# Deformation Theory and Operads

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Let me give the simplicial interpretation of the Hochschild complex. Let's start with a prelude on simplicial objects.

**Definition 1** *Let $\Delta$ be the category with objects $\bar{n} = \{0, \ldots, n\}$ for $n \geq 0$ and morphisms ar maps $f : \{0, \ldots, n\} \to \{0, \ldots, n\}$, non-decreasing. Composition is composition of set maps.*

The most important maps are the maps $\delta_i : \bar{n} \to n \bar{+} 1$ which skips $i$ and is injective. Let $\sigma_i : \bar{n} \to n \bar{-} 1$ be the surjective map that hits $i$ twice.

As homework, every non-decresaing map $f : \bar{n} \to \bar{m}$ can be written as $f = \delta_{i_1} \circ \cdots \delta_{i_r} \circ \sigma_{j_1} \circ \cdots \sigma_{j_s}$.

This is unique if $i_r \leq \cdots i_1$ and $j_s \leq \cdots \leq j_1$. We have the relations $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$ and $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i \leq j$ along with $\sigma_j \delta_i = \delta_i \sigma_{j-1}$ for $i < j$, $id_{\bar{n}}$ for $i = j$ or $i = j + 1$, and $\delta_{i-1} \sigma_j$ for $i > j + 1$

**Definition 2** *Let $\mathscr{C}$ be a category. A cosimplicial object in $\mathscr{C}$ is a covariant functor $X^\cdot : \Delta \to \mathscr{C}$ A simplicial object in $\mathscr{C}$ is a contravariant functor $X_. : \Delta \to \mathscr{C}$*

**Lemma 1** *a A cosimplicial object in $\mathscr{C}$ is equivalent to a sequence of objects $X^i \in Obj(\mathscr{C})$ and morphisms in $\mathscr{C}$ $D_i : X^n \to X^{n+1}$ for $0 \leq i \leq n + 1$, $S_i : X^n \to X^{n-1}$ for $0 \leq i \leq n - 1$ satisfying the corresponding commutation relations.*

*b A simplicial object is a sequence of objects $X_0, X_1, \ldots$ and morphisms $d_i : X_{n+1} \to X_n$ for $0 \leq i \leq n + 1$ ando $s_i : X_{n-1} \to X_n$ for $0 \leq i \leq n - 1$ satisfying $d_i d_j = d_{j-1} d_i$ for $i < j$, $s_i s_j = s_{j+1} s_i$ for $i \leq j$ and then*

$$d_i s_j = \begin{cases} s_{j-1} d_i & , \quad i < j \\ id_{X_n} & , \quad i = j, i = j + 1 \\ s_j d_{i-1} & , \quad i > j + 1 \end{cases}$$

The proof for the first part is, take $X^n = X^\cdot(\bar{n})$ and $D_i = X^\cdot(\delta_i)$,$S_i = X^\cdot(\sigma_i)$. This works both forward and backward.