Problem 1. *Find all the solutions in integers of* $71x - 50y = 1$.

Answer. First we notice that 71 and 50 are coprime (indeed 71 is prime, and 50 is not a multiple of it). Therefore we know that the problem has an infinity of solutions. Let's find one of it by Euclid's algorithm:

$$\begin{aligned} 71 &= 1.50 + 21 \\ 50 &= 2.21 + 8 \\ 21 &= 2.8 + 5 \\ 8 &= 1.5 + 3 \\ 5 &= 1.3 + 2 \\ 3 &= 1.2 + 1 \end{aligned}$$

And then we undo what we did:

$$\begin{array}{ccccc} 1 & = & 3-2 & = & 3-(5-3) \\ & & & = & 2.3-5 \\ & & & = & 2.(8-5)-5 \\ & & & = & 2.8-3.5 \\ & & & = & 2.8-3.(21-2.8) \\ & & & = & 8.8-3.21 \\ & & & = & 8.(50-2.21)-3.21 \\ & & & = & 8.50-19.21 \\ & & & = & 8.50-19.(71-50) \\ & & & = & 27.50-19.71 \end{array}$$

Therefore we know that the solutions are exactly of the form

$$\{(x,y)/x = -19 + 50t, y = -27 + 71t, t \in \mathbb{Z}\}.$$

Problem 2. *If a and b are any positive integers* > 2, *then prove that* $2^a + 1$ *is not divisible by* $2^b - 1$.

Answer. By Euclidian division, one can write $a = b.s + r$. So if we write $m = 2^b - 1$, one has

$$2^a + 1 \equiv 2^r.(2^b)^s + 1 \equiv 2^r + 1 (\text{mod } m)$$

Now notice that $b > 2$ implies that $m > 3$ therefore if $r = 0, 1$ then $0 \leq 2^r + 1 < m$ and m does not divide $2^r + 1$. Thus we can assume $r \geq 2$. Since we know $r + 1 \leq b$ we deduce $2.2^r \leq 2^b$ and so $2^r + 1 \leq 2^b - 2^r + 1 \leq 2^b - 2$ because $r \leq 2$ implies $1 - 2^r \leq -2$.

Problem 3. *Show that if* $G.C.D(a,b) = 1$ *then* $G.C.D.(a+b, a^2 - a.b + b^2) = 1$ *or* 3.

Answer. Let's call d the $G.C.D.(a+b, a^2 - a.b + b^2)$. Then d must also divide $(a+b)^2$ and $a^2 - a.b + b^2$, so it must divide their difference $3ab$. I claim that d and ab are relatively prime: indeed if p was a prime factor common to ab and d, then p should divide a or b, but also $a+b$, so it would divide both a and b (absurd). Therefore d must divide 3, which means that d is ± 1 or ± 3.

Problem 4. *Using congruences, show that 7 divides $(3^{2n+1} + 2^{n+2})$ for all $n \geqslant 1$.*

Answer. It's simply a matter of seeing that

$$(3^{2n+1} + 2^{n+2}) \equiv 3.2^n + 4.2^n \equiv 7.2^n \equiv 0(\bmod 7)$$

because $(3^2 \equiv 2)$.

Problem 5. *Find all the solutions of the congruence $x^2 + 4x + 2 \equiv 0(mod 7)$.*

Answer. There are many different ways of solving this. One way is to compute the values taken by the polynomial $X^2 + 4X + 2$ at the points $\{0, \ldots, 6\}$, and find out when this is 0 mod 7. Or you can notice that $x^2 + 4x + 2 \equiv x^2 - 3x + 2 \equiv (x-1)(x-2)$. But since $\mathbb{Z}/7\mathbb{Z}$ is a field $(x-1)(x-2) \equiv 0$ is equivalent to $x \equiv 1$ or $x \equiv 2$, which is the answer.

Problem 6. *Consider a polygon centered on the origin and that is regular with m sides (regular means that all the sides have same length). You can go in the counterclockwise direction and number all the vertices from 1 to m. Consider now a counterclockwise rotation (with center the origin) that brings the vertex 1 to the vertex $1+k$, where k and m are coprime. Can you show that by iterating this same rotation you will visit all the vertices of the polygon?*

Answer. By a rotation-dilation centered at the origin, one can bring the vertices of our regular polygon to the $m-$th roots of the unity $\{1, e^{i\frac{2\pi}{m}}, \ldots, e^{i\frac{2\pi}{m}(m-1)}\}$. Therefore the rotation we consider corresponds to the multiplication by $e^{i\frac{2\pi}{m}(k)}$. Now we have seen that $G.C.D(k, m) = 1$ implies that there exists integers s, t such that $1 = s.k + t.m$. Therefore

$$e^{i\frac{2\pi}{m}} = (e^{i\frac{2\pi}{m}(k)})^s.(e^{i\frac{2\pi}{m}(m)})^t = (e^{i\frac{2\pi}{m}(k)})^s$$

But we are done now because we know that by multiplying by $\frac{2\pi}{m}$ we can get all the $m-$th roots of the unity.