(X-2)$ (just notice that $P(1) = a + b, P(2) = 2a + b$). Therefore immediately one knows that $ker f$ is the principal ideal generated by $(X-1)(X-2)$.

Problem 2. *As in the class, given a ring R, we define $SpecR$ as the set of all prime ideals of R, distinct from R itself. $SpecR$ is a topological space, once we define the closed sets as follows: the closed sets are all the sets of prime ideals of the form $V(I)$, where I is an ideal and $V(I)$ is the set of all prime ideals in $SpecR$ that contain I.*

1. *show that if $Z_1 = V(I_1), Z_2 = V(I_2)$ are two closed sets, then $Z_1 \cap Z_2 = V(I_1 + I_2)$ and $Z_1 \cup Z_2 = V(I_1 \cap I_2)$;*

2. *prove that the intersection of any collection of closed sets Z_i is still a closed set.*

Answer.

1. If a prime ideal $\mathcal{P}$ is in $Z_1 \cap Z_2$, then it contains both I and J, hence it contains their sum $I + J$, hence it is in $V(I + J)$. Conversely a prime ideal containing the sum must contain each of the two ideals. Now if $\mathcal{P}$ is in $Z_1 \cup Z_2$ then it contains either I_1 or I_2, and in both cases it contains $I_1 \cap I_2$, so it is in $V(I_1 \cap I_2)$. Now assume that you have a proper prime ideal $\mathcal{P}$ in $V(I_1 \cap I_2)$: then either it contains I_1 (and then we are done because $\mathcal{P}$ will be in $V(I_1) \cup V(I_2)$), or it doesn't contain I_1, therefore there exists $i \in I_1$ that is not in $\mathcal{P}$. Now pick any $j \in I_2$: the product $i.j \in I_1 \cap I_2$, so it is in $\mathcal{P}$, but this ideal is prime and doesn't contain i, therefore it must contain j, for any $j \in I_2$ hence $\mathcal{P}$ contains I_2 and we are done.

2. The same argument works for any family of ideals: the intersection of any family of $V(I_\alpha)$ is simply $V(\sum_\alpha I_\alpha)$, where the sum of any family of ideals is defined as the set of all finite sums of elements taken in these ideals.

Problem 3. *Let A be a ring and I, J two ideals in A. Let's write the "reduction map" $\rho : A \longrightarrow A/I$ that takes any $a \in A$ and returns $a \bmod I$ (it can be written as $\bar{a}$ if you prefer).*

1. *Show that $\rho(J)$ is an ideal in A/I.*
2. *Show that $A/(I+J)$ is isomorphic to $(A/I)/(\rho(J))$.*
3. *Application: show that $\mathbb{Z}[X]/((3)+(X^2+5))$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})[\sqrt{-5}]$.*

Answer.

1. We can notice that $\rho(J)$ is just J mod I: it's clearly an additive group (because $\overline{j_1} + \overline{j_2} = \overline{j_1 + j_2}$) and if you multiply any $\bar{j}$ by $\bar{a}$ (where a is any element of A), then you get $\overline{a.j}$ which is in $\rho(J)$ because J is an ideal and therefore $a.j \in J$.

2. Consider the map $A \to A/I \to (A/I)/(\rho(J))$.It is surjective (composition of two surjective maps). What about the kernel? Well the kernel is the set of all elements $a \in A$ such that a mod $I = j$ mod I for some $j \in J$, but this means that $a - j \in I$ so $a - j = i$ for some $i \in I$, or if you prefer $a = i + j$. Thus the kernel of the map is included in $I + J$. Conversely $I + J$ is in the kernel. By the isomorphism theorem we know that $A/(I+J)$ is then isomorphic to $(A/I)/(\rho(J))$.

3. Application: replace I by (3) and J by (X^2+5).