Problem 1. *Characterize the set of positive integers n such that* $\phi(2n) > \phi(n)$.

Answer. If n is odd then $\phi(2n) = \phi(2).\phi(n) = \phi(n)$. If n is even, it can be written as $n = 2^k.m$ with m odd. Thus $\phi(2n) = \phi(2^{k+1}.m) = \phi(2^{k+1}).\phi(m) = (2^{k+1} - 2^k).\phi(m) = 2.\phi(n) > \phi(n)$.Therefore the set of positive integers such that $\phi(2n) > \phi(n)$ coincides with the set of even integers.

Problem 2. *What are the last two digits, that is the tens and units digits of* $2^{1000}, 3^{1000}$?

Answer. For the first one, you can notice that $2^{12} \equiv -4(\bmod 100)$ so $2^{12.12} \equiv 2^4$. Therefore $2^{1000} \equiv 2^{6.144+136} \equiv 2^{6.4}.2^{136} \equiv 2^{12.12+16} \equiv 2^4.2^{16} = 2^{12+8} \equiv (-4).2^8 \equiv -24 \equiv 76$ Therefore the last two digits are 76. For 3, things are much simpler: $3^{\phi(100)} \equiv 1$ so $3^{40} \equiv 1$. Thus $3^{1000} \equiv 3^{40.25} \equiv 1$, so the last digits are 01.

Problem 3. *Prove that for* $n \geq 2$ *the sum of all the positive integers less than n and coprime with n is* $\frac{n}{2}.\phi(n)$.

Answer. The answer I proposed to you was a bit long (you can still ask me if you tried it). Instead here is a simpler one, by one of you, Kevin Donahue. First, realize that $n/2$ is never coprime with n. Then if you pick any integer a coprime with n and less than $n/2$, the integer $n - a$ is coprime with n too and is between $n/2$ and n. Therefore the $\phi(n)$ integers that are coprime with n and less than n can be partitioned into two collections A and B with the same number of elements ($\phi(n)/2$), and each $a \in A$ can be paired with $n - a \in B$.Thus when you sum all of these pairs you get $n.\frac{\phi(n)}{2}$, which is the answer.

Problem 4. *Find all the primes p such that p divides* $2^p + 1$.

Answer. The prime 2 has not the property, so we can assume now that p is odd, but then Fermat's theorem implies that $2^p + 1 \equiv 3$ modulo p, therefore p must divide 3. Now 3 actually divides 9, so 3 is the only prime with this property.

Problem 5. *Show that* $x^2 - 2y^2 + 8z = 3$ *has no solutions* $(x, y, z) \in \mathbb{Z}^3$. (**Hint:** *reduce modulo 8).*

Answer. Clearly x must be odd (therefore congruent to $1, 3, 5, 7$ whose squares are all congruent to 1). So necessarily, $2y^2 \equiv -2$. But this is impossible because $2t^2$ only takes the values 0 or 2 modulo 8

Problem 6. *For any n show that* $\phi(n) = n.(1 - \frac{1}{p_1}) \ldots (1 - \frac{1}{p_k})$ *where the* p_i *are the prime factors present in the prime decomposition of n.* (**Hint:**: *compute first* $\phi(p^s)$, *for any prime p.)*

Answer. First, $\phi(p^s) = p^s - p^{s-1} = p^s(1 - \frac{1}{p})$ because the only integers less than p^s and not coprime to p^s are the multiples of p. Since we proved that $(G.C.D.(m, n) = 1 \Rightarrow \phi(m.n) = \phi(m).\phi(n))$, we can consider the decomposition of n in prime factors, $n = p_1^{r_1} \dots . p_k^{r_k}$ and thus obtain

$$\phi(n) = \phi(p_1^{r_1}) \dots . \phi(p_k^{r_k}) = (p_1^{r_1}.(1 - \frac{1}{p_1})) \dots .. (p_k^{r_k}.(1 - \frac{1}{p_k})).$$