Math 311 — Homework 7 due on Monday 03/26

Problem 1. *Consider the map* $f : \mathbb{Z}[X] \to \mathbb{Z} \times \mathbb{Z}$*, given by* $P(X) \mapsto (P(1), P(2))$*. Is it a surjective map? What is the kernel of it?*

Problem 2. *As in the class, given a ring* R*, we define* $SpecR$ *as the set of all prime ideals of* R*, distinct from* R *itself.* $SpecR$ *is a topological space, once we define the closed sets as follows: the closed sets are all the sets of prime ideals of the form* $V(I)$*, where* I *is an ideal and* $V(I)$ *is the set of all prime ideals in* $SpecR$ *that contain* I*.*

1. *show that if* $Z_1 = V(I_1), Z_2 = V(I_2)$ *are two closed sets, then* $Z_1 \cap Z_2 = V(I_1 + I_2)$ *and* $Z_1 \cup Z_2 = V(I_1 \cap I_2)$*;*
2. *prove that the intersection of any collection of closed sets* Z_i *is still a closed set.*

Problem 3. *Let* A *be a ring and* I, J *two ideals in* A*. Let's write the "reduction map"* $\rho : A \longrightarrow A/I$ *that takes any* $a \in A$ *and returns* $a \bmod I$ *(it can be written as* $\bar{a}$ *if you prefer).*

1. *Show that* $\rho(J)$ *is an ideal in* A/I*.*
2. *Show that* $A/(I + J)$ *is isomorphic to* $(A/I)/(\rho(J))$*.*
3. *Application: show that* $\mathbb{Z}[X]/((3) + (X^2 + 5))$ *is isomorphic to* $(\mathbb{Z}/3\mathbb{Z})[\sqrt{-5}]$*.*