THE DIRICHLET PROBLEM WITH PRESCRIBED INTERIOR SINGULARITIES

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ABSTRACT

In this paper we solve the nonlinear Dirichlet problem (uniquely) for functions with prescribed asymptotic singularities at a finite number of points, and with arbitrary continuous boundary data, on a domain in $\mathbf{R}^n$. The main results apply, in particular, to subequations with a Riesz characteristic $p \geq 2$. It is shown that, without requiring uniform ellipticity, the Dirichlet problem can be solved uniquely for arbitrary continuous boundary data with singularities asymptotic to the Riesz kernel $\Theta_j K_p(x - x_j)$ where

$$K_p(x) \;=\; \begin{cases} -\frac{1}{|x|^{p-2}} & \text{for } \; 2 < p < \infty, \\ \log|x| & \text{if } \; p = 2. \end{cases}$$

at any prescribed finite set of points $\{x_1, ..., x_k\}$ in the domain and any finite set of positive real numbers $\Theta_1, ..., \Theta_k$. This sharpens a previous result of the authors concerning the discreteness of high-density sets of subsolutions.

Uniqueness and existence results are also established for finite-type singularities such as $\Theta_j|x - x_j|^{2-p}$ for $1 \leq p < 2$.

The main results apply similarly with prescribed singularities asymptotic to the fundamental solutions of Armstrong-Sirakov-Smart (in the uniformly elliptic case).

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1. Introduction and Statement of Some Main Results.

The aim of this paper is to study the Dirichlet problem for functions with prescribed asymptotic singularities on a domain in $\mathbf{R}^n$. We shall adopt the notation and definitions in our previous work [HL$_{1,3,6,10}$].

Throughout the paper $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ will denote a closed set which satisfies the weakest possible ellipticity condition:

(F1). $A \in F$ and $P \geq 0 \quad \Rightarrow \quad A + P \in F$,

so that solutions can be taken in the viscosity sense (cf. [C], [CIL], [CC]). Said differently, F is a constant coefficient, pure second-order subequation in $\mathbf{R}^n$.

In addition we will always assume that:

(F2). F is a cone with vertex at the origin (i.e., $tF = F$ for $t > 0$).

We will refer to an F satisfying (F1) and (F2) succinctly as a **cone subequation**.

For certain existence results we shall also make the mild requirement that

(F3). $-P_e \notin F$ for all unit vectors $e \in \mathbf{R}^n$

where P_e denotes orthogonal projection onto the line $\mathbf{R} \cdot e$. Said differently, quadratic functions such as $u(x) = -x_1^2$ are not allowed to be F-subharmonic. This property (F3) is equivalent to the following condition:

(B3). All smooth boundaries are strictly $\widetilde{F}$-convex.

See Proposition 1.10 below for the proof of this equivalence and further discussion.

Our asymptotic singularities will be prescribed by functions of the following type.

Definition 1.1. A **downward-pointing singular F-subharmonic** at a point $x_0 \in \mathbf{R}^n$ is a continuous $[-\infty, \infty)$-valued F-subharmonic function ψ defined on a neighborhood U of x_0 with $\psi(x) > \psi(x_0)$ for $x \in U - \{x_0\}$, such that either:

(Polar Case): $\psi(x_0) = -\infty$, or

(Finite Case): $\psi(x_0) > -\infty$, and ψ has no test functions at x_0.

If in addition ψ is F-harmonic on $U - \{x_0\}$, then ψ will be referred to as a **downward-pointing singular F-harmonic** at x_0, and will usually be denoted by h instead of ψ.

Note that in the polar case ψ also has no test functions at x_0.

The problem we want to address is the following, which we will refer to as the **(DPPS)**.

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Let Ω be a bounded domain in $\mathbf{R}^n$, and fix a finite number of points $x_1, ..., x_k \in \Omega$.

Boundary Data: This consists of a function $\varphi \in C(\partial\Omega)$ along with a downward-pointing singular F-harmonic function h_j at each point x_j.

A function H is a **solution** of the corresponding (DPPS) if

(1) $H \in C(\overline{\Omega} - \{x_1, ..., x_k\})$

(2) H is F-harmonic on $\Omega - \{x_1, ..., x_k\}$,

(3) $H\big|_{\partial\Omega} = \varphi$,

(4) H is **asymptotically equivalent to** h_j **at** x_j. By definition this means that:

(4a) (In the Polar Case). There exists a constant $C > 0$ such that for each $j = 1, ..., k$

$$h_j(x) - C \ \leq \ H(x) \ \leq \ h_j(x) + C \qquad \text{near } x_j.$$

(4b) (In the Finite Case). For each $j = 1, ..., k$

$$\lim_{x \to x_j} \frac{H(x) - H(x_j)}{h_j(x) - h_j(x_j)} \ = \ 1.$$

Remark 1.2. One easily verifies that asymptotic equivalence is indeed an equivalence relation (in fact, on the larger space of upper semi-continuous functions defined in a neighborhood of the origin). Each equivalence class is invariant under the change of functions by an additive constant. We will denote asymptotic equivalence by $H \approx h$ in the polar case, and by $H \sim h$ in the finite case.

The following result is an immediate consequence of the comparison Theorems 5.2 and 5.4 proved in Section 5.

THEOREM 1.3. (Uniqueness). *For any cone subequation F there is at most one function H with properties (1) through (4).*

An existence construction is presented in Section 6 and then completed in the polar case by proving the following theorem. Our existence results in the finite case are stated and proved in Section 7.

THEOREM 1.4. (Existence in the Polar Case). *Suppose that F is a cone subequation satisfying Condition (F3) = (B3), and that the boundary of Ω is smooth and strictly F-convex, Assume that each prescribed singularity h_j at x_j is a downward-pointing F-harmonic of polar type. Finally assume:*

Hypothesis (H): *There exists a continuous F-subharmonic function ψ on Ω, finite except at $x_1, ..., x_k \in \Omega$, with $\psi \approx h_j$ at x_j for each j.*

Then the Dirichlet Problem with Prescribed Singularities described above has a solution. Moreover, it is uniquely determined as

$$H(x) \;=\; \sup_{v \in \mathcal{F}} v(x)$$

where

$$\mathcal{F} \equiv \{v \in \mathrm{USC}(\overline{\Omega}) : v \text{ is } F\text{-subharmonic on } \Omega,\ v \leq \varphi \text{ on } \partial\Omega, \\ \text{and } v - h \text{ is bounded above near each } x_j\}.$$

Typically the functions h_j arise from a single global F-subharmonic function h on $\mathbf{R}^n$ which is F-harmonic outside the origin and has a downward-pointing singularity at 0. Such an h will be called a *generalized fundamental solution* for the subequation F. It determines the data in the (DPPS) by taking $h_j(x) = \Theta_j h(x - x_j)$ with constants $\Theta_j > 0$ for $j = 1, ..., k$, and setting

$$\psi(x) \;=\; \sum_{j=1}^{k} \Theta_j h(x - x_j). \tag{1.1}$$

In the polar case (where $h(0) = -\infty$) the condition $\psi \approx h_j$ at x_j is automatic since h is continuous outside the origin. Thus the hypothesis (H) reduces to

Hypothesis (H)′: $\psi(x) \;=\; \sum_j \Theta_j h(x - x_j)$ is F-subharmonic on $\mathbf{R}^n$.

There are two cases where this hypothesis is easily satisfied.

A Single Point Singularity at $x_0 \in \Omega$: Then $\psi(x) \equiv \Theta h(x - x_0)$ ($\Theta > 0$) is F-subharmonic on $\mathbf{R}^n$ – in fact, F-harmonic on $\mathbf{R}^n - \{x_0\}$.

The subequation F is convex: Then ψ is F-subharmonic on $\mathbf{R}^n$ because sums and positive multiples of F-subharmonic functions are also F-subharmonic.

This yields two special cases where the (DPPS) is uniquely solvable.

Corollary 1.5. *Let F and Ω be as in Theorem 1.4, and suppose h is a generalized fundamental solution for F.*

(a) For each $x_0 \in \Omega$ and $\Theta > 0$ there exists a unique solution to the (DPPS) having boundary values φ on $\partial\Omega$ and asymptotic to $\Theta h(x - x_0)$ at x_0.

(b) If, in addition, F is convex, then the multi-pole (DPPS) with ψ as in (1.1), has a unique solution.

In the case of just one point singularity with $\Theta = 1$ and the outer boundary function $\varphi \equiv 0$, the solution provided by Corollary 1.5(a) will be denoted $G_\Omega(x;\, x_0, h)$ and referred to as the **nonlinear Green's function** for the subequation F on the domain Ω with asymptotic singularity determined by the generalized fundamental solution h.

In the multi-pole case where F is required to be convex, we again take $\varphi \equiv 0$. Then the solution to the (DPPS) given by Corollary 1.5(b) will be called the **multi-pole nonlinear Green's function** and denoted by $G_\Omega(x;\, x_1, ..., x_k;\, \Theta_1, ..., \Theta_k;\, h)$.

These functions extend the classical pluri-complex Green's function with logarithmic singularities (cf. [Lem], [K$_*$], [Le], [Z]) where $h(x) = \log|x|$ on $\mathbf{R}^{2n} = \mathbf{C}^n$. (See Corollary 3.6 ff. for further discussion.)

Remark 1.6. (The Boundary Convexity Hypothesis). The hypothesis that $\partial\Omega$ is strictly F-convex in Theorem 1.4 is necessary for existence in the finite cases as well as the polar case. On course there are many domains with smooth boundary where this is true. For example, if $\partial\Omega$ is strictly convex, then $\partial\Omega$ is also strictly F-convex for any cone subequation F because $\mathcal{P} \subset F$. On the other hand, there are many cone subequations F with the property that every smooth boundary is strictly F-convex, allowing $\partial\Omega$ to be arbitrary. This is true if and only if $P_e \in \mathrm{Int}F$ for all $|e| = 1$ by Proposition 1.10 below, applied to $\widetilde{F}$ instead of F.

The importance of Condition (F3) arises because of boundary $\widetilde{F}$-convexity.

Proposition 1.10. (Concerning Condition (F3)). *For a cone subequation F the conditions:*

(F3) $-P_e \notin F$ *for any unit vector e, and*

(B3) All boundaries are strictly $\widetilde{F}$-convex

are equivalent. Furthermore, if F is a convex subequation, then (F3) and (B3) are also equivalent to:

(F3)$'$ The subequation F is complete, i.e., it cannot be defined using the variables in a proper linear subspace of $\mathbf{R}^n$.

Finally, if F is an invariant cone subequation as in Lemma 3.4, then these conditions are equivalent to

(F3)$''$ F has finite Riesz characteristic p (see Definition 3.2).

We shall use condition (B3), while condition (F3) provides the simplest test for this assumption.

Proof. For cone subequations the equivalence of (F3) and (B3) was established in Section 5 of the earlier paper [HL$_1$], but was not stated explicitly. The argument goes as follows. First, by Lemma 5.3 (ii)$'$ in [HL$_1$] condition (B3) is equivalent to

(F3)$'''$ For all $B \in \mathrm{Sym}^2(\mathbf{R}^n)$ and all unit vectors e

$$B + tP_e \in \mathrm{Int}\widetilde{F} \text{ for all } t \geq \text{some } t_0.$$

(This can be considered to be the definition of strict $\widetilde{F}$-convexity.) Second, by the Elementary Property (5) in [HL$_1$, §3] with $B' \equiv 0$, this is equivalent to

(F3)$^{(iv)}$ $P_e \in \mathrm{Int}\widetilde{F}$ for all unit vectors e.

Finally, by definition $-\mathrm{Int}\widetilde{F} =\sim F$, so that $(\mathrm{F3})^{(iv)} \iff (\mathrm{F3})$.

The equivalence of the two structural conditions (F3) and (F3)′, for F convex, was established in Proposition 3.6 of [HL_2].

For the equivalence of (F3) and (F3)″ see Lemma 3.4. ∎

2. Examples of Downward-Pointing F-Harmonics.

Example 2.1. (Riesz Kernels). Perhaps the most important such examples are the classical Reisz kernels K_p (with $p \geq 1$) which are defined and discussed in the next section. Each has a downward-pointing singularity. They play a fundamental role in standard potential theory (cf [L]). Moreover, each K_p is actually a punctured harmonic for large family of subequations – those of *Riesz characteristic p*. For such subequations F the Riesz kernels are central to the study of tangents and densities in the associated F-potential theories [HL_{10}], [HL_{11}]. In particular, they often arise as the unique tangent to any F-subharmonic function.

A large and important class of subequations with characteristic-p, which are convex cones but not uniformly elliptic, come from:

Geometrically Defined Subequations: These are the convex cone subequations $\mathcal{P}(\mathbb{G})$ determined by a closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$, of the Grassmannian of p-dimensional subspaces of $\mathbf{R}^n$, by the requirement

$$A \in F \qquad \Longleftrightarrow \qquad \mathrm{tr}\left(A\big|_W\right) \geq 0 \qquad \forall\, W \in \mathbb{G}.$$

The following **Fullness Condition on $\mathbb{G}$**:

$$\text{Each unit vector } \ e \in \mathbf{R}^n \ \text{ is contained in a subspace } \ W \in \mathbb{G}, \tag{2.1}$$

is equivalent to the p^{th} Riesz kernel being a punctured $\mathcal{P}(\mathbb{G})$-harmonic on $\mathbf{R}^n - \{0\}$. Moreover, if $\mathbb{G}$ satisfies (2.1), then Condition (F3) = (B3) also holds for $\mathcal{P}(\mathbb{G})$.

These examples contain the subequations naturally associated to many calibrations. They also include the Lagrangian subequations on $\mathbf{C}^n$. If $\mathbb{G} = G(p, \mathbf{R}^n)$, the resulting subequation $\mathcal{P}(\mathbb{G})$ (denoted $\mathcal{P}_p$ in Example 3.7(1)) is basic in geometry (cf. [Wu], [Sha] and [HL_5,] for example). See Section 2 and Appendix A in [HL_{11}] for many more examples.

Perhaps it deserves mentioning here that the potential theory associated with the subequation $\mathcal{P}(\mathbb{G})$ is more appropriately called the $\mathbb{G}$-*pluripotential theory* because of the fact that: u is $\mathcal{P}(\mathbb{G})$-subharmonic $\Leftrightarrow u\big|_W$ is Δ-subharmonic for every affine $W \in \mathbb{G}$ (proved in [HL_4]).

Example 2.2. (Homogeneous Singularities). Suppose that the function ψ in Definition 1.1 is homogeneous. This means (assuming $x_0 = 0$ for simplicity) that

$$\psi(x) = |x|^\alpha \psi\left(\frac{x}{|x|}\right) \qquad \text{for some } \ \alpha \neq 0$$

or, in the case "$\alpha = 0$", $\psi(x) = \psi(\frac{x}{|x|}) + \Theta \log|x|$ where $\sup_{|x|=1} \psi = 0$ and $\Theta > 0$. Note that the Riesz kernel K_p has homogeneity $\alpha = 2 - p$.

Suppose that ψ is a homogeneous F-subharmonic on $\mathbf{R}^n - \{0\}$. Then ψ is downward-pointing (Definition 1.1) as follows.

Case: $0 < \alpha \leq 1$**.** Then ψ has a strict minimum at $x_0 = 0$ if and only if $\psi(x) > 0$ for $x \neq 0$, in which case $\psi(0) = 0$ and $c|x|^\alpha \leq \psi(x)$ with $c = \inf_{\partial B} \psi$.. The condition of having no test functions at 0 follows easily from this inequality since $0 < \alpha \leq 1$. Thus ψ has a downward-pointing singularity at $0 \iff \psi(x) > 0$ on $|x| = 1$.

Case: $\alpha < 0$**.** Then ψ has a strict minimum at $x_0 = 0$ if and only if $\psi(x) < 0$ for $x \neq 0$, in which case $\psi(0) = -\infty$. Thus ψ has a downward-pointing singularity at $0 \iff \psi(x) < 0$ on $|x| = 1$.

Case: $\alpha = 0$**.** For any $\psi(\frac{x}{|x|})$ continuous we have $\psi(0) = -\infty$, while $\psi(x)$ is finite for $x \neq 0$. Thus ψ always has a downward-pointing singularity at 0.

Example 2.3. (The Special Cases $\mathcal{P}$, $\mathcal{P}^{\mathbf{C}}$ and $\mathcal{P}^{\mathbf{H}}$). These are the geometrically defined subequations obtained by taking $\mathbb{G}$ to be $G(1, \mathbf{R}^n)$, $G^{\mathbf{C}}(1, \mathbf{C}^n) \subset G(2, \mathbf{R}^{2n})$ and $G^{\mathbf{H}}(1, \mathbf{H}^n) \subset G(4, \mathbf{R}^{4n})$ respectively. Here there are huge families of downward-pointing F-harmonic functions h. Those which are homogeneous can be classified as follows. (The differential inequalities are in the viscosity sense.)

The $\mathcal{P}$ Case:

$$h(x) \ = |x| g\left(\frac{x}{|x|}\right) \qquad \text{with} \quad g \in C(S^{n-1}) \qquad \text{satisfying}$$
$$\text{Hess}_{S^{n-1}} g + gI \ \geq \ 0, \quad \text{and} \quad \inf_{S^{n-1}} g \ > \ 0$$

The $\mathcal{P}^{\mathbf{C}}$ Case:

$$h(x) \ = g\left([x]\right) + \Theta \log|x| \qquad \text{with} \quad g \in C(\mathbf{P}(\mathbf{C}^n)) \quad \text{and} \quad \Theta > 0 \quad \text{satisfying}$$
$$i\partial\overline{\partial} g + \Theta\omega \ \geq \ 0, \quad \text{and} \quad \sup_{\mathbf{P}(\mathbf{C}^n)} g \ = \ 0$$
$$\text{(equivalently} \quad \text{Hess}^{\mathbf{C}}_{\mathbf{P}(\mathbf{C}^n)} g + \Theta I \ \geq \ 0, \quad \text{and} \quad \sup_{\mathbf{P}(\mathbf{C}^n)} g \ = \ 0).$$

The $\mathcal{P}^{\mathbf{H}}$ Case:

$$h(x) \ = \frac{1}{|x|^2} g\left([x]\right) \qquad \text{with} \quad g \in C(\mathbf{P}(\mathbf{H}^n)) \qquad \text{satisfying}$$
$$\text{Hess}^{\mathbf{H}}_{\mathbf{P}(\mathbf{H}^n)} g - 2gI \ \geq \ 0, \quad \text{and} \quad \sup_{\mathbf{P}(\mathbf{C}^n)} g \ < \ 0$$

See Section 5 in [HL$_{11}$] for details and proofs.

Example 2.4. (Bedford-Taylor Singularities). Downward-pointing F-harmonics need not be homogeneous. The $\mathcal{P}^{\mathbf{C}}$-harmonic (i.e., maximal plurisubharmonic) functions on $\mathbf{C}^n - \{0\}$ given by

$$h(z,w) \;=\; \log(|z|^2 + |w|^4)$$

and each of its unitary rotates, define a family of distinct asymptotic equivalence classes at the origin different from the basic harmonic $\log(|z|^2 + |w|^2)$. There are also the $\mathcal{P}^{\mathbf{C}}$-harmonic functions $\log(|z|^\alpha + |w|^\beta)$ for $\alpha, \beta > 0$. These examples go back to Bedford and Taylor [BT].

Example 2.5. (Armstrong-Sirakov-Smart Singularities). In the very interesting paper [AS_1] the authors consider pure second-order cone subequations exactly as in this paper but with the additional hypothesis of uniform ellipticity (where (F3) is automatic). Under this hypothesis they establish the existence of a canonical *fundamental solution* Φ which is F-harmonic on $\mathbf{R}^n - \{0\}$, has the homogeneity property $\Phi(tx) = t^{2-p}\Phi(x), \forall\, t > 0$ for some $1 < p < \infty$, and has a downward-pointing singularity at 0. In stark contrast to the three special cases $\mathcal{P}, \mathcal{P}^{\mathbf{C}}$ and $\mathcal{P}^{\mathbf{H}}$ in Example 2.3 above, in this uniformly elliptic case there is precisely one asymptotic equivalence class of downward-pointing F-harmonic, namely Φ, up to a positive scale (see [AS_1]).

Now start with any cone subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ and consider the family $F(\delta), \delta > 0$ of elliptic regularizations consisting of uniformly elliptic cone subequations converging to F as $\delta \to 0$ (See Appendix A in [HL_{11}]). The paper [AS_1] applies to each $F(\delta)$ presenting us with a large set of functions $h_\delta = \Phi$, other than the Riesz kernels, which satisfy the conditions of Definition 1.1.

Now suppose that F is convex, so that for small δ, $F(\delta)$ is also convex. The smallest possible subequation F_δ for which the function $h_\delta = \Phi$ is an entire subharmonic is constructed by taking the closed convex cone on $\{D_x^2\Phi : |x| = 1\} + \mathcal{P}$. See Example 3.6 where this is carried out in the cases $F = \mathcal{P}, \mathcal{P}^{\mathbf{C}}$ and $\mathcal{P}^{\mathbf{H}}$. If Φ is also a harmonic for this smaller F_δ (outside the origin), then Φ is a downward-pointing harmonic for all cone subequations F such that $F_\delta \subset F \subset F(\delta)$. For each such F, Theorems 1.3 and 1.4 apply (as well as Theorem 7.1 in the finite case).

Other examples of this phenomenon can be constructed by applying a non-orthogonal linear transformation to any of the many convex subequations studied in [HL_{10}] and [HL_{11}].

3. Riesz Kernels and the Riesz Characteristic.

Of particular importance for the study of isolated singularities of subsolutions are the classical *Riesz kernels*:

$$K_p(x) \;=\; \begin{cases} -\frac{1}{p-2}\frac{1}{|x|^{p-2}} & \text{for } \; 1 \le p < \infty, \; p \neq 2 \\ \\ \log|x| & \text{if } \; p = 2. \end{cases}$$

One sees easily that

$$D_x K_p \;=\; \frac{x}{|x|^p} \qquad \text{and} \qquad D_x^2 K_p \;=\; \frac{1}{|x|^p}\big(P_{x^\perp} - (p-1)P_x\big)$$

where $P_{x^\perp}$ and P_x denote orthogonal projection onto the hyperplane perpendicular to x and the line through x respectively. From this one sees the following.

Proposition 3.1. *Given a cone subequation F, the p^{th} Riesz kernel*

$$K_p \;\; \textit{is} \;\; F \textit{ harmonic on } \; \mathbf{R}^n - \{0\} \qquad \Longleftrightarrow \qquad P_{e^\perp} - (p-1)P_e \in \partial F \quad \forall\, |e| = 1. \;\; (3.1)$$

Definition 3.2. A (not necessarily convex) cone subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is said to have a **finite Riesz characteristic** $p_F = p$ if

$$P_{e^\perp} - (p-1)P_e \;\in\; \partial F \qquad \text{for all unit vectors } \; e \in \mathbf{R}^n.$$

If F has finite Riesz characteristic p, then $1 \le p$ since $\mathrm{Int}\mathcal{P} \subset \mathrm{Int}F$. In addition, the Riesz kernel K_p and each of its translates is **F-harmonic** outside its singularity. Thus

$$h(x) \equiv K_p(x) \;\; \text{is a downward pointing, singular } F \text{ harmonic on } \mathbf{R}^n - \{0\} \qquad (3.2)$$

for any cone subequation with Riesz characteristic p. This is perhaps the most important example satisfying the conditions in Definition 1.1. It is discussed in detail after a few additional comments on the polar case $p \ge 2$.

Remark 3.3. It is natural to extend the definition of Riesz characteristic. We say that F **has Riesz characteristic** $p_F = \infty$ if

$$-P_e \in \partial F, \;\; \text{of equivalently} \;\; -P_e \in F, \quad \forall\, |e| = 1. \qquad (3.3)$$

The equivalence is because $-P_e \notin \mathrm{Int}F$ for a cone subequation F unless $F = \mathrm{Sym}^2(\mathbf{R}^n)$.

Of course, there are cone subequations which do not have a Riesz characteristic. However, for "invariant" cone subequations the Riesz characteristic is very easy to compute, and condition (F3) holds $\Longleftrightarrow$ p is finite.

Lemma 3.4. *Suppose that F is invariant under a subgroup $H \subset \mathrm{O}(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbf{R}^n$. Then*

$$p_F \;=\; \sup\{q : P_{e^\perp} - (q-1)P_e \;\in\; F\} \tag{3.4}$$

for some (and therefore any) unit vector $e \in \mathbf{R}^n$. In particular, $p_F = \infty \iff$ (F3) fails for all $|e| = 1$.

The proof is straightforward. Note that in appropriate coordinates $P_{e^\perp} - (q-1)P_e$ is diagonal matrix with one eigenvalue $-(q-1)$ and the remaining entries all equal to 1.

The principal Corollary 1.5 has as its most important special case the following.

THEOREM 3.5. (The Polar Case). *Suppose F is a cone subequation with finite Riesz characteristic $p \geq 2$ and property (F3), and let Ω be a domain with a smooth boundary which is strictly F-convex.*

(a) For each $\varphi \in C(\partial\Omega)$, $x_0 \in \Omega$, and $\Theta > 0$, there exists a unique solution H to the (DPPS) having boundary values φ on $\partial\Omega$ and asymptotic singularity $H(x) \approx \Theta K_p(x - x_0)$ at x_0.

(b) If, in addition, F is convex, then the multi-pole (DPPS) with asymptotics $\Theta_j K_p(x - x_j)$ has a unique solution H.

Moreover, in either of these cases, if $\varphi \equiv 0$, this provides the existence and uniqueness of a nonlinear Green's function G_Ω with

$$G_\Omega(x) \;\approx\; \Theta K_p(x - x_0) \qquad \text{at} \;\; x_0$$

in the case of a single pole, and, provided that F is convex,

$$G_\Omega(x) \;\approx\; \Theta_j K_p(x - x_j) \qquad \text{at each} \;\; x_j$$

in the multi-pole case.

The existence and uniqueness of the multi-pole Green's function for the subequation $F = \mathcal{P}^{\mathbf{C}}$ was proved by Lelong in 1989 ([Le]). This built on previous work for single-point Green's functions (Lempert [Lem] and Klimek [K$_*$]). An even more general version was established by A. Zeriahi [Z].

Theorem 3.5 includes all invariant cone subequations whose Riesz characteristic is finite (by Lemma 3.4).

There are many more subequations F which have a finite Riesz characteristic than one might at first imagine. We start by mentioning four extreme examples of characteristic p subequations. The first explains our choice of normalization in the definitions of K_p and characteristic p. It coincides with the subequation $\mathcal{P}(G(p, \mathbf{R}^n)$ in Example 1.7 if p is an integer.

Examples. 3.6.

(1) $\mathcal{P}_p = \{A : \lambda_1(A) + \cdots + \lambda_{[p]}(A) + (p - [p])\lambda_{p+1}(A) \geq\}$ where $\lambda_1(A) \leq \lambda_1(A) \leq \cdots$ are the ordered eigenvalues of A and $1 \leq p \leq n$.

(2) $\mathcal{P}(\delta_p) = \Big\{A : A + \frac{\delta_p}{n}\mathrm{tr}(A)I \ \geq \ 0\Big\}$, where $\delta_p = \frac{n(p-1)}{n-p}$.

(3) $\mathcal{P}_p^{\min/\max} = \{A : \lambda_{\min}(A) + (p-1)\lambda_{\max}(A) \geq 0\}$.

(4) $\mathcal{P}_p^{\min/2} = \{A : \lambda_{\min}(A) + (p-1)\lambda_2(A) \geq 0\}$.

Note that $\mathcal{P}_p^{\min/2} \subset \mathcal{P}_p \subset \mathcal{P}(\delta_p) \subset \mathcal{P}_p^{\min/\max}$ and that each subequation F with $\mathcal{P}_p^{\min/2} \subset F \subset \mathcal{P}_p^{\min/\max}$ has finite Riesz characteristic p. It is somewhat surprising that, under mild restriction, there exists both a "largest" and a "smallest" characteristic p subequation. More precisely, *every invariant (as in Lemma 3.4) cone subequation F with finite Riesz characteristic p satisfies*

$$\mathcal{P}_p^{\min/2} \ \subset \ F \ \subset \ \mathcal{P}_p^{\min/\max} \tag{3.5}$$

and if F is $O(n)$-invariant and convex, then

$$\mathcal{P}_p \ \subset \ F \ \subset \ \mathcal{P}(\delta_p). \tag{3.6}$$

This is proved in Appendix A of [HL_{11}] where many more examples of characteristic p subequations are given. These include subequations of Monge-Ampère type arising from Gårding operators. Among these is the following *Hessian equation*, which has been studied by Trudinger-Wang [TW_*], Labutin [La_*] and others. We have drawn heavily from [La_3] in this paper.

(5) $\Sigma_k = \{A : \sigma_1(A) \geq 0, ..., \sigma_k(A) \geq 0\}$, where $p_F = \frac{n}{k}$.

One also has

(6) $F = \{A : \mathrm{tr}(A^q) \geq 0\}$ where $p_F = 1 + (n-1)^{\frac{1}{q}}$ for $q \in \mathbf{Z}$ odd. (This F is not convex.)

Suppose F is an $\mathrm{O}(n)$-invariant subequation with $p_F = p$, and let $F(\mathbf{C}), F(\mathbf{H})$ be the complex and quaternionic analogues given by the same conditions on the eigenvalues of their hermitian symmetric components. Then $p_{F(\mathbf{C})} = 2p_F$ and $p_{F(\mathbf{H})} = 4p_F$. These examples contain the complex and quaternionic Monge-Ampère equations [BT], [A_*], [AV], as well as the complex and quaternionic hessian equations. There are also the complex (as well as quaternionic) analogues:

$$\mathcal{P}_p^{\min/2}(\mathbf{C}) \ \subset \ \mathcal{P}_p(\mathbf{C}) \ \subset \ \mathcal{P}(\delta_p)(\mathbf{C}) \ \subset \ \mathcal{P}_p^{\min/\max}(\mathbf{C}). \tag{3.7}$$

with

$$\mathcal{P}_p^{\min/2}(\mathbf{C}) \ \subset \ F(\mathbf{C}) \ \subset \ \mathcal{P}_p^{\min/\max}(\mathbf{C}) \qquad \text{and} \qquad \mathcal{P}_p^{\min/2}(\mathbf{C}) \ \subset \ F(\mathbf{C}) \ \subset \ \mathcal{P}_p^{\min/\max}(\mathbf{C})$$

as in (3.5) and (3.6).

4. The Dirichlet Problem with Prescribed Densities.

Assume, as in the previous section, that F is a cone subequation with a finite Riesz characteristic p. For an arbitrary F-subharmonic function u, the density of u at a point x_0 in its domain, is the limit

$$\begin{aligned} \Theta(u, x_0) &\equiv \lim_{r\to 0} \frac{\sup_{B_r(x_0)} u}{K_p(x - x_0)} \qquad \text{for } \ p \geq 2 \qquad \text{and} \\ \Theta(u, x_0) &\equiv \lim_{r\to 0} \frac{\sup_{B_r(x_0)} u - u(x_0)}{K_p(x - x_0)} \quad \text{for } \ 1 \leq p < 2, \end{aligned} \tag{4.1}$$

which always exists by [HL$_{10}$].

The **Dirichlet Problem with Prescribed Densities**, denoted by (DPPD), is the same as the (DPPS) except that the asymptotic requirement

$$H \ \approx \ \Theta_j K_p(x - x_j) \quad \text{for } p \geq 2 \qquad \text{or} \qquad H \ \sim \ \Theta_j K_p(x - x_j) \quad \text{for } 1 \leq p < 2 \tag{4.2}$$

is replaced by prescribing the density

$$\Theta(H, x_j) \ = \ \Theta_j \tag{4.3}$$

at each point x_j.

Lemma 4.1. *For any F-subharmonic function u, the condition (4.2) implies (4.3).*

Proof. Assume $x_j = 0$. The notion $u \sim \Theta K_p$ at the origin is defined for $p \geq 2$ by requiring that

$$\lim_{x\to 0} \frac{u(x)}{K_p(x)} \ = \ \Theta$$

(see (A.1) and also (4a) in (DPPS)). By Proposition A.3 the condition $u \approx \Theta K_p$ at 0 implies that $u \sim \Theta K_p$ at 0, so even when $p \geq 2$, we can assume $u \sim \Theta K_p$. Finally, the limits in (4.1) equal Θ since the limits in (A.1) as $x \to 0$ equal Θ. ∎

Corollary 4.2. *If existence holds for the (DPPS), then existence holds for the (DPPD). Moreover, uniqueness holds for the (DPPD) if and only if for any downward-pointing singular F-harmonic H with density $\Theta(H, 0) = \Theta$ we have*

$$H \ \approx \ \Theta K_p \ \text{ at } 0 \quad \text{if } 2 \leq p \qquad \text{or} \qquad H \ \sim \ \Theta K_p \ \text{ at } 0 \quad \text{if } 1 \leq p < 2 \tag{4.4}$$

The Polar Case ($p \geq 2$)

Existence for the (DPPS) provided by Theorem 3.5 implies existence for the (DPPD) because of Corollary 4.2.

THEOREM 4.3. (Polar Case $p \geq 2$) *Let F, Ω and φ be as in Theorem 3.5. Then there exits a solution H to the (DPPD) with prescribed positive density Θ at any given*

point $x_0 \in \Omega$. Moreover, if F is convex, there exists a solution H with prescribed positive densities $\Theta_1, ..., \Theta_k$ at an arbitrary collection of distinct points $x_1, ..., x_k \in \Omega$.

Uniqueness of the non-linear Green's function implies uniqueness in the general (DPPD).

THEOREM 4.4. (Polar Case $p \geq 2$). *Suppose that the Riesz kernel K is the only solution to the (DPPD) on a ball about the origin with the same boundary values and asymptotic behavior as K. Then uniqueness holds for the general Dirichlet problem with prescribed densities (DPPD) in Theorem 4.3.*

Proof. Suppose H is a solution to the (DPPD) in Theorem 4.3. By the definition of the (DPPD) we know that H has density Θ_j at x_j. It suffices to prove (4.4), i.e., to show that $H \approx \Theta_j K(x - x_j)$ because then the uniqueness part of Theorem 3.5 applies. As in [La$_3$, Thm. 3.6] here is how (4.4) can be proved for H. Normalize so that $x_j = 0$, $\Theta_j = 1$ and $B \subset\subset \Omega$ is a ball about the origin. By the existence part of Theorem 4.3, we obtain $h \in C(\overline{B} - \{0\})$, which is F-harmonic on $B - \{0\}$, equal to the constant K on ∂B, and satisfies $h \approx H$ at 0. Now $h \approx H$ at 0 implies that h also has density 1 at 0. By the hypothesis, this proves $h = K$. ∎

Corollary 4.5. (Polar Case $p \geq 2$). *Suppose F is an orthogonally invariant subequation of finite Riesz characteristic p. Then both existence and uniqueness hold for the (DPPD) in Theorem 4.3.*

Proof. The classical moving plane argument shows that if h is F-harmonic on B, constant on ∂B and $h(0) = -\infty$, then h is a radial function (see [La$_3$], [GLN]). Hence, $h(x) = \Theta K_p(x) + k$ by [HL$_{10}$, Prop. 3.5]. ∎

Final Note. For orthogonally invariant subequations F (and many others as well) it has been shown ([HL$_{10}$], [HL$_{11}$]) that for any F-**sub**harmonic function u on $\Omega \subset \mathbf{R}^n$ and $c > 0$,

$$\textit{the set} \quad E_c(u) \;\equiv\; \{x \in \Omega : \Theta(u, x) \geq c\} \quad \textit{is discrete.}$$

The results here show that any finite set $E \subset \Omega$ can occur as $E_c(H)$ for an F-harmonic function H on Ω.

5. Comparison.

In this section we prove the Uniqueness Theorem 1.3 for the (DPPS) by establishing a comparison theorem in the setting of prescribed singularities. The key idea is contained in a local result. First we consider the polar case.

Lemma 5.1. (The Polar Case). *Suppose v is F-subhamronic and w is $\widetilde{F}$-subharmonic in a deleted neighborhood of a point $x_0 \in \mathbf{R}^n$, and that h is a downward-pointing F-harmonic at x_0. Assume that for some constants c and k*

$$v \ \leq \ h + c \qquad \text{and} \qquad w \ \leq \ -h + k \qquad \text{near} \ \ x_0. \tag{5.1}$$

Then, with $u \equiv v + w$ extended to x_0 by setting

$$u(x_0) \ \equiv \ \overline{\lim_{x \to x_0}}(v(x) + w(x)) \tag{5.2}$$

we have that

$$u \ \ \textit{is subaffine on a neighborhood of} \ \ x_0. \tag{5.3}$$

Proof. We recall the notion of subaffine functions [HL_1]. These are the functions $u \in \mathrm{USC}(\Omega)$ such that for any affine function a and $K \subset\subset \Omega$, $u \leq a$ on $\partial K \quad \Rightarrow \quad u \leq a$ on K. It turns out that this is a local condition on the function, namely that it satisfy the subequation $\widetilde{\mathcal{P}}$ dual to $\mathcal{P}$ (see [HL_1]). Comparison follows from (5.3) since subaffine functions clearly satisfy the maximum principle.

Consider the sum $u = v + w$ which defines an upper semi-continuous $[-\infty, \infty)$-valued function in a deleted neighborhood of x_0. This function has the following two properties.

$$u \ \ \text{is bounded above across} \ \ x_0, \tag{5.4}$$

$$u \ \ \text{is subaffine on a deleted neighborhood of} \ \ x_0. \tag{5.5}$$

Obviously (5.1) implies (5.4). For (5.5) recall (see [] or []) that for any subequation F, the sum of an F-subharmonic function and an $\widetilde{F}$-subharmonic function is $\widetilde{\mathcal{P}}$-subharmonic. The condition (5.4) implies that the extension of u defined by (5.2) is upper semi-continuous, with values in $[-\infty, \infty)$, in a neighborhood of x_0. It remains to show that u is $\widetilde{\mathcal{P}}$-subharmonic. (Note that the function $-|x - x_0|$ satisfies (5.4) and (5.5) and, even though it is continuous across x_0, it is not $\widetilde{\mathcal{P}}$-subharmonic across x_0. Said differently, Lemma 5.1 is not simply a removable singularity theorem for $\widetilde{\mathcal{P}}$.)

To prove that u is subaffine across x_0 we approximate u by

$$u_\epsilon \ = \ u + \epsilon v \ = \ (1 + \epsilon)v + w. \tag{5.6}$$

Since u is bounded above and $v(x_0) = -\infty$, if we define $u_\epsilon(x_0) = -\infty$, then u_ϵ is upper semi-continuous on a neighborhood of x_0. Note that u_ϵ has no test functions at x_0, so to prove that u_ϵ is subaffine on a neighborhood V of x_0, we need only prove that u_ϵ is subaffine on $V - \{x_0\}$. However, u_ϵ is the sum of the F-subharmonic function $(1 + \epsilon)v$

and the $\widetilde{F}$-subharmonic function w. This implies, as above, that u_ϵ is $\widetilde{\mathcal{P}}$-subharmonic on $V - \{x_0\}$ as desired. (Here we have used that F is a cone.)

Now on a neighborhood of x_0 the subaffine functions u_ϵ increase pointwise to u (since $v < 0$). By the "families bounded above" property (see [HL$_1$]) and the fact that u is upper semi-continuous with values in $[-\infty, \infty)$, this proves that u is $\widetilde{\mathcal{P}}$-subharmonic. ∎

Comparison can be stated as follows.

THEOREM 5.2. (Comparison in the Polar Case). *Suppose Ω is a domain and $h_1, ..., h_k$ are downward-pointing F-harmonic at $x_1, ..., x_k$ respectively. Given $v, w \in \mathrm{USC}(\overline{\Omega} - \{x_1, ..., x_k\})$ with v F-subharmonic and w $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$, suppose that near each x_j, $j = 1, ..., k$ we have*

$$v \;\le\; h_j + c_j \qquad \text{and} \qquad w \;\le\; -h_j + k_j \qquad \text{for some constants } c_j \text{ and } k_j. \tag{5.7}$$

Then comparison holds on Ω, that is,

$$\text{If } \; v + w \;\le\; 0 \;\text{ on }\; \partial\Omega, \;\text{ then }\; v + w \;\le\; 0 \;\text{ on }\; \Omega - \{x_1, ..., x_k\}. \tag{5.8}$$

Proof. By Lemma 5.1 the function $u \equiv v + w$, defined on $\overline{\Omega} - \{x_1, ..., x_k\}$, extends to an upper semi-continuous $[-\infty, \infty)$-valued function u on $\overline{\Omega}$ which is subaffine on Ω. Hence, $\sup_{\overline{\Omega}} u \le \sup_{\partial\Omega} u$ by the maximum principle. ∎

Proof of the Uniqueness Theorem 1.3 in the Polar Case. It suffices to prove that

$$\text{If } \; H \; \text{ is a solution to the (DPPS), then } \; H(x) \;=\; \sup_{v \in \mathcal{F}} v(x). \tag{5.9}$$

where $\mathcal{F}$ is the family defined in Theorem 1.3. We can apply the Comparison Theorem 5.2 to $v \in \mathcal{F}$ and $w \equiv -H$. On $\partial\Omega$, we have $v \le \varphi$ and $w = -\varphi$, and so $v + w \le 0$ (or $v \le H$) on $\overline{\Omega}$ by (5.8). Since $H \in \mathcal{F}$, this proves (5.9). ∎

Now we turn to the finite case.

Lemma 5.3. (The Finite Case). *Suppose v is F-subharmonic and w is $\widetilde{F}$-subharmonic on a deleted neighborhood of x_0, and that h is a downward-pointing (singular) F-harmonic at x_0. Assume that*

$$\liminf_{x \to x_0} \frac{v(x) - v(x_0)}{h(x) - h(x_0)} \;\ge\; 1 \qquad \text{and} \qquad \liminf_{x \to x_0} \frac{w(x) - w(x_0)}{h(x) - h(x_0)} \;\ge\; -1 \tag{5.10}$$

Then with $u \equiv v + w$ extended to x_0 by setting

$$u(x_0) \;\equiv\; \limsup_{x \to x_0} u(x) \tag{5.11}$$

we have that

$$u \;\text{ is subaffine on a neighborhood of }\; x_0. \tag{5.12}$$

Proof. To prove that u is subaffine, we approximate $\overline{u} \equiv u - v(x_0) + w(x_0)$ by

$$\overline{u}_\epsilon \;=\; \overline{u} + \epsilon(v(x) - v(x_0)) \;=\; (1+\epsilon)(v(x) - v(x_0)) + (w(x) - w(x_0)) \tag{5.13}$$

and prove that $\overline{u}$ is subaffine. Note that as in Lemma 5.1, $\overline{u}_\epsilon$ is subaffine on a deleted neighborhood of x_0 since F is a cone. To show that $\overline{u}_\epsilon$ is subaffine on a neighborhood of x_0, we need only show that at x_0, $\overline{u}_\epsilon$ has no test functions.

The hypothesis (5.10) on v implies that for $1 < \alpha < 1+\epsilon$, there exists a neighborhood of x_0 with $v(x) - v(x_0) \geq \frac{\alpha}{1+\epsilon}(h(x) - h(x_0))$.

The hypothesis (5.10) on w implies that for $1 < \beta < \alpha$, we have $w(x) - w(x_0) \geq (\alpha - \beta)(h(x) - h(x_0))$ near x_0. Hence, $\overline{u}_\epsilon \geq (\alpha - \beta)(h(x) - h(x_0))$, which proves that $\overline{u}_\epsilon$ has no test functions at x_0, since h has no test functions at x_0 (note that $\overline{u}_\epsilon(x_0) = 0$.) This proves that each $\overline{u}_\epsilon$ is subaffine in a neighborhood of x_0.

Finally the fact that $v(x) - v(x_0)$ is bounded below by a positive multiple of $h(x) - h(x_0)$ implies that $\overline{u}_\epsilon$ is decreasing pointwise as $\epsilon \to 0$ in a neighborhood of x_0. However, outside x_0, $\overline{u}$ is the sum of the F-subharmonic function $v(x) - v(x_0)$ and the $\widetilde{F}$-subharmonic function $w(x) - w(x_0)$, which implies that $\overline{u}$ is subaffine. ∎

THEOREM 5.4. (Comparison in the Finite Case). *Suppose that Ω is a domain and $h_1, ..., h_k$ are downward-pointing F-harmonics at $x_1, ..., x_k \in \Omega$ respectively. Given $v, w \in \mathrm{USC}(\overline{\Omega} - \{x_1, ..., x_k\})$ with v F-subharmonic and w $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$, suppose that for $j = 1, ..., k$ we have*

$$\liminf_{x \to x_j} \frac{v(x) - v(x_j)}{h(x) - h(x_j)} \;\geq\; 1 \qquad \text{and} \qquad \liminf_{x \to x_j} \frac{w(x) - w(x_j)}{h(x) - h(x_j)} \;\geq\; -1. \tag{5.14}$$

Then (with $u = v + w$ defined at x_j by $(v+w)(x_j) \equiv \limsup_{x \to x_j}(v+w)(x)$),

$$\textit{If} \quad v + w \;\leq\; 0 \quad \textit{on} \quad \partial\Omega, \quad \textit{then} \quad v + w \;\leq\; 0 \quad \textit{on} \quad \overline{\Omega}. \tag{5.15}$$

Proof. This follows from Lemma 5.3 as in the proof of Theorem 5.2. ∎

Corollary 5.5. (Uniqueness in the Finite Case). *Let $\mathcal{F}$ denote the family of $v \in \mathrm{USC}(\overline{\Omega})$ satisfying: v is F-subharmonic on Ω, $v \leq \varphi$ on $\partial\Omega$, and for $j = i, ..., k$*

$$\liminf_{x \to x_j} \frac{v(x) - v(x_j)}{h(x) - h(x_j)} \;\geq\; 1.$$

If H is a solution to the (DPPS), then

$$H(x) \;=\; \sup_{v \in \mathcal{F}} v(x). \tag{5.16}$$

Proof. Note that if $v \in \mathcal{F}$ and H is a solution, then comparison implies that $v \leq H$. Since $H \in \mathcal{F}$, this proves (5.16). ∎

6. A Basic Construction and the Proof of Existence in the Polar Case.

In this section we describe the basic construction of the solution to the (DPPS) in some generality, and then complete the existence proof in the polar case. The finite case is finished in Section 7. Our starting point is the standard (DP) on the domain with a neighborhood of the singular points removed.

The Dirichlet Problem on the Perforated Domain.

We fix $r_0 > 0$ so that the closed balls $\overline{B}_{r_0}(x_j) = \{|x - x_j| \leq r_0\},\ j = 1, ..., k$, are mutually disjoint and contained in Ω. Let

$$D_r \ \equiv \ \bigcup_{j=1}^{k} \overline{B}_r(x_j) \qquad \text{for } 0 < r \leq r_0.$$

Consider the perforated domain

$$\Omega_r \ \equiv \ \Omega - D_r$$

whose oriented boundary is the sum of the *outer boundary* $\partial\Omega$ and the *inner boundary* $-\partial D_r$.

Consider the Dirichlet Problem for a subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ on Ω_r with given boundary functions $\underline{\varphi} \in C(\partial D_r)$ and $\varphi \in C(\partial\Omega)$. As discussed in the proof of Lemma 5.1 comparison and hence uniqueness holds on any domain. For existence, consider the **Perron family** $\mathcal{F}$ consisting of all $u \in \mathrm{USC}(\overline{\Omega_r})$ such that

$$u\big|_{\Omega_r} \in F(\Omega_r), \qquad u\big|_{\partial D_r} \leq \underline{\varphi}, \qquad \text{and} \qquad u\big|_{\partial\Omega} \leq \varphi.$$

along with its **Perron function**

$$H_r(x) \ \equiv \ \sup_{u \in \mathcal{F}} u(x).$$

By [HL$_3$, §12] the Perron function will solve the Dirichlet problem for the given boundary values provided that, at each point of $\partial\Omega_r = \partial\Omega - \partial D_r$, one can construct barriers as in Propositions F and $\widetilde{F}$, [HL$_3$, p. 453].

We list our assumptions.

Assumption (B1): The outer boundary $\partial\Omega$ is strictly F-convex.

Note that $\partial\Omega$ is strictly $\widetilde{F}$-convex by (F3) = (B3). Thus barriers, as in Propositions F and $\widetilde{F}$ exist for each point $x_0 \in \partial\Omega$.

Note similarly that the inner boundary $-\partial D_r$ is also strictly $\widetilde{F}$-convex by This provides a barrier as in Proposition $\widetilde{F}$ at points $x_0 \in \partial D_r$.

To obtain an F-barrier at a point $x_0 \in \partial D_r$ we assume that the inner boundary function $\underline{\varphi}$ is of a special nature, namely,

Assumption (B2): $\underline{\varphi} \equiv \psi\big|_{\partial D_r}$ where ψ is F-subharmonic on $\overline{\Omega}$, and either

Polar Case: ψ is continuous on $\overline{\Omega}$ and $= -\infty$ precisely at the points $x_1, ..., x_k$, or

Finite Case: $\psi \in C(\overline{\Omega})$ with each x_j, $j = 1, ..., k$ a strict local minimum point with no test functions.

Assumption (B4): $\psi\big|_{\partial\Omega} \leq \varphi$.

Since ψ and φ satisfy (B2) and (B4), we can use the F-subharmonic function ψ to construct barriers as in Proposition F. This is done by setting $\underline{u}(x) \equiv \psi(x) - \delta + \epsilon|x - x_0|^2$ for $\epsilon > 0$ sufficiently small.

This establishes the following existence result.

THEOREM 6.1. *Assume that F is a cone subequation which satisfies Condition (F3) = (B3). Suppose $\varphi \in C(\partial\Omega)$. Suppose that $\partial\Omega$ is strictly F-convex (B1), and that ψ satisfies (B2) and (B4). Then the Perron function H_r solves the Dirichlet Problem:*

(a) $H_r \in C(\overline{\Omega}_r)$

(b) H_r *is F-harmonic on* Ω_r

(c) $H_r\big|_{\partial D_r} = \psi\big|_{\partial D_r}$ and $H_r\big|_{\partial\Omega} = \varphi$.

The Candidate for the Solution to the (DPPS)

We continue with the same notation and hypotheses.

The proposed solution to the (DPPS) is constructed as a pointwise increasing limit

$$H(x) \equiv \lim_{r\to 0} \overline{H}_r(x) \qquad \text{on } \overline{\Omega} \tag{6.1}$$

of functions $\overline{H}_r$ which extend the functions H_r to $\overline{\Omega}$.

Lemma 6.2. *The continuous functions*

$$\overline{H}_r \equiv \begin{cases} H_r & \text{on } \overline{\Omega}_r \\ \psi & \text{on } D_r \end{cases} \tag{6.2}$$

are F-subharmonic on Ω and pointwise increasing on $\overline{\Omega}$ as $r \to 0$.

Proof. The function ψ is in the Perron family on Ω_r since $\psi\big|_{\partial D_r} = \underline{\varphi}$ and $\psi\big|_{\partial\Omega} \leq \varphi$. This proves that

$$\psi \leq H_r \quad \text{on } \overline{\Omega}_r. \tag{6.3}$$

Consequently,

$$\overline{H}_r(x) \equiv \begin{cases} \max\{\psi(x), H_r(x)\} & \text{on } \overline{\Omega}_r \\ \psi(x) & \text{on } D_r. \end{cases} \tag{6.4}$$

To see that

$$\overline{H}_r \quad \text{is } F\text{ subharmonic on } \Omega, \tag{6.5}$$

note that $\overline{H}_r^{\epsilon} = \max\{\psi + \epsilon, H_r\}$ defines an F-subharmonic function on Ω since $\psi + \epsilon > H_r$ on ∂D_r, and then note that $\overline{H}_r^{\epsilon}$ decreases pointwise as $\epsilon \to 0$ to $\overline{H}_r$ on $\overline{\Omega}$.

Finally we show that for all ρ with $0 < \rho < r \leq r_0$, one has

$$\overline{H}_r \ \leq \ \overline{H}_\rho \qquad \text{on } \ \overline{\Omega}. \tag{6.6}$$

On D_r we have $\overline{H}_r = \overline{H}_\rho = \psi$. In particular, on ∂D_ρ, $\overline{H}_r = \psi$, while on the outer boundary $\partial\Omega$, $\overline{H}_r = \varphi$. Thus $\overline{H}_r$ is in the Perron family for H_ρ on this larger domain Ω_ρ, proving that $\overline{H}_r \leq H_\rho$ on $\overline{\Omega}_\rho$, which establishes (6.6). ∎

Let H^{DP} denote the solution to the standard Dirichlet Problem (DP) on Ω. That is, $H^{\mathrm{DP}} \in C(\Omega)$, H^{DP} is F-harmonic on Ω, and $H^{\mathrm{DP}}\big|_{\partial\Omega} = \varphi$. This function exists since the outer boundary is assumed to be strictly F-convex, and by (F3) = (B3) it is also strictly $\widetilde{F}$-convex.

Proposition 6.3. *The proposed solution H to the (DPPS) defined by (6.1) satisfies:*

(1) *(a)* $H^* \in \mathrm{USC}(\overline{\Omega})$ and *(b)* $-H \ = \ (-H)^* \in \mathrm{USC}(\overline{\Omega} - \{x_1, ..., x_k\})$,

(2) *(a) H^* is F-subharmonic on Ω,* *(b) $-H$ is $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$*

(3) $H^*\big|_{\partial\Omega} \ = \ H\big|_{\partial\Omega} \ = \ \varphi$

(4) $\psi \ \leq \ H \ \leq \ H^{\mathrm{DP}}$ *on $\overline{\Omega}$.*

Proof. Assertion (1) is immediate while (2a) follows from the "families bounded above property" and (2b) by the "decreasing limit property". To prove (4) first note that (6.3) implies that $\psi \leq H$. Now the function H^{DP} is the Perron function for the standard Dirichlet Problem on Ω with boundary values φ. Therefore, H^{DP} is larger than h by assumption (B4). Thus $H_r \leq H^{\mathrm{DP}}$ on $\partial\Omega_r$, and this remains true on $\overline{\Omega}_r$. Therefore, $H \leq H^{\mathrm{DP}}$, which implies that

$$H^* \ \leq \ H^{\mathrm{DP}}. \tag{6.7}$$

In turn this implies (3). ∎

Existence in the Polar Case – The Proof of Theorem 1.4

First note that in the polar case the notion of asymptotic equivalence is preserved by subtracting a constant. This implies that the hypothesis (B4) in Theorem 6.1 and Proposition 6.3 can always be satisfied.

Now (4) in Proposition 6.3 yields the left hand inequality in Part (4a) of the (DPPS) stated in the introduction. For the right hand inequality in (4a) we prove the following.

Lemma 6.4. *Near each x_j we have $H \leq h_j + c_j$ for some constant c_j.*

Proof. For $0 < \rho \leq r$ we have that

$$H_\rho - h_j \quad \text{is } \ \widetilde{\mathcal{P}} \text{ subharmonic on } \ A_{\rho,r} \ = \ B_r - \overline{B}_\rho.$$

On the inner boundary ∂B_ρ we have, since $\psi \approx h_j$ implies $\psi - h_j \leq C$, that

$$H_\rho - h_j \ = \ \psi - h_j \ \leq \ C,$$

and on the outer boundary ∂B_r we have

$$H_\rho - h_j \ \leq \ U - h_j \ \leq \ C(r)$$

independent of ρ. Hence,

$$H - h_j \ \leq \ \max\{C, C(r)\} \ \equiv \ C' \qquad \text{on} \ \ B_r(x_j). \qquad \blacksquare$$

Now we apply the Comparison Theorem 5.2 to $v \equiv H^*$ and $w \equiv -H$. By (1a) H^* takes values in $[-\infty, \infty)$ on Ω, while by (1b) w takes values in $[-\infty, \infty)$ except at $x_1, ..., x_k$ where w equals $+\infty$. The inequality

$$v \ \equiv \ H^* \ \leq \ h_j + c_j \qquad \text{near} \ \ x_j$$

is immediate from Lemma 6.4. Observe now that by combining (4) $\psi \leq H$ $(\equiv -w)$ from Proposition 6.3 with the inequality $h_j - k_j \leq \psi$ near x_j, which is part of Hypothesis (H) in Theorem 1.4, gives

$$w \ \leq \ -h_j + k_j \qquad \text{near} \ \ x_j.$$

This establishes the hypothesis (5.7) in Theorem 5.2. Finally, by (3) in Proposition 6.3, $v = \varphi$ and $w = -\varphi$ on $\partial\Omega$, so that $v + w = 0$ on $\partial\Omega$. We can now apply Theorem 5.2 to conclude that $v + w \leq 0$ on $\overline{\Omega} - \{x_1, ..., x_k\}$, i.e., $H^* \leq H$ on $\overline{\Omega} - \{x_1, ..., x_k\}$. This proves that $H^* = H$ on $\overline{\Omega} - \{x_1, ..., x_k\}$. Since we already have $H = H_*$ (because it is an increasing limit of continuous functions), we conclude that H is continuous on $\overline{\Omega} - \{x_1, ..., x_k\}$, and Conditions (1) and (2) are proved. This completes the proof of Theorem 1.4. $\blacksquare$

The (DPPS) with Singularities on a Compact Polar Set

The arguments given above adapt to prove a version of Theorem 1.4 with $\{x_1, ..., x_k\}$ replaced by a compact polar set.

THEOREM 6.5. *Let F be a cone subequation satisfying Condition (F3). Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with smooth boundary $\partial\Omega$ which is strictly F-convex, and let $\Sigma \subset \Omega$ be a compact subset. Suppose there exists a continuous function $h : \overline{\Omega} \to [-\infty, \infty)$ such that $\Sigma = h^{-1}(-\infty)$ and h is F-harmonic on $\Omega - \Sigma$.*

Then for any $\varphi \in C(\partial\Omega)$ we have the following.

Existence. *There exists $H \in C(\overline{\Omega} - \Sigma)$ such that:*

(1) *H is F-harmonic on $\Omega - \Sigma$,*

(2) *$H\big|_{\partial\Omega} = \varphi$,*

(3) *H is asymptotically equivalent to h, i.e., there exists $c, C \in \mathbf{R}$ such that*

$$h(x) + c \ \leq \ H(x) \ \leq \ h(x) + C \qquad \text{on} \ \ \Omega - \Sigma$$

Uniqueness. *There is at most one function $H \in C(\overline{\Omega} - \Sigma)$ satisfying (1), (2) and (3). (Here the F-convexity of $\partial\Omega$ is not required.)*

Proof. The uniqueness is proved as in Section 5. For existence we choose a sequence of regular values $\{r_j\}$ of h with $r_j \downarrow -\infty$. Again b adjusting h with an additive constant we can assume that (B4) is satisfied. For each j define

$$D_j \;\equiv\; \{x \in \Omega : h(x) > r_j\} \qquad \text{and} \qquad \Omega_j \;\equiv\; \Omega - \overline{D}_j.$$

Let H_j be the solution to the Dirichlet Problem on Ω_j for F-harmonic functions with boundary values

$$H_j\big|_{\partial\Omega} \;=\; \varphi \qquad \text{and} \qquad H_j\big|_{\partial D_j} \;=\; h\big|_{\partial D_j}$$

on the outer and inner boundaries respectively. As before this solution exists due to the assumption of F-convexity on $\partial\Omega$, the hypothesis (F3) = (B3), the fact that h is in the Perron family by (B4), and the F-harmonicity of h outside Σ.

Now define $\overline{H}_j(x)$ as in Lemma 6.2 and note that the assertions of Lemma 6.3 hold by exactly the same arguments.

It follows that the increasing limit

$$H \;\equiv\; \lim_{j\to\infty} \overline{H}_j$$

has the properties that H^* is F-subharmonic and $-H$ is $\widetilde{F}$-subharmonic on $\Omega - \Sigma$. Using the solution H^{DP} to the Dirichlet Problem on Ω with boundary values φ the same arguments as above show that

$$H^*\big|_{\partial\Omega} = \varphi. \tag{6.8}$$

We now prove (3). The left-hand inequality in (3) holds since h is in the Perron family for H_j on Ω_j for all j. On the other hand, for $C > 0$ sufficiently large we will have $h + C > \varphi$ on $\partial\Omega$. Since $h + C$ is F-harmonic on $\Omega - \Sigma$ and greater than H_j on $\partial\Omega_j$, we have $h + C > H_j$ on Ω_j by comparison. This establishes the right-hand inequality in (3).

As noted above we have

$$H^* \in F(\Omega - \Sigma) \qquad \text{and} \qquad -H \in \widetilde{F}(\Omega - \Sigma).$$

As before this implies that $H^* - H \in \widetilde{\mathcal{P}}_p(\Omega - \Sigma)$. Now Condition (3) implies Condition (3) for H^* and therefore $H^* - H \leq C - c$ on $\Omega - \Sigma$. We now apply the removable singularity argument in [HL$_7$] using the polar function h (as before) to conclude that $H^* - H$ is F-subharmonic on Ω. By (6.8) $H^* - H$ is ≤ 0 on $\partial\Omega$. Hence, we have $H^* - H \leq 0$ on Ω. We have proved that $H^* = H$ on Ω. This proves $H \in C(\overline{\Omega} - \Sigma)$ and condition (1), and we are done. ∎

7. Existence in the Finite Case.

We now take up the proof of existence in the finite case. It proceeds in two stages. The first is a construction which provides a family of F harmonics which only satisfy a weakened form of asymptotic equivalence in a range. The second stage shows that one member of the family has the desired asymptotic singularity at the given point.

Throughout this section we assume that F is a cone subequation with the property that all boundaries are strictly $\widetilde{F}$-convex, and that Ω is a domain with smooth strictly F-convex boundary. We limit the discussion to the case of a single singular point $x_0 \in \Omega$. The following assumption replaces the hypothesis (H) in Theorem 1.4.

Hypothesis (H1) (Finite Case). We are given a function $h \in C(\overline{\Omega})$ which is F-harmonic on $\Omega - \{x_0\}$ and has a downward-pointing singularity at x_0. Moreover, for convenience, we assume by rescaling that

$$h(x_0) \;=\; 0 \qquad \text{and} \qquad \sup_{\partial\Omega} h \;=\; 1. \tag{7.1}$$

The Construction

It is similar to the construction in the polar case. However, for each $t \geq 0$ we construct a function H^t which is a candidate for the solution with $H^t \sim th$ at x_0. When $t = 0$, the construction will yield the solution to the standard Dirichlet Problem on Ω, namely:

$$H^0 = H^0 \in C(\overline{\Omega}), \quad H^0 \text{ is } F \text{ harmonic on } \Omega, \quad \text{and} \quad H^0\big|_{\overline{\Omega}} \;=\; \varphi. \tag{7.2}$$

The construction of H^t is based on using the function

$$\underline{\psi}_t \;\equiv\; H^0(x) + t(h(x) - 1). \tag{7.3}$$

to prescribe the boundary values on the inner boundary $\partial B_r(x_0)$. An upper bound for H^t is provided by the function

$$\overline{h}_t \;\equiv\; \lambda h(x) + H^0(x_0) + t(h(x) - 1) \tag{7.4}$$

for λ sufficiently large. Note that $\overline{h}_t$ is F-harmonic on $\Omega - \{x_0\}$ by the hypothesis (H1). We need the following "monotonicity" hypothesis to ensure that $\underline{\psi}_t$ is F-subharmonic.

Hypothesis (H2). The function $H^0 + th$ is F-subharmonic on Ω, and therefore, so is each $\underline{\psi}_t$. (Of course (H2) is satisfied if F is a convex subequation.)

Let H^t_r denote the solution, given by Theorem 6.1, to the Dirichlet Problem on $\Omega_r = \Omega - \overline{B_r(x_0)}$ with boundary values φ on $\partial\Omega$ and boundary values $\underline{\psi}_t$ on $\partial B_r(x_0)$. As in Section 6, we will show that these functions H^t_r are pointwise increasing as $r \to 0$. We define H^t by

$$H^t(x) \;\equiv\; \lim_{r\to 0} H^t_r(x) \;\text{ on } \overline{\Omega} - \{x_0\} \qquad \text{and} \qquad H^t(x_0) \;\equiv\; H^0(x_0) - t. \tag{7.5}$$

This is our candidate for a "solution" to the (DPPS) with singularity prescribed by th at x_0. Next we show that H^t satisfies all the required conditions with the exception of the asymptotic equivalence $H^t \sim th$ at x_0.

THEOREM 7.1. *For each $t \geq 0$,*

(1) $H^t \in C(\overline{\Omega})$,

(2) H^t *is F-harmonic on* $\Omega - \{x_0\}$,

(3) $H^t\big|_{\partial\Omega} = \varphi$.

In addition,

(4a) $\underline{\psi}_t \leq H^t$ *with equality at* x_0,

(4b) $H^t \leq \overline{h}_t$ *(for λ large) with equality at* x_0.

Proof. First note that $\underline{\psi}_t \in C(\overline{\Omega})$ has the properties

$$\underline{\psi}_t \text{ is } F\text{subharmonic on } \Omega, \quad \underline{\psi}_t \leq \varphi, \quad \text{and } \underline{\psi}_t(x_0) = H^0(x_0) - t \tag{7.6}$$

because of (H2), $h\big|_{\partial\Omega} \leq 1$ and $h(x_0) = 0$ respectively. Therefore $\underline{\psi}_t$ is in the Perron family for H_r^t, which proves that

$$\underline{\psi}_t \;\leq\; H_r^t \quad \text{on } \Omega_r. \tag{7.7}$$

Define

$$\overline{H}_r^t \;\equiv\; \begin{cases} \max\{\underline{\psi}_t, H_r^t\} = H_r^t & \text{on } \overline{\Omega}_r \\ \underline{\psi}_t & \text{on } \overline{B}_r. \end{cases} \tag{7.8}$$

Then exactly as in Section 6 one proves that

$$\overline{H}_r^t \text{ is } F\text{subharmonic on } \Omega \text{ and increasing in } r \text{ as } r \to 0. \tag{7.9}$$

Define

$$H^t(x) \;\equiv\; \lim_{r\to 0} \overline{H}_r^t(x) \qquad \text{for all } x \in \overline{\Omega}. \tag{7.10}$$

Now Property (3) is immediate since $H_r^t\big|_{\partial\Omega} = \varphi$ independent of r. Next note that $\underline{\psi}_t \leq \overline{H}_r^t \leq H^t$ by (7.7). Together with the equality $\underline{\psi}_t(x_0) = \overline{H}_r^t(x_0) = H^t(x_0) - t$ at x_0, this proves (4a).

We begin the proof of (1) and (2). The family $\{\overline{H}_r^t\}_{r>0}$ of functions on $\overline{\Omega}$ is bounded. The upper bound $\overline{H}_r^t \leq H^0$ implies that $H^t \leq (H^t)^* \leq H^0$, and hence (3) can be strengthened to

$(3)'$ $\quad H^t\big|_{\partial\Omega} \;=\; (H^t)^*\big|_{\partial\Omega} \;=\; \varphi.$

Thus,

$$(H^t)^* \in \mathrm{USC}(\overline{\Omega}) \qquad \text{and} \qquad H^t = (H^t)_* \in \mathrm{LSC}(\overline{\Omega}) \tag{7.11}$$

are finite-valued. Moreover, by the "families bounded above" property and the "decreasing limit" property, we have

$$(a)\;\; (H^t)^* \text{ is } F\text{subharmonic on } \Omega, \quad (b)\;\; -H^t \text{ is } \widetilde{F}\text{subharmonic on } \Omega - \{x_0\}. \tag{7.12}$$

Hence $u \equiv (H^t)^* - H^t \in \mathrm{USC}(\overline{\Omega})$ satisfies $u \geq 0$ on $\overline{\Omega}$, and $u\big|_{\partial\Omega} = 0$ by $(3)'$. As discussed in the proof of Lemma 5.1, u is subaffine on $\Omega - \{x_0\}$. Therefore, by the Maximum Principle for subaffine functions on th domain $\Omega - \{x_0\}$ we have

$$(H^t)^*(x) - H^t(x) \;\leq\; (H^t)^*(x_0) - H^t(x_0) \;=\; u(x_0) \quad \text{on } \overline{\Omega}. \tag{7.13}$$

It remains to show that $u(x) \equiv (H^t)^*(x) - H^t(x)$ equals zero at x_0, i.e.,

$$(H^t)^*(x_0) \;=\; H^0(x_0) - t. \tag{7.14}$$

Once this is established, we will have $(H^t)^* = H^t$, which implies both (1) and (2). Now (7.14) follows immediately from the upper bound (4b) since $\overline{h}^t(x_0) = H^0(x_0) - t$.

Thus it remains to prove (4b). As noted, equality in (4b) holds at x_0. Hence it suffices to prove that $H^t \leq \overline{h}_t$ on $\overline{\Omega} - \{x_0\}$, or that

$$H^t_r \;\leq\; \overline{h}_t \qquad \text{on } \Omega_r \text{ for small } r. \tag{7.15}$$

Since $\overline{h}_t$ is F-harmonic on Ω_r by (H1), it suffices, by comparison, to show that

$$H^t_r \;\leq\; \overline{h}_t \qquad \text{on } \partial\Omega_r. \tag{7.16}$$

That is, we must show that

$$\varphi \;\leq\; \overline{h}_t \qquad \text{on the outer boundary } \partial\Omega, \text{ and} \tag{7.16a}$$

$$\underline{\psi}_t \;\leq\; \overline{h}_t \qquad \text{on the inner boundary } \partial B_r. \tag{7.16b}$$

It is straightforward to see that (7.16a) holds if and only if λ is chosen greater than or equal to the positive quantity

$$\frac{\sup \varphi - H^0(x_0) + t\,\mathrm{osc}_{\partial\Omega}(h)}{\inf_{\partial\Omega} h} \tag{7.17}$$

In fact, since $\underline{\psi}_t(x_0) = \overline{h}_t(x_0)$, $\underline{\psi}_t\big|_{\partial\Omega} \leq \varphi \leq \overline{h}_t\big|_{\partial\Omega}$, and $u \equiv \underline{\psi}_t - \overline{h}_t \in \widetilde{\mathcal{P}}(\Omega - \{x_0\})$ (here we use (H1)), the Maximum Principle on $\Omega - \{x_0\}$ applies to u to prove that:

$$\underline{\psi}_t \;\leq\; \overline{h}_t \qquad \text{on } \overline{\Omega}, \tag{7.16b}'$$

which completes the proof of (4b) and hence Theorem 7.1. ∎

Prescribing Densities

We make two additional assumptions on h.

Hypothesis (H3). For each H which is F-subharmonic near $x_0 \in \Omega$ and F-harmonic on a deleted neighborhood of x_0,

$$\lim_{x \to x_0} \frac{H(x) - H(x_0)}{h(x)} \;\equiv\; \Theta \qquad \text{exists and } \Theta \geq 0, \tag{7.18}$$

that is, $H \sim \Theta h$ for some $\Theta \geq 0$.

There are many examples where this is true. They will be discussed later in this section.

Definition 7.2. Under the hypothesis (H3) the **h-density of H at** x_0, denoted $\Theta_{x_0}(H)$ is defined to be the limit in (7.18).

The second additional assumption on h is

Hypothesis (H4). If H is F-harmonic in a neighborhood of x_0, then $\Theta_{x_0}(H) = 0$, i.e.,

$$\lim_{x \to x_0} \frac{H(x) - H(x_0)}{h(x)} \;=\; 0.$$

.

As in Theorem 7.1, we assume (H1). We shall assume that F is convex, so (H2) is unnecessary. In addition we assume (H3) and (H4).

THEOREM 7.3. (Existence for Prescribed Densities). *Assume that F is a convex cone subequation. Then for each $\Theta \geq 0$ there exists a (unique) solution H to the (DPPS) with $H\big|_{\partial\Omega} = \varphi$ and $\Theta_{x_0}(H) = \Theta$.*

A briefer description is that under the hypotheses of Theorem 7.3, the Dirichlet Problem with Prescribed Densities, denoted (DPPD), has a unique solution.

Proof. Let $H^t,\ t \geq 0$ denote the family of downward-pointing F-harmonics at x_0 on Ω with $H^t\big|_{\partial\Omega} = \varphi$, constructed in Theorem 7.1. Let

$$f(t) \;\equiv\; \Theta(H^t).$$

To prove Theorem 7.3 it is enough to show that $f([0,\infty)) = [0,\infty)$, and then choose $t \in f^{-1}(\Theta)$ and $H = H^t$. The next lemma supplies this needed fact.

Lemma 7.4. *The function $f(t)$ satisfies:*

(A) $f(0) = 0$,

(B) $f(s) + (t-s) \leq f(t)$ *for* $0 \leq s \leq t$,

(C) $f(t)$ *is concave.*

Proof. Part (A) is immediate from the assumption (H4) since H^0 is F-harmonic on Ω. To prove (B) we show

(B)′ $\quad H^s(x) - H^s(x_0) + (t-s)h(x) \;\leq\; H^t(x) - H^t(x_0).$

The functions

$$u \;\equiv\; H^s_r(x) + (t-s)(h(x)-1) \qquad \text{and} \qquad v \;\equiv\; H^t_r(x)$$

have the same boundary values on ∂B_r since $\underline{\psi}_s + (t-s)(h-1) = \underline{\psi}_t$, while on $\partial\Omega$ we have $u = \varphi + (t-s)(h(x)-1) \leq \varphi = v$. By the convexity assumption on F, u is F-subharmonic

on $\Omega - B_r$. Since v is F-harmonic on $\Omega - B_r$, comparison implies that $u \le v$ on $\Omega - B_r$. Taking $r \downarrow 0$ gives

$$H^s(x) + (t-s)(h(x)-1) \;\le\; H^t(x)$$

which implies (B)′ since $H^s(x_0) = H^0(x_0) - s$, $H^t(x_0) = H^0(x_0) - t$, and $h(x_0) = 0$.

For (C) we show that for $0 \le \sigma \le 1$ and $0 < s < t$,

$$(C)' \quad \sigma\,(H^s(x) - H^s(x_0)) + (1-\sigma)\,\big(H^t(x) - H^t(x_0)\big) \;\le\; H^{\sigma s+(1-\sigma)t}(x) - H^{\sigma s+(1-\sigma)t}(x_0)$$

on $\overline{\Omega}$, which implies (C) by (H3).

Since $H^s(x_0) = H^0(x_0) - s$ for all s, the inequality (C)′ is equivalent to

$$(C)'' \quad \sigma H^s(x) + (1-\sigma)H^t(x) \;\le\; H^{\sigma s+(1-\sigma)t}(x) \quad \text{on } \overline{\Omega}.$$

This follows from

$$\sigma H^s_r(x) + (1-\sigma)H^t_r(x) \;\le\; H^{\sigma s+(1-\sigma)t}_r(x) \quad \text{on } \overline{\Omega} - B_r, \tag{7.19}$$

which is true because the LHS and the RHS have the same boundary values on $\partial(\Omega - \overline{B}_r)$ and, by the hypothesis that F is a convex cone, the sum of the two F-harmonics on the LHS is F-subharmonic and therefore in the Perron family for $H^{\sigma s+(1-\sigma)t}_r$. ∎

Remark 7.4. The inequality (4b) implies $f(t) \le t + \lambda(t)$. When does $f(t) = t$?

There are many interesting examples of subequations F and singularities h for which the hypotheses (H3) and (H4) hold.

Case 7.5. (Strong Uniqueness of Tangents). Here we assume that the subequation F is invariant under $\mathrm{O}(n)$. Then F has a well-defined Riesz characteristic p which we assume to satisfy $1 \le p < 2$. The function $h(x) = |x|^{2-p}$ provides the asymptotic singularity Every F-subharmonic function u defined near a point $x_0 \in \mathbf{R}^n$ has a well-defined density $\Theta = \Theta(u, x_0)$ at x_0 defined to be $\lim_{r\to 0} \sup_{B_r(x_0)} u/r^{2-p}$. We say that F satisfies **Strong Uniqueness of Tangents** if, for each u, every tangent to u at x_0 is $\Theta(u, x_0)|x - x_0|^{2-p}$. This Strong Uniqueness holds for F with the exception of $F = \mathcal{P}$, which is the only case where $p = 1$ (see [HL$_{10}$], [HL$_{11}$]).

Proposition 12.6 in [HL$_{10}$] states that for any F-subharmonic (not just a punctured F-harmonic) function u, strong uniqueness of the tangent to u holds if and only if (H3) holds, i.e., $u \sim \Theta K_p$. In particular, we have the following.

THEOREM 7.6. (Existence in the Finite Case). *Suppose F is $O(n)$-invariant with Riesz characteristic $1 < p < 2$ Take $h(x) = |x - x_0|^{2-p}$. Then h satisfies the hypotheses (H3) and (H4), and Theorem 7.3 applies, proving that the (DPPD) is uniquely solvable.*

Case 7.7. (Armstrong-Sirakov-Smart Fundamental Solutions) Now we assume instead that the subequation F in Theorem 7.1 is uniformly elliptic, and let $h = \Phi$ be the fundamental solution discussed in Example 2.5.

THEOREM 7.8. (Armstrong-Sirakov-Smart [AS_1]). *Every F-harmonic function H on $B_r(x_0) - \{x_0\}$, which has a downward-pointing singularity at x_0 is asymptotically equivalent to $\Theta\Phi(x - x_0)$ for some $\Theta \geq 0$.*

Corollary 7.9. *Suppose the subequation F is uniformly elliptic, and that the downward-pointing fundamental solution Φ of Armstrong, Sirakov and Smart is of finite type (i.e.. of homogeneity > 0). Then Theorem 7.3 applies with $h = \Phi$.*

↓ **To be rewritten.**

Finally we note that the Dirichlet Problem with multiple prescribed singularities, as stated at the beginning of Section 1, also has a solution.

THEOREM 7.9. (Existence in the Finite Case, II). *Let F be a cone subequation satisfying (F3), and suppose that φ, h_j and x_j are given as in the Dirichlet Problem with Prescribed Singularities in Section 1. Furthermore, assume that*

(i) the boundary of Ω is smooth and strictly F-convex,

(ii) each of the prescribed singularities h_j is of finite type, and has a strict local minimum at x_j,

(iii) there exists a function $h \in C(\overline{\Omega})$ which is F-subharmonic on $\Omega - \{x_1, ..., x_k\}$ with $h \sim h_j$ at x_j for each j and with $h\big|_{\partial\Omega} \leq \varphi$.

Then there exists a function H such that

(1) $H \in C(\overline{\Omega} - \{x_1, ..., x_k\})$,

(2) H is F-harmonic on $\Omega - \{x_1, ..., x_k\}$,

(3) $H\big|_{\partial\Omega} = \varphi$,

(4) There exist constants $0 < a < b$ such that

$$a \ \leq \ \frac{H(x) - H(x_j)}{h_j(x) - h_j(x_j)} \ \leq \ b \qquad \text{for } x \text{ near } x_j, \quad j = 1, ..., k.$$

In particular, $H(x_j) = h_j(x_j), \ \forall\, j$.

The proof is essentially the same as the one above and is left to the reader.

Corollary 7.9. Suppose F is as in Case 7.3 above, and F is convex. Then as in Theorem 7.4, the solution H in Theorem 7.8 satisfies

$$H \ \sim \ \Theta(H, x_j)|x - x_j|^{p-2} \qquad \text{for each } j.$$

Proof. Here we take $h_j(x) = |x - x_j|^{p-2}$ and $h(x) = \sum_k |x - x_j|^{p-2} + C$ with the constant C chosen so that $h\big|_{\partial\Omega} \leq \varphi$. ∎

We note that in the case where $p = 1$ (i.e., $F = \mathcal{P}$), multiple singularities cannot exist, because a convex function cannot have more than one strict local minimum. The theorem does not apply since the global function h does not exist when $k > 1$.

However, for characteristic $1 < p < 2$, the theorem does apply. The singularities of $h(x) = \sum_k |x - x_j|^{p-2} + C$ are equivalent to $|x - x_j|^{p-2}$ at each point. (To see this note that if $h(x) = \Theta|x|^\alpha + g(x)$ where g is smooth and $0 < \alpha < 1$, then $(h(x) - h(0))/|x|^\alpha \to 0$ as $x \to 0$.) The simplest subequation to which this applies is $\mathcal{P}_p$.

Appendix A. Asymptotic Equivalences and Tangent Flows (for p $\neq$ 2).

The asymptotic equivalence classes can be related to the tangent flow very generally. Suppose u is an upper semi-continuous function defined in a neighborhood of the origin. Recall the tangent flow and the Riesz kernel:

$$u_r(x) \equiv \begin{cases} r^{p-2}u(rx) & \text{if } p \geq 2 \\ \frac{u(rx)-u(0)}{r^{2-p}} & \text{if } 1 \leq p < 2. \end{cases} \qquad \text{and} \qquad K(x) \equiv \begin{cases} -\frac{1}{|x|^{p-2}} & \text{if } p \geq 2 \\ |x|^{2-p} & \text{if } 1 \leq p < 2. \end{cases}$$

Note that $K_r = K$ for the full range $p \geq 1,\ p \neq 2$.

Fix $\Theta > 0$. We define the first notion of asymptotic equivalence at 0 as follows.

Definition A.1. $u \sim \Theta K$ at 0 if

$$\begin{aligned} \lim_{x\to 0} \frac{u(x)}{K(x)} &= \Theta \qquad \text{when } p > 2 \\ \lim_{x\to 0} \frac{u(x) - u(0)}{K(x)} &= \Theta \qquad \text{when } 2 > p \geq 1 \end{aligned} \tag{A.1}$$

We have a second alternative definition of asymptotic equivalence when $p > 2$.

Definition A.2. $u \approx \Theta K$ at 0 if

$$u - \Theta K \quad \text{is bounded in a neighborhood of the origin.} \tag{A.2}$$

This notion is stronger. (When $2 > p \geq 1$ it is weaker, in fact too weak to be useful.)

Proposition A.3. ($p > 2$). *If $u \approx \Theta K$, then $u \sim \Theta K$. However, the converse is false.*

Proof. Note that

$$\begin{aligned} u \approx \Theta K \qquad &\Longleftrightarrow \qquad |u(x) - \Theta K(x)| \leq C \ \text{ near } x = 0 \\ &\Longleftrightarrow \qquad \left| \frac{u(x)}{K(x)} - \Theta \right| \leq \frac{C}{|K(x)|} \ \text{ near } x = 0 \end{aligned}$$

This implies that $\lim_{x\to 0} u(x)/K(x) = \Theta$ since $K(0) = -\infty$. See Example A.6 for a counterexample to the converse. ∎

These asymptotic equivalences are related to the tangent flows as follows.

Proposition A.4. $(p > 2)$**.**

$$\begin{array}{lll} (a) & u \approx \Theta K & \Rightarrow \quad u_r \to \Theta K \text{ uniformly on } \overline{B}_R \\ (b) & u_r - \Theta K \text{ bounded near } 0 & \Rightarrow \quad u \approx \Theta K. \end{array}$$

Proposition A.5.

$$\begin{array}{lll} (a) \ (1 \le p < 2) & u \sim \Theta K & \Rightarrow \quad u_r \to \Theta K \text{ uniformly on } \overline{B}_R \\ (b) \ (2 < p) & u \sim \Theta K & \Rightarrow \quad u_r \to \Theta K \text{ uniformly on } \overline{A}_{s,R} \\ (c) \ (1 \le p < \infty) & u_r \to \Theta K \text{ uniformly on some sphere } \partial B_R & \Rightarrow \quad u \sim \Theta K \end{array}$$

Proof of Proposition A.4(a). Assume $u \approx \Theta K$. The inequality

$$|u(y) - \Theta K(y)| \ \le \ C \tag{A.3}$$

can be rewritten with $y = rx$ as

$$|u_r(x) - \Theta K(x)| \ \le \ C r^{p-2} \tag{A.3$'$}$$

by applying the tangent flow to both sides. Thus if (A.3) holds for all $|y| \le \delta$, then (A.3)$'$ holds for all $|x| \le R$ and $r \le \delta/R$, which suffices to prove that u_r converges uniformly to ΘK on B_R.

Proof of Proposition A.5(a), (b). With y, r, x related by $y = rx$, the inequalities

$$(a) \quad \left| \frac{u(y)}{K(y)} - \Theta \right| \ \le \ \epsilon \qquad \text{and} \qquad (b) \quad |u_r(x) - \Theta K(x)| \ \le \ \epsilon |K(x)| \tag{A.4}$$

are equivalent. If (a) holds for $|y| \le \delta$, then (b) holds for $|x| \le R$ and $r \le \delta/R$.

Case: $1 \le p < 2$**.** Note that $|x| \le R \Rightarrow |K(x)| = K(x) \le K(R)$. This proves that (A.4a) for $|y| \le \delta$ implies $|u_r(x) - \Theta K(x)| \le \epsilon K(R)$ for $|x| \le R$ and $r < \delta/R$. Thus u_r converges uniformly to ΘK on $\overline{B}_R$.

Case: $2 < p$**.** Note that $s \le |x| \Rightarrow |K(x)| \le |K(s)|$. This proves that (A.4a) for $|y| \le \delta$ implies that $|u_r(x) - \Theta K(x)| \le \epsilon |K(s)|$ for $s \le |x| \le R$ and $r < \delta/R$, since $|y| = r|x| \le \frac{\delta}{R} R = \delta$. Thus u_r converges uniformly to ΘK on $A_{s,R}$.

Proof of Proposition A.4(b). With y, r, x related by $y = rx$, the inequalities

$$\begin{array}{lll} (a) & |u_\delta(x) - \Theta K(x)| \ \le \ C & \forall \ |x| \le R \qquad \text{and} \\ (b) & \delta^{p-2} |u(y) - \Theta K(y)| \ \le \ C & \forall \ |y| \le \delta R \end{array} \tag{A.5}$$

are equivalent. If (a) holds for all $|x| \le R$, then (b) holds for all $|y| \le \delta R$. This proves that $u \approx \Theta K$ if (a) is true for some $\delta, C > 0$ and all $|x| \le R$

Proof of Proposition A.5(c). Again if $y = rx$, then

$$(a) \quad |u_r(x) - \Theta K(x)| \ \leq \ \epsilon \qquad \text{and} \qquad (b) \quad \left| \frac{u(y)}{K(y)} - \Theta \right| \ \leq \ \frac{\epsilon}{|K(x)|} \tag{A.6}$$

are equivalent. To see this divide both sides of (A.6a) by $|K(x)|$ and note that $\frac{u_r(x)}{K(x)} = \frac{u_r(x)}{K_r(x)} = \frac{u(rx)}{K(rx)}$. Suppose (A.6a) holds for all $|x| = R$ and $r \leq \delta$. Then (A.6b) holds for all $y = rx$ with $|x| = R$ and $r \leq \delta$, or for all $|y| \leq \delta R$. Since $K(x) = K(R)$ this is enough to prove that $\lim_{y\to 0} u(y)/K(y) = \Theta$.

Example A.6. (Counterexamples to: $u \sim \Theta K \ \Rightarrow \ u \approx \Theta K$). Take

$$F = \Delta, \qquad K(x) = -\frac{1}{|x|^{n-2}}, \qquad \text{and} \qquad u(x) \equiv -\frac{\Theta}{|x|^{n-2}} - \frac{1}{|x|^{p-2}}$$

with $p < n$. Then $\lim_{x\to 0} \frac{u(x)}{K(x)} \ = \ \lim_{x\to 0} (\Theta + |x|^{n-p}) \ = \ \Theta$ so that $u \sim K$. However, $u - \Theta K = -\frac{1}{|x|^{p-2}}$ is not bounded near the origin, i.e., $u \approx \Theta K$ is not true. Note that u is Δ-subharmonic.

In this example we could also replace n with any $q \leq n$ and take $p < q$. In this case the corresponding function u is $\mathcal{P}_q$-subharmonic since $\mathcal{P}_p \subset \mathcal{P}_q$.

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