WANDERING DOMAINS

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Abstract. This expository article sketches a proof of Sullivan's no-wandering-domains theorem for polynomials, and summarizes some of what is known about wandering domains for entire functions. The proof is intended to be self-contained, except for references to standard results in complex analysis, real analysis and topology.

1. Introduction

One of the most famous questions in complex dynamics, asked by Fatou, is whether the Fatou set of a rational map f can have a wandering component, i.e., a connected component of the Fatou set whose orbit under f is not periodic or pre-periodic. In [58] Dennis Sullivan famously showed the answer is no: rational functions do not have wandering components. The result was quickly extended to entire functions that have only finitely many singular values by Alex Eremenko and Misha Lyubich [27], and by Lisa Goldberg and Linda Keen [33], with a special case also being given by Irvine Noel Baker [9]. On the other hand, Baker [8] had earlier shown that transcendental (i.e., non-polynomial) entire functions can have wandering domains, and there are now variety of such examples. Indeed, the existence of wandering domains is one of the primary distinctions between polynomial and transcendental dynamics and transcendental wandering domains are currently the subject of intense investigation.

Sullivan's theorem has had a huge impact. Besides completing the characterization of periodic Fatou components started by Fatou and Julia, it introduced quasiconformal methods into the subject, a powerful idea that has had many other applications. On August 25, 2021, the online version of *Mathematical Reviews* listed 177 papers referring to his paper [58]. However, these 177 did not include the papers [9], [27], [33], mentioned above, since these were written before *Mathematical Reviews* started to routinely list references are part of the review. No doubt the true list of citations is substantially longer. At the end of this paper I give the list of citations from *Math. Reviews*, as well as few more that I am aware of. [1]

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[1] Readers are welcome to send other citations to me; I will update this paper on my website.

In this note we shall give a proof of Sullivan's theorem that is as self-contained as seems reasonable; all proofs require some version of the measurable Riemann mapping theorem, but we will attempt to replace some of the more technical analytic aspects of this result with (equally involved, but perhaps more elementary) topological arguments. We omit the case of general rational functions since this simplifies the argument (an extra argument is needed to reduce to the case of simply connected domains; see Appendix F of Jack Milnor's book [50]). For other treatments of Sullivan's theorem see [5], [23], [27], [33], [37], [50], [58], [62], [65].

Sullivan's proof was motivated by the proof of Ahlfors' Finiteness Theorem: if G is a finitely generated Kleinian group acting discontinuously on an open set Ω of the 2-sphere, then Ω/G is a (possibly branched) Riemann surface of finite area. Ahlfors' theorem can now be proved without using quasiconformal mappings, as a consequence of the solution of the tameness conjecture for hyperbolic 3-manifolds ([1], [24]; see [25] for an expository account) but is seems doubtful that such methods can easily be adapted to the case of polynomial or transcendental dynamics.

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2. Polynomials have no wandering domains

The Fatou set, $\mathcal{F}(f)$, of a polynomial or a transcendental entire function f is the union of open disks on which $\{f^n\}$ forms a normal family. It is also clear that $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$ (forward invariance) but equality need not hold for general entire functions, e.g., $e^z/10$ has a Fatou component that contains 0, but $0 \notin f(\mathcal{F}(f))$. However, if Ω is a bounded Fatou component of f, then the image is a full component (Lemma D.9). In general, if U, V are Fatou components of f so that $f(U) \subset V$, then $V \setminus U$ can contain at most one point. This is Theorem 4' of [35] by Maurice Heins: if f is entire, V is open and connected and U is a connected component of $f^{-1}(V)$, then $f^{-1}(z) \cap U$ either has finite constant size for $z \in V$ or is finite for at most one point. See also the papers by M.E. Herring [36] and Walter Bergweiler and Steffen Rohde [20]. Similarly, $f^{-1}(\mathcal{F}(f)) \subset \mathcal{F}(f)$ (backwards invariance). Thus we can think of f as inducing a map between Fatou components and a wandering domain is a component of the Fatou set all of whose forward images are disjoint. The grand orbit of a Fatou component is the union of all its forward and backwards images. The complement $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$ is called the Julia set of f and is clearly a closed, totally invariant set. One can show $\mathcal{J}(f) = \mathcal{J}(f^n)$ for every $n \in \mathbb{N}$. For polynomials of degree ≥ 2 and all transcendental entire functions, the Julia set is non-empty, indeed, has uncountably many points. We shall see later (Corollary D.8) that for transcendental entire functions, the Julia set always contains a non-trivial continuum (this is due to Baker, [7]).

Lemma 2.1. *A wandering domain for a polynomial must be simply connected.*

Proof. For a polynomial, a neighborhood of ∞ is attracted to ∞, so any wandering domain Ω must be bounded and have a bounded orbit. By the maximum principle, the iterates of f are bounded in the interior of any closed curve in the wandering domain and hence form a normal family inside the curve. Thus the curve does not surround any Julia points and so Ω must be simply connected. □

Lemma 2.2. *Suppose X is connected and $\{\psi_t\}_{t\in X}$ is a family of homeomorphisms of $\mathbb{C}$ so that $\psi_t(z) : X \to \mathbb{C}$ is continuous in t for each fixed z. Suppose also that ψ_{t_0} is the identity for some $t_0 \in X$ and that f is a polynomial with the property that $\psi_t \circ f = f \circ \psi_t$ for all $t \in X$. Then $\psi_t(z) = z$ for all $t \in X$ and all $z \in \mathcal{J}(f)$, i.e., every ψ_t is the identity when restricted to the Julia set of f.*

Proof. A periodic point z for f is a point such that $f^n(z) = z$ for some $n \geq 1$. A point is pre-periodic if some iterate of it is periodic. For a non-constant polynomial, the periodic points are clearly a finite set for each n, hence the sets of all periodic or pre-periodic points are countable. Because ψ_t conjugates the action of f to itself, pre-periodic points are mapped to pre-periodic points. Since there are only countable many such points, $\{\psi_t(z) : t \in X\} \subset \psi_{t_0}(X)$ must be a single point, since X is connected. Since one of these maps is the identity, every map must fix every pre-periodic point. Finally, since the Julia set is contained in the closure of the pre-periodic points (Theorem A.9), each map ψ_t must fix every point in $\mathcal{J}(f)$. □

Theorem 2.3. *Polynomials have no wandering domains.*

Proof. Choose a smooth function $h : \mathbb{C} \to [0, \frac{1}{2}]$ supported in $\mathbb{D}$ with gradient bounded by 1 and such that $h(0) > 0$. Define a family of mappings of the upper half-plane to itself by $\Phi_t(z) = z + th(z)$, for $|t| \leq 1$. It is easy to check that these are quasiconformal self-maps of $\mathbb{H}^2_+ = \{x + iy : y > 0\}$ (the definition of a QC map is given in Appendix B), at least if we restrict t to a small enough interval $[0, \epsilon]$ and that Φ_0 is the identity. If $t \neq 0$, then the mapping is definitely not the identity since $\Phi_t(0) = t \cdot h(0) \neq 0$. Choose N disjoint intervals $I_k = \{[4k - 1, 4k + 1]\}_1^N$ and define an N-dimensional family of maps by $\mathbf{t} = (t_1, \dots, t_N)$, and

$$\Phi_{\mathbf{t}}(z) = z + \sum_{k=1}^{N} t_k h(z - 4k).$$

The main point we need below is that $\mathbf{t}$ is uniquely determined by knowing the cross ratio of the images of all quadruples on the boundary; in particular by knowing this for quadruples of the form $(0, 1, 4k, \infty)$, $k = 1, \dots, N$.

Suppose p is a polynomial of degree d and suppose Ω is a wandering domain for p. Since p has only finitely many critical values, we can replace Ω, if necessary, by an iterate of itself so that the forward orbit contains no critical points. Therefore we may assume p is univalent on Ω and on all forward orbits of Ω.

By Lemma 2.1, Ω is simply connected, so we can map it conformally by f to $\mathbb{H}^2_+$ and define a quasiconformal map $\varphi_{\mathbf{t}} = f^{-1} \circ \Phi_{\mathbf{t}} \circ f$ that maps Ω to itself. The dilatation of $\varphi_{\mathbf{t}}$ defines a smooth dilatation $\mu_{\mathbf{t}}$ on Ω that we can extend to the grand orbit of Ω using the composition rule for dilatations, Equation B.1. A version of the measurable Riemann mapping theorem (Theorem B.19) implies there is a family of quasiconformal maps $\Psi_{\mathbf{t}}$ that conjugate p to a function

$$p_{\mathbf{t}} = \Psi_{\mathbf{t}}^{-1} \circ p \circ \Psi_{\mathbf{t}},$$

that is entire and d-to-1, hence a polynomial of degree d. Doing the extension backwards is always possible; extending to the forward iterates uses the assumption that p and all its iterates are univalent on Ω. Moreover, we will show $p_{\mathbf{t}}(z)$ moves continuously as a function of $\mathbf{t}$ for each fixed z. See Theorem B.19.

Theorem C.1 says that given any continuous map of an open set in $\mathbb{R}^n$ into $\mathbb{R}^k$ for $k < n$ there is a point $z \in \mathbb{R}^k$ whose preimage has topological dimension ≥ 1 and hence contains a connected set X. Apply this to the mapping $\mathbf{t} \to p_{\mathbf{t}}$ from the N-dimensional set of parameters $\mathbf{t}$ to the $(d+1)$-dimensional space of degree d polynomials. If we take $N > d+1$, then we obtain a connected set X of parameters that all map to the same polynomial p.

Choose some $\mathbf{s} \in X$ and consider the maps $\psi_{\mathbf{t}} = \Psi_{\mathbf{t}} \circ \Psi_{\mathbf{s}}^{-1}$ for $\mathbf{t} \in X$. The maps $p \to \psi_{\mathbf{t}}^{-1} \circ p \circ \psi_{\mathbf{t}}$ all conjugate p to itself, and $\psi_{\mathbf{s}}$ is the identity. Thus by Lemma 2.2, for $\mathbf{t} \in X$, we have that $\psi_{\mathbf{t}}$ is the identity on $\mathcal{J}(p)$, hence these maps are all the identity on $\partial\Omega$ (a subset of the Julia set). Thus $\Psi_{\mathbf{t}} = \Psi_{\mathbf{s}}$ on $\partial\Omega$ for all $\mathbf{t} \in X$. Note that $\Psi_{\mathbf{t}}$ and $\varphi_{\mathbf{t}}$ are both quasiconformal maps of the wandering domain Ω to itself and that they have the same dilatation inside Ω by definition. Thus they differ by a conformal self-map of Ω. Since we have just seen that $\Psi_{\mathbf{t}}$ is the identity on $\partial\Omega$ for $\mathbf{t} \in X$, we deduce that $\varphi_{\mathbf{t}}$ is the boundary value of a conformal self-map of Ω, and hence $\Phi_{\mathbf{t}} \circ \Phi_{\mathbf{s}}$ agrees with a Möbius transformation on the boundary of the upper half-plane. Thus for any $\mathbf{t}, \mathbf{s} \in X$, the map boundary values of $\Phi_{\mathbf{t}}$ preserve the cross ratio of any four points on the boundary. However, this is manifestly false by construction; the boundary maps don't preserve all cross ratios unless $\mathbf{t} = \mathbf{s}$. The contradiction proves that a polynomial can have no wandering domains. □

This completes the proof of Sullivan's theorem, except for the following facts:

- Theorem A.9: the Julia set is contained in the closure of the pre-periodic points,
- Theorem B.19: a weak version of the measurable Riemann mapping theorem,
- Theorem C.1: if $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous, then f maps some non-trivial continuum to a point.

We will prove these in the Appendices A-C (thus, while Sullivan's theorem is not as easy as abc, it is as easy as ABC). In Appendix D we will sketch some of what is know about wandering domains for entire functions. Our goal in Appendices A-C is to be as self-contained as possible, attempting to prove everything not found in standard first year graduate courses in real analysis, complex analysis and topology. Our main sources for these topics will be

- [30]: Gerald Folland's *Real Analysis*,
- [44]: Don Marshall's *Complex Analysis*,
- [51]: James Munkres' *Topology: a first course.*

The most advanced result we use without proof is the uniformization theorem: every non-compact simply connected Riemann surface is either $\mathbb{D}$ or $\mathbb{C}$. A proof of this is given in Marshall's book above.

Appendix A. Normal families and extremal length

The theory of covering spaces says that every Riemann surface has a universal covering surface that is also a Riemann surface. Koebe's uniformization theorem says that there are only three simply connected Riemann surfaces (up to conformal isomorphism): $\mathbb{D}$, $\mathbb{C}$ and the 2-sphere. Any other Riemann surface (and there are many) is the quotient of one of these by a discrete group of Möbius transformations. An element of such group can't have a fixed point, and this implies that the sphere covers only itself and the plane covers only genus one tori and the once punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Every other Riemann surface is the quotient of the disk by a Fuchsian group (i.e., a discrete group of Möbius transformations acting on $\mathbb{D}$). There are proofs of this in several recent textbooks, e.g., Don Marshall's [44] or Dror Varolin's [61]. A planar domain Ω is called hyperbolic if $\mathbb{C} \setminus \Omega$ has at least two points. Thus the following is a special case of the uniformization theorem.

Theorem A.1. *Every hyperbolic plane domain Ω is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to Ω).*

For us, the most important example is the twice punctured plane $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$. The fact that its universal cover is the unit disk (which can also be proven by a direct construction of the covering map) implies several very useful facts.

Theorem A.2 (Picard's little theorem)**.** *If f is a non-constant entire function, then $E = \mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.*

Proof. If E contains two points $\{a, b\}$, then using the covering map $p : \mathbb{D} \to \mathbb{C} \setminus \{a, b\}$, f can be lifted to a holomorphic map $f : \mathbb{C} \to \mathbb{D}$. By Liouville's theorem, the lift is constant and hence so must f. $\square$

A family $\mathcal{F}$ of meromorphic functions on a planar domain Ω is a normal family if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact set or converges uniformly to ∞ on every compact set. The following can be found in several texts, e.g., Folland's [30].

Theorem A.3 (Arzela-Ascoli)**.** *A family $\mathcal{F}$ of continuous functions from a planar domain Ω to a metric space (X, d) is normal if and only if*

(1) *$\mathcal{F}$ is equicontinuous on every compact $E \subset \Omega$.*
(2) *For any $z \in \Omega$, $\{f(z) : f \in \mathcal{F}\}$ is pre-compact (lies in a compact subset).*

By the Cauchy estimates, a holomorphic map f from a planar domain Ω to the unit disk satisfies

$$|f'(z)| \leq C/\text{dist}(z, \partial\Omega).$$

By the Arzela-Ascoli Theorem, the family of such functions is normal; we call this the "first version" of Montel's theorem. More generally, we have the following.

Theorem A.4 (Montel's theorem)**.** *If $\mathcal{F}$ is a family of holomorphic functions on a planar domain Ω all taking values in $W = \mathbb{C} \setminus \{a, b\}$ for some $a \neq b$, then $\mathcal{F}$ is a normal family.*

Proof. Since $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$ is covered by by the disk, each map $f : \Omega \to \mathbb{C}^{**}$ can be lifted to a map $F : \Omega \to \mathbb{D}$. The family of lifted maps is normal by the first version of Montel's theorem. Thus any sequence $\{f_n\}$ in $\mathcal{F}$ can be lifted to a sequence $\{F_n\}$ that has a convergent subsequence $\{F_{n_k}\}$ and $\{f_{n_k}\} = \{p \circ F_{n_k}\}$ is convergent in the original family. □

Thus omitting two values has two consequences: it implies normality when applied to functions on a hyperbolic domain and it implies constancy when applied to functions on $\mathbb{C}$. Bloch's principle says that a property of a family of functions implies one of these conclusions if and only if it implies the other. This is not always true, but it does hold for a number of interesting cases (such as families that are uniformly bounded or omitting two values, as above) and it can be made into a precise mathematical statement. See Bergweiler's paper [17].

Theorem A.5 (Picard's great theorem)**.** *If f is meromorphic on $A_R = \{R < |z| < \infty\}$ and has an essential singularity at ∞, then for every $r \geq R$, $E = \mathbb{C} \setminus f(A_r)$ contains at most one point.*

Proof. Let $D_r = \{|z| > r\}$. For r sufficiently large, $f(D_r)$ omits two points, and therefore f has a lift to a map $D_r \to \mathbb{D}$. This is a bounded holomorphic function on D_r, which is conformally a punctured disk. So by the Riemann removability theorem (Corollary 5.10, [44]) the lift extends holomorphically across the puncture. Thus f cannot have an essentially singularity at ∞, a contradiction. □

A.1. **Pre-periodic points.** The following was known to Fatou [29], and made explicit by Rosenbloom [54].

Lemma A.6. *If g is entire and $h(z) = (g(g(z)) - z)/(g(z) - z)$ is constant then g is constant or linear.*

Proof. If $h \equiv 0$, then $g(g((z)) = z$ implying g is 1-to-1, hence linear. If $h \equiv 1$, then $g \circ g = g$ so g is constant or $g(z) = z$. So assume h is a constant $c \neq 0, 1$, i.e,

$$g^2(z) - z = c(g(z) - z),$$

and differentiate to get

$$g'(g(z))g'(z) - 1 = c(g'(z) - 1),$$

or

$$g'(z)(g'(g(z)) - c) = 1 - c.$$

Since $c \neq 1$, the left side is never zero, hence both factors are never zero. Thus g' omits 0. It also omits c, for if g covers the whole plane this is obvious; if $g' = c$ only at the single possible omitted value of g, then g' takes the values $0, c$ only finitely often; by the great Picard theorem g' is a polynomial omitting 0, hence constant. Thus g is linear. □

We leave it to the reader to show that if h is rational, then g must be rational too.

Theorem A.7. *If g is entire and not constant or linear then it has at least two pre-periodic points.*

Proof. Consider the function

$$h(z) = (g(g(z)) - z)/(g(z) - z),$$

as in Lemma A.6. Our assumption implies that h is a non-constant meromorphic function. If $h(z) = \infty$ then $g(z) = z$, so every such point is a fixed point of g. If $h(z) = 0$ then $g^2(z) = z$ so every such point is periodic or period 2. If $h(z) = 1$, then $g^2(z) = g(z)$ so $g(z)$ is a fixed point of g and hence z is pre-periodic.

If h is a rational of degree $d \geq 1$, then each of $\{0, 1\}$ has at least one preimage and hence g has at least two pre-periodic points. Otherwise h has an essential singularity at ∞ and then Picard's great theorem says that it takes on at least one of the values $\{0, 1, \infty\}$ infinitely often. Hence g has infinitely many pre-periodic points. □

Lemma A.8. *The Julia set is contained in the accumulation set of the backwards orbits $\cup_n f^{-n}(z)$, except possibly for one exceptional point z.*

Proof. Suppose $z \in \mathcal{J}(f)$ and V is a neighborhood of z. Then $\{f^n\}$ is not normal on V, so takes every complex value except possibly one (Theorem A.4). Thus given any two points, at least one of them is eventually covered by $f^n(V)$. □

Thus the Julia set is contained in the closure of any backwards invariant set with at least two elements. For polynomials of degree ≥ 2 or transcendental entire functions, this includes the set of pre-periodic points.

Theorem A.9. *The Julia set is contained in the closure of the pre-periodic points.*

In fact, a much stronger statement is true: the Julia set is the closure of the repelling periodic points of f (all of which must be contained in the Julia set). For entire functions this is due to Baker [6], and to Baker, Kotus and Lu [10] for meromorphic functions. The early proofs of this depended on the Ahlfors islands theorem; a deep result giving the normality of meromorphic families satisfying certain geometric conditions. See Bergweiler's paper [16] for an illuminating discussion of Ahlfors' theorem and its applications to dynamics. More elementary proofs of the density of repelling fixed points have been found by Bargmann [12], Berteloot and Duval [21], and Schwick [56]. The proof in [21] is particularly short and elementary, depending on Picard's theorem and an extremely useful characterization of non-normality known as Zalcman's lemma [63].

In general, the Julia set need not be the whole accumulation set of a backwards orbit. For example, there can be simply connected Fatou components where f is conjugate to an irrational rotation (Siegel disks) and the accumulation set of a point in such a component contains a closed curve inside the Fatou component.

A.2. **Modulus and extremal length.** Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is admissible for Γ (and write $\rho \in \mathcal{A}(\Gamma)$) if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma\in\Gamma} \int_\gamma \rho ds \geq 1,$$

and define the modulus of Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. This is a well known conformal invariant whose basic properties are discussed in many sources such as Ahlfors' book [3]. Its reciprocal is called the extremal length of the path family and is denoted

$$\lambda(\Gamma) = 1/M(\Gamma).$$

Modulus and extremal length satisfy several properties that are helpful in estimating these quantities.

Lemma A.10 (Conformal invariance)**.** *If $\mathcal{F}$ is a family of curves in a domain Ω and f is a one-to-one analytic mapping from Ω to Ω' then $M(\mathcal{F}) = M(f(\mathcal{F}))$.*

Proof. This is just the change of variables formulas

$$\int_\gamma \rho\circ f|f'|ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho\circ f)^2|f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

These imply that if $\rho \in \mathcal{A}(f(\mathcal{F}))$ then $|f'|\cdot\rho\circ f^{-1} \in \mathcal{A}(f(\mathcal{F}))$, and thus $M(f(\mathcal{F})) \leq M(\mathcal{F})$. We get the other direction by considering f^{-1}. □

Lemma A.11 (Monotonicity)**.** *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are collections such that every $\gamma \in \mathcal{F}_1$ contains some curve in $\mathcal{F}_2$ then $M(\mathcal{F}_1) \leq M(\mathcal{F}_2)$ and $\lambda(\mathcal{F}_1) \geq \lambda(\mathcal{F}_2)$.*

The proof is immediate since $\mathcal{A}(\mathcal{F}_1) \supset \mathcal{A}(\mathcal{F}_2)$.

Lemma A.12 (Grötzsch Principle). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) = M(\mathcal{F}_1) + M(\mathcal{F}_2)$.*

Proof. Suppose ρ_1 and ρ_2 are admissible for $\mathcal{F}_1$ and $\mathcal{F}_2$. Take $\rho = \rho_1$ and $\rho = \rho_2$ in their respective domains. Then it is easy to check that ρ is admissible for $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$ so domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) \leq M(\mathcal{F}_1) + M(\mathcal{F}_2)$. By restricting an admissible metric ρ to each domain, a similar argument proves the other direction. □

Lemma A.13 (Series Rule). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains and every curve of $\mathcal{F}$ contains both a curve from $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2)$.*

Proof. If $\rho_i \in \mathcal{A}(\mathcal{F}_i)$ for $i = 1, 2$, then $\rho = t\rho_1 + (1-t)\rho_2$ is admissible for $\mathcal{F}$. Since the domains are disjoint we may assume $\rho_1\rho_2 = 0$ everywhere so for $0 \leq t \leq 1$,

$$\rho^2 = t^2\rho_1^2 + (1-t)^2\rho_2^2.$$

Integrating ρ^2 then shows

$$M(\mathcal{F}) \leq t^2 M(\mathcal{F}_1) + (1-t)^2 M(\mathcal{F}_2),$$

for each t. To find the optimal t set $a = M(\mathcal{F}_1)$, $b = M(\mathcal{F}_2)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$\begin{aligned} M(\mathcal{F}) &\leq t^2 a + (1-t)^2 b = \frac{b^2 a + a^2 b}{(a+b)^2} \\ &= \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a} + \frac{1}{b}} \end{aligned}$$

or

$$\frac{1}{M(\mathcal{F})} \geq \frac{1}{M(\mathcal{F}_1)} + \frac{1}{M(\mathcal{F}_2)},$$

which, by definition, is the same as

$$\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2),$$

□

Next we actually compute the modulus of some path families. The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates. So suppose $R = [0, b] \times [0, a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a. Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2} ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fgdx)^2 \leq (\int f^2 dx)(\int g^2 dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible, the Cauchy-Schwarz inequality gives

$$1 \leq \int_0^b (1 \cdot \rho(x,y))dx \leq \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus we must have equality. Let $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$ denote the unit circle.

Lemma A.14. *If $A = \{z : r < |z| < R\}$ then the modulus of the path family connecting the two boundary components is $\frac{1}{2\pi}\log\frac{R}{r}$. More generally, if $\mathcal{F}$ is the family of paths connecting $r\mathbb{T}$ to a set $E \subset R\mathbb{T}$, then $M(\mathcal{F}) \geq |E| \log\frac{R}{r}$.*

Proof. By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for $\mathcal{F}$. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho dr)^2 \leq (\int_1^R \frac{dr}{r})(\int_1^R \rho^2 r dr) = \log R \int_1^R \rho^2 r dr$$

so

$$\int_0^{2\pi}\int_1^R \rho^2 r dr d\theta \quad \geq \quad \int_E \int_r^R \rho^2 r dr d\theta \geq |E| \int_1^R \rho^2 r dr \geq |E| \log R \quad \square$$

A quadrilateral Q is a Jordan curve in the plane with two distinguished, disjoint, closed subarcs. The modulus of Q is the modulus of the path family in Q connecting these two boundary arcs. We will use without proof that there is a conformal map of the interior of Q to a rectangle that extends homeomorphically to the boundary with the four marked points mapping to the four corners of the rectangle. If the rectangle has side lengths $a, b > 0$, and the distinguished arcs of Q map to the then the modulus of the quadrilateral is a/b.

Lemma A.15. *Suppose Q is a quadrilateral with opposite pairs of sides E, F and C, D. Assume*

(1) *E and F can be connected in Q by a curve of diameter $\leq \epsilon$,*

(2) *any curve connecting* C *and* D *in* Q *has diameter at least* 1.

Then the modulus of the path family connecting E *and* F *in* Q *is larger than* $M(\epsilon)$ *where* $M(\epsilon) \to \infty$ *as* $\epsilon \to 0$.

Proof. There is a segment $(a,b) \subset Q$ with $|a-b| \leq \epsilon$ and $a \in E$ and $b \in F$. Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\epsilon)$ for $\epsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross S and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero. As before this shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. See Figure 1. □

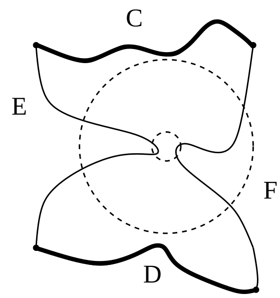

FIGURE 1. Proof of Lemma A.15.

Lemma A.16. *Suppose* $\Omega \subset \mathbb{C}$ *is a topological annulus of modulus* M *whose boundary consists of two Jordan curves* γ_1, γ_2 *with* γ_2 *separating* γ_1 *from* ∞. *Then* $\operatorname{diam}(\gamma_1) \leq (1-\epsilon)\operatorname{diam}(\gamma_2)$ *where* $\epsilon > 0$ *depends only on* M.

Proof. Rescale so $\operatorname{diam}(\gamma_2) = \operatorname{diam}(\Omega) = 1$ and suppose $\operatorname{diam}(\gamma_1) > 1-\epsilon$. Then there are points $a \in \gamma_1$ and $b \in \gamma_2$ with $|a-b| \leq \epsilon$. Let ρ be the metric on Ω defined by $\rho(z) = \frac{1}{|z-a|\log(1/2\epsilon)}$ for $\epsilon < |z-a| < 1/2$. Then any curve $\gamma \subset \Omega$ that separates γ_1 and γ_2 satisfies $\int_\gamma \rho ds \geq 1$ and

$$M \leq \int \rho^2 dxdy \leq \frac{\pi}{4} \log^{-2} \frac{1}{2\epsilon}.$$

Thus the modulus of the path family separating the boundary components is bounded above by the right hand side, and the modulus of the reciprocal family connecting the boundary components is bounded below by $\frac{\pi}{4}\log^2\frac{1}{2\epsilon}$. Thus $\epsilon \geq \frac{1}{2}\exp(-\sqrt{\pi M/4})$. □

A.3. **Koebe's $\frac{1}{4}$-theorem.** In this sub-section we give a proof of Koebe' theorem using extremal length (the usual proof uses the area theorem, e.g. Theorem I.4.1 of [31]).

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let $\gamma^+ = (\gamma \cap \mathbb{H}^2_+) \cup \overline{\gamma \cap \mathbb{H}_l}$, where $\mathbb{H}^2_+, \mathbb{H}_l$ denote the upper and lower half-planes. If Γ is a path family in the plane then $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma^+ = \{\gamma^+ : \gamma \in \Gamma\}$.

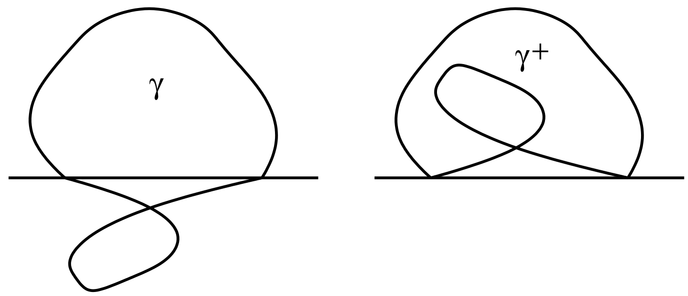

FIGURE 2. The curves γ and γ^+

Lemma A.17. *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma^+)$.

Proof. We start by proving $M(\Gamma) \leq 2M(\Gamma^+)$. Given a metric ρ, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\int +\gamma^+ \sigma ds \geq \int_{\gamma^+} \rho ds \geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds.$$

Thus if ρ admissible for Γ^+, then σ is admissible for Γ Thus, since $\max(a,b)^2 \leq a^2+b^2$,

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z) dxdy + \int \rho^2(\bar{z}) dxdy \leq 2 \int \rho^2(z) dxdy.$$

Taking the infimum over admissible ρ's for Γ^+ makes the right hand side equal to $2M(\Gamma^+)$, proving the claim.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}^2_+$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

$$\begin{aligned}
\int_{\gamma^+} \sigma ds &= \int_{\gamma^+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma \cap \mathbb{H}^2_+} \rho(z) ds + \int_{\gamma \cap \mathbb{H}^2_+} \rho(\bar{z}) ds + \int_{\gamma \cap \mathbb{H}_l} \rho(z) + \int_{\gamma \cap \mathbb{H}_l} \rho(\bar{z}) ds \\
&= \int_\gamma \rho(z) ds + \int_{\bar{\gamma}} \rho(z) ds \geq 2 \inf_\rho \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ^+. Hence, since $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned}
M(\Gamma^+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy = \frac{1}{4} \int_{\mathbb{H}^2_+} (\rho(z) + \rho(\bar{z}))^2 dxdy \\
&\leq \frac{1}{2} \int_{\mathbb{H}^2_+} \rho^2(z) dxdy + \int_{\mathbb{H}^2_+} \rho^2(\bar{z}) dxdy \\
&= \frac{1}{2} \int \rho^2 dxdy.
\end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. □

Lemma A.18. *Let $\mathbb{D}^* = \{z : |z| > 1\}$ and $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ for some $R > 1$. Let $\Omega = \mathbb{D}^* \setminus K$, where K is a closed, unbounded, connected set in $\mathbb{D}^*$ which contains the point $\{R\}$. Let Γ_0, Γ denote the path families in these domains with separate the two boundary components. Then $M(\Gamma_0) \leq M(\Gamma)$.*

Proof. We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma^+) = \frac{1}{2}M(\Gamma_0)$. The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0. Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$. Suppose $\gamma \in \Gamma_R$. Since γ has

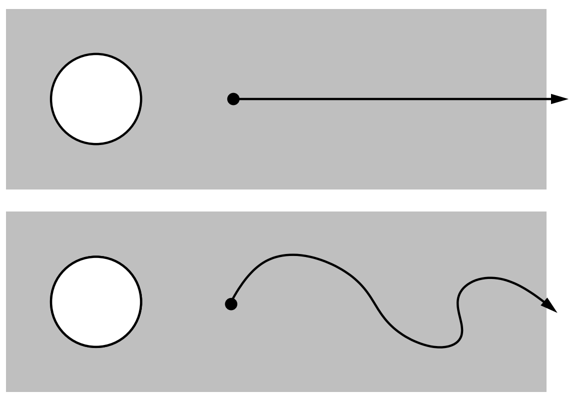

FIGURE 3. The annulus on top has smaller modulus than any other annulus formed by connecting R to ∞.

non-zero winding around 0 it must cross both the negative and positive real axes. If it never crossed $(0, R)$ then the winding around 0 and R would be the same, which false, so γ must cross $(0, R)$ as well. Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide γ into two subarcs γ_1 and γ_2. Then $\gamma^+ = \gamma_1^+ \cup \gamma_2^+$. But if we reflect γ_2^+ into the lower half-plane and join it to γ_1^+ it forms a closed curve γ_0 that is in Γ_0 and $\gamma_0^+ = \gamma^+$. Thus $\gamma^+ \in \Gamma_0^+$, as desired. □

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$. Thus $\Omega_{1,R}$ is the domain considered in the previous lemma. We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$. Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius$\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R})$. Thus the modulus of $\Omega_{\epsilon,R}$ is $2\pi \log \frac{4R}{\epsilon} + O(\frac{\epsilon}{R})$.

Lemma A.19. *Suppose $z, w \in \mathbb{D}$ and K is a compact connected set in $\mathbb{D}$ which contains both these points. Let Γ be the path family that separates K and $\mathbb{T}$. Then the modulus of this family is maximized when K is the hyperbolic geodesic between z and w in which case the modulus is $2\pi \log \frac{4}{\rho}(z,w) + O(\rho(z,w))$, where ρ denotes the hyperbolic distance.*

Proof. By conformal invariance we may use a Möbius transformation to move z to 0 and move w onto the positive axis. Applying an inversion, the path family is mapped to one as in Lemma A.18, showing that the radial line from z to w maximizes the modulus. The estimate of the modulus follows from our previous remarks. □

We now give an elegant proof of the Koebe $\frac{1}{4}$-theorem due to Mateljevic [47].

Theorem A.20 (The Koebe $\frac{1}{4}$ Theorem)**.** *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof. Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components). Let $R = \text{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1})).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$. Since $f'(0) = 1$, $\delta = \epsilon + O(\epsilon^2)$. Note that $f(\mathbb{D}) \setminus D(0, \delta) \supset f(A_{\epsilon,1})$, so

$$M(f(\mathbb{D}) \setminus D(0, \delta)) \geq M(f(A_{\epsilon,1})).$$

By Lemma A.18

$$M(f(\mathbb{D}) \setminus D(0, \delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon},$$

or

$$\log 4R - \log(\epsilon + O(\epsilon^2)) + O(\frac{\epsilon}{R}) \geq -\log \epsilon.$$

Taking $\epsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. □

A.4. **The Gehring-Hayman theorem and radial limits of conformal maps.** In [32] Gehring and Hayman proved the following fundamental inequality that says that the hyperbolic geodesic is (up to a constant factor) the most efficient way to connect two points in a simply connected plane domain. This is a fundamental (and very useful) property of the hyperbolic metric that has been generalized in many directions, e.g., [34], [41], [60].

Theorem A.21 (Gehring-Hayman inequality)**.** *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in Ω, let γ be the hyperbolic geodesic connecting these two points and let γ' be any other curve in Ω connecting them. Then $\ell(\gamma) \leq C\ell(\gamma')$.*

Proof. Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z-1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z-1| = 2^{-n}\},$$
$$z_n = \gamma_n \cap [0,1).$$

Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0,r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = z_{N+1}$ for some N (if not we truncate a segment of the form $J = [z_{N+1}, r]$ and use Koebe's theorem to compare the lengths of $f(J)$ and $\gamma' \cap f(Q_{N+1})$).

Let $Q'_n \subset Q_n$ be the sub-quadrilateral of points with $|\arg(z-1)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart. In particular, this means that any curve in $f(Q_n)$ separating γ_n and γ_{n+1} must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma A.15 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$. Thus any curve γ in Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\gamma')).$$

□

Given $E \subset \mathbb{T}$ we will denote the capacity of E to be the modulus of the path family in the annulus $\{\frac{1}{2} < |z| < 1\}$ that has one endpoint on $\{|z| = \frac{1}{2}\}$ and one endpoint on E. This definition of capacity is non-standard, and is a substitute for the logarithmic capacity $\operatorname{cap}(E)$ of E. By Pfluger's theorem (e.g., see [31]) If $K \subset \mathbb{D}$ is a compact connected set, the logarithmic capacity satisfies the estimates

$$\frac{1}{\operatorname{cap}(E)} + C_1 \leq \frac{\pi}{M(\mathcal{F}_E)} \leq \frac{1}{\operatorname{cap}(E)} + C_2,$$

where $\mathcal{F}_E$ is the path family connecting $\{|z| = \frac{1}{2}\}$ to E. Here C_1, C_2 are universal constants. If $\{|z| = 1/2\}$ was replaced by some other nontrivial, compact, connected subset K of the unit disk, these constants would only depend on K. In particular, one of these quantities is zero iff the other is. We will not use this connection between logarithmic capacity and modulus, but we will need the following (rather weak) statement that zero capacity sets are small. We leave it to the reader to check that replacing $\{|z| < 1/2\}$ by some other compact subset of $\mathbb{D}$ does not change whether the corresponding capacity of a boundary set is zero or not.

Lemma A.22. *If E has zero capacity, then it has zero length.*

Proof. We prove the contrapositive. If E has positive length, suppose ρ is an admissible metric for the corresponding path family. Considering the radial segments connecting E to $\{|z| = 1/2\}$, we see

$$\begin{aligned} |E| &\leq 2\int_E \int_{1/2}^1 \rho(z)drd\theta \leq 4\int_E \int_{1/2}^1 \rho(z)rdrd\theta \\ &\leq 4\left(\int_E \int_{1/2}^1 \rho^2(z)dxdy\right)^{1/2} \cdot \left(\int_E \int_{1/2}^1 1dxdy\right)^{1/2} \\ &\leq 2\left(\int_E \int_{1/2}^1 \rho^2(z)dxdy\right)^{1/2} \cdot \sqrt{|E|}. \end{aligned}$$

Hence $\int \rho^2 dxdy \geq \frac{1}{4}|E|$. □

Lemma A.23. *Suppose $f : \mathbb{D} \to \Omega$ is conformal, and for $R \geq 1$,*

$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R\,\mathrm{dist}(f(0), \partial\Omega)\}.$$

Then E_R has capacity $O(1/\log R)$ if R is large enough.

Proof. Assume $f(0) = 0$ and $\mathrm{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then ρ is admissible for the path family $\mathcal{F}$ connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$. By definition $M(\mathcal{F}) \leq 2\pi/\log R$ and $\lambda(\mathcal{F}) \geq (\log R)/2\pi$. By the Koebe distortion theorem $K = f^{-1}(D(0, 1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω, one can show that the extremal length connecting K to the E is comparable to the extremal length connecting $\{|z| = 1/2\}$ to E. □

Corollary A.24. *Suppose $f : \mathbb{D} \to \Omega$ is conformal and $a \in \mathbb{C} \cup \{\infty\}$. Then the set where f has radial limit a has zero capacity.*

Proof. When $a = \infty$, this is immediate from the previous result. If $a \in \partial\Omega \setminus \{\infty\}$, we can reduce to the case $a = \infty$ by applying the conformal transformation $z \to 1/(a-z)$. The cases $a \notin \partial\Omega$ are trivial. □

Lemma A.25. *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let $E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta$ for some $s \in (r, 1)\}$. Then the extremal length of the path family $\mathcal{P}$ connecting $D(0, r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof. Suppose $z, w \in \Omega$, suppose γ is the hyperbolic geodesic connecting z and w and suppose $\tilde{\gamma}$ is any path in Ω connecting these points. By the Gehring-Hayman inequality (Theorem A.21) there is a universal $C < \infty$ such that $\ell(\gamma) \leq C\ell(\tilde{\gamma})$ (here $\ell(\gamma)$ denotes the length of γ).

Now suppose we apply this with $z = f(sx)$ and $w \in f(D(0,r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them. But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathcal{P}) \geq \frac{\delta^2}{C^2 a(r)}$. □

Corollary A.26. *If $f : \mathbb{D} \to \Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

Proof. Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\text{diam}(f(rx, x)) > \delta$, and let $E_\delta = \cap_{0<r<1} E_{r,\delta}$. If f does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by Lemma A.25. Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma A.23, so we deduce f has finite radial limits except on zero capacity. □

Appendix B. Quasiconformal mappings

B.1. **Continuity of modulus.** A quadrilateral Q is a Jordan curve in the plane with two distinguished, disjoint, closed subarcs. We will use without proof that there is a conformal map of the interior of Q to a rectangle that extends homeomorphically to the boundary with the four marked points mapping to the four corners of the rectangle. If the rectangle has side lengths $a, b > 0$, and the distinguished arcs of Q map to the then the modulus of the quadrilateral is a/b. Our first goal is to show this is continuous under perturbation, at least for an appropriate version of convergence of quadrilaterals.

Lemma B.1. *Suppose $\{f_n\}$ are conformal maps of $\mathbb{D} \to \Omega_n$ that converge uniformly on compact subsets of $\mathbb{D}$ to a conformal map $f : \mathbb{D} \to \Omega$. Suppose that the boundary of each Ω_n is the homeomorphic image $\partial\Omega_n = \sigma_n(\mathbb{T})$ and that $\{\sigma_n\}$ converges uniformly on $\mathbb{T}$ to a homeomorphism $\sigma : \mathbb{T} \to \partial\Omega$. Then $f_n \to f$ uniformly on the $\overline{\mathbb{D}}$.*

Proof. Fix $\epsilon > 0$ and choose n so large that if we divide $\mathbb{T}$ into n equal sized intervals $\{J_j\}_1^n$, then σ maps each of them to a set I_j of diameter at most $\epsilon/2$. Let $I_j^k = f_k(J_j)$. Because $\sigma_k \to \sigma$ uniformly, the sets I_j all have diameter at most ϵ, if k is large enough.

Next choose $\eta > 0$ so small that if $k, m > 1/\eta$ and $\sigma_m(J_j)$ and $\sigma_k(J_i)$ contain points at most distance $C\eta$ apart, then J_i and J_k are the same or adjacent to each other.

We can do this because of the uniform convergence and the fact that σ is 1-to-1. By passing to the limit the same property holds if we replace σ_m by σ.

Next choose m so large that $f(\mathbb{D}) \setminus f(\{|z| < 1 - \frac{1}{m}\})$ is contained in an η-neighborhood of $\partial\Omega$. Choose m points $\{z_j\}$ equally spaced on the circle $|z| = 1 - \frac{1}{m}$, and let $K_j \subset \mathbb{T}$ be the arc centered at $z_j/|z_j|$ of length $4\pi/m$. Fix a small number $\delta > 0$ (δ will be determined below, depending only on η). By Lemmas A.22 and A.23, we may choose a point $w_j \in K_j$ so that $|w_j - z_j| \le 2/m$ and

$$|f(w_j) - f(w_j(1 - \frac{1}{m}))| \le C\delta.$$

Similarly, choose points $w_j^k \in K_j$ so that

$$|f_k(w_j^k) - f_k(z_j)| \le 2C\delta.$$

This is possible since $f_k \to f$ uniformly on the compact set $\{|z| \le 1 - \frac{1}{m}\}$ and thus $\partial f_k(\mathbb{D})$ is contained in an 2δ-neighborhood of $\partial\Omega$ for k large enough and $\partial\Omega_k$ is contained in a δ-neighborhood of $\partial\Omega$ because of the uniform convergence of the parameterizations.

By taking m larger, if necessary, we can also arrange that each I_j contains at least one of the points $f(z_m/|z_m|)$. Thus each $f(K_j)$ is mapped into the union of at most 2 of the I_j and hence its image has diameter at most 2ϵ. Also, the points $f(w_p^k)$ and $f(w_{p+1}^k)$ are at most $C\delta$ apart, so belong to the same or adjacent sets I_j. Thus $f_k(K_p)$ is a union of at most 4 such adjacent sets and hence has diameter $O(\epsilon)$.

For each w_p^k there is an arc J_j so that $f_k(w_p^k) \subset \sigma_k(J_j)$. Similarly, there is an arc J_i so that $f(w_p) \in I_i = \sigma(J_i)$. Since $f_k \to f$ uniformly on the finite set $\{z_n\}$, we have, for k sufficiently large

$$\begin{aligned} |f_k(w_n^k) - f(w_n)| &\le |f_k(w_n^k) - f_k(z_n)| + |f_k(z_n) - f(z_n)| \\ &\qquad + |f(z_n) - f(w_n)| \\ &\le (2C + 1 + C)\delta. \end{aligned}$$

This is less than η if δ is small enough.

Since I_i and I_j each have diameter at most ϵ, their union has diameter $< 2\epsilon$ and the union of the intervals adjacent to these is at most 4ϵ. Similarly for I_i^k and J_j^k. Thus $f_k(K_p)$ and $f(K_p)$ are contained in $O(\epsilon)$-neighborhoods of each other. Thus $f_k \to f$ uniformly on $\mathbb{T}$. By the maximum principle, this implies uniform convergence on the closed disk, as desired. □

Corollary B.2. *Suppose $\{f_n\}$ are homeomorphisms $\mathbb{C} \to \mathbb{C}$ that converge uniformly to a homeomorphism f and that Q is a quadrilateral. If $Q_n = f_n(Q)$, then the moduli of Q_n converge to the modulus of $f(Q)$*

B.2. **Angle distortion of linear maps.** Conformal maps preserves angles; quasiconformal maps can distort angles, but only in a controlled way. To make this distinction more precise we must have a way to measure angle distortion and we start with a discussion of linear maps.

Consider the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cx + dy).$$

Let M^T denote the transpose of the real matrix M, i.e., its reflection over the main diagonal. Then

$$M^T \cdot M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is positive and symmetric and hence has two positive eigenvalues λ_1, λ_2, assuming M in non-degenerate. The square roots $s_1 = \sqrt{\lambda_1}$, $s_2 = \sqrt{\lambda_2}$ are the singular values of A (without loss of generality we assume $s_1 \geq s_2$). Then

$$M = U \cdot \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \cdot V,$$

where U, V are rotations. Thus M maps the unit circle to an ellipse whose major and minor axes have length s_1 and s_2. Thus M preserves angles iff it maps the unit circle to a circle iff $s_1 = s_2$. Otherwise M distorts angles and we let $D = s_1/s_2$ denote the dilatation of the linear map M. This is the eccentricity of the image ellipse and is ≥ 1, with equality iff M conformal.

The inverse of a linear map with singular values $\{s_1, s_2\}$ has singular values $\{\frac{1}{s_2}, \frac{1}{s_1}\}$ and hence dilatation $D = (1/s_2)/(1/s_1) = s_1/s_2$. Thus the dilatation of a linear map and its inverse are the same.

Given two linear maps M, N with singular values $s_1 \geq s_2$ and $t_1 \geq t_2$ respectively, the singular values of the composition MN are trapped between $s_1 t_1$ and $s_2 t_2$ (this occurs for the maximum singular values since they give the operator norms of the matrices and these are multiplicative; a similar argument works for the minimum singular values and the inverse maps). Thus the dilation is less than $(s_1 t_1)/(s_2 t_2)$ i.e., dilatations satisfy

$$D_{M \circ N} \leq D_M \cdot D_N.$$

The dilatation D can be computed in terms of a, b, c, d as follows. The eigenvalues λ_1, λ_2 are roots of the

$$0 = \det(M^T \cdot M - \lambda I),$$

which is the same as

$$0 = (E - \lambda)(G - \lambda) - F^2 = EG - F^2 - (E + G)\lambda + \lambda^2.$$

Thus

$$\begin{aligned}\lambda_1\lambda_2 &= EG - F^2\\ &= (a^2+c^2)(b^2+d^2)-(ab+cd)^2\\ &= a^2b^2+a^2d^2+c^2b^2+d^2c^2-(a^2b^2+2abcd+c^2d^2)\\ &= a^2d^2+c^2b^2-2abcd\\ &= (ad-bc)^2\end{aligned}$$

Similarly,

$$\lambda_1+\lambda_2 = E+G = a^2+b^2+c^2+d^2.$$

The values of λ_1, λ_2 can be found using the quadratic formula:

$$\begin{aligned}\{\lambda_1,\lambda_2\} &= \frac{1}{2}[E+G\pm\sqrt{(E+G)^2-4(EG-F^2)}]\\ &= \frac{1}{2}[E+G\pm\sqrt{(E-G)^2+4F^2)}].\end{aligned}$$

Thus

$$\begin{aligned}\frac{\lambda_1}{\lambda_2} &= \frac{E+G+\sqrt{(E-G)^2+4F^2}}{E+G-\sqrt{(E-G)^2+4F^2}}\\ &= \frac{(E+G+\sqrt{(E-G)^2+4F^2})^2}{(E+G)^2-(E-G)^2-4F^2}\\ &= \frac{(E+G+\sqrt{(E-G)^2+4F^2})^2}{4(EG+F^2)}.\end{aligned}$$

and hence

$$D = \frac{s_1}{s_2} = \sqrt{\frac{\lambda_1}{\lambda_2}} = \frac{E+G+\sqrt{(E-G)^2+4F^2}}{2\sqrt{EG+F^2}}.$$

This formula can be made simpler by complexifying. Think of the linear map M on $\mathbb{R}^2$ as a map f on $\mathbb{C}$:

$$x+iy \to ax+by+i(cx+dy) = u(x,y)+iv(x,y) = f(x+iy)$$

Then

$$M = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and we define

$$f_z = \frac{1}{2}(f_x - if_y) = \frac{1}{2}(u_x+v_y)+\frac{i}{2}(v_x-u_y), f_{\overline{z}} = \frac{1}{2}(f_x+if_y) = \frac{1}{2}(u_x-v_y)+\frac{i}{2}(v_x+u_y).$$

Some tedious arithmetic now shows that

$$\begin{aligned} 4|f_z|^2 &= (u_x + v_y)^2 + (v_x - u_y)^2 \\ &= u_x^2 + 2u_xv_y + v_y^2 + v_x^2 - 2v_xu_y + u_y^2 \end{aligned}$$

$$\begin{aligned} 4|f_{\overline{z}}|^2 &= (u_x - v_y)^2 + (v_x + u_y)^2 \\ &= u_x^2 - 2u_xv_y + v_y^2 + v_x^2 + 2v_xu_y + u_y^2 \end{aligned}$$

so

$$(|f_z| + |f_{\overline{z}}|)(|f_z| - |f_{\overline{z}}|) = |f_z|^2 - |f_{\overline{z}}|^2 \quad = \quad u_xv_y - v_xu_y = s_1s_2 = \det(M).$$

In particular, if we assume M is orientation preserving and full rank, then $\det(M) > 0$ and we deduce $|f_z| > |f_{\overline{z}}|$. Similarly,

$$\begin{aligned} (|f_z| + |f_{\overline{z}}|)^2 + (|f_z| - |f_{\overline{z}}|)^2 &= 2(|f_z|^2 + |f_{\overline{z}}|^2) \\ &= u_x^2 + v_x^2 + u_y^2 + v_x^2 \\ &= E + G \\ &= \lambda_1 + \lambda_2 \\ &= s_1^2 + s_2^2. \end{aligned}$$

From these equations and the facts $s_1 \geq s_2$, $|f_z| > |f_{\overline{z}}|$ we can deduce

$$s_1 = |f_z| + |f_{\overline{z}}|, \qquad s_2 = |f_z| - |f_{\overline{z}}|,$$

and hence

$$D = \frac{s_1}{s_2} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}.$$

Note that $D \geq 1$ with equality iff f is a conformal linear map.

B.3. **Dilatations.** It is often more convenient to deal with the complex number

$$\mu = \frac{f_{\overline{z}}}{f_z},$$

which is called the complex dilatation (although sometimes we abuse notation and just call thus the dilatation, if the meaning is clear from context). Since $|f_{\overline{z}}| < |f_z|$, we have $|\mu| < 1$ and it is easy to verify that

$$D = \frac{1 + |\mu|}{1 - |\mu|}, \qquad |\mu| = \frac{D-1}{D+1},$$

so that either D or $|\mu|$ can be used to measure the degree of non-conformality.

We leave it to the reader to check that the map

$$x + iy \to (ax + by) + i(cx + dy)$$

can also be written as

$$(z, \overline{z}) \to \alpha z + \beta \overline{z},$$

where $z = x + iy$, $\overline{z} = x - iy$ and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, satisfy

$$\alpha_1 = \frac{a+d}{2}, \quad \alpha_2 = \frac{a-d}{2}, \quad \beta_1 = \frac{c-b}{2}, \quad \beta_2 = \frac{b+c}{2},$$

In this notation $\mu = \beta/\alpha$ and

$$D = \frac{|\beta| + |\alpha|}{|\alpha| - |\beta|}.$$

As noted above, the linear map f sends the unit circle to an ellipse of eccentricity D. What point on the circle is mapped furthest from the origin? Since

$$s_1 = |f_z| + |f_{\overline{z}}|,$$

the maximum stretching is attained when $f_z z$ and $f_{\bar{z}}\bar{z}$ have the same argument, i.e., when

$$0 < \frac{f_z z}{f_{\overline{z}} \overline{z}} = \frac{z^2}{\mu |z|^2},$$

or

$$\arg(z) = \frac{1}{2} \arg(\mu),$$

Thus $|\mu|$ encodes the eccentricity of the ellipse and $\arg(\mu)$ encodes the direction of its major axis.

If we follow f by a conformal map g, then the same infinitesimal ellipse is mapped to a circle, so we must have $\mu_{g \circ f} = \mu_f$. If f is preceded by a conformal map g, then the ellipse that is mapped to a circle is the original one rotated by $-\arg(g_z)$, so $\mu_{f\circ g} = (|g_z|/g_z)^2 \mu_f$. To obtain the correct formula in general we need to do a little linear algebra. Consider the composition $g \circ f$ and let $w = f(z)$ so that the usual chain rule gives

$$(g \circ f)_z = (g_w \circ f) f_z + (g_{\bar{w}} \circ f) \bar{f}_z,$$
$$(g \circ f)_{\overline{z}} = (g_w \circ f) f_{\overline{z}} + (g_{\bar{w}} \circ f) \bar{f}_{\overline{z}}.$$

or in vector notation

$$\begin{pmatrix} (g \circ f)_z \\ (g \circ f)_z \end{pmatrix} = \begin{pmatrix} f_z & \bar{f}_z \\ f_{\overline{z}} & \bar{f}_{\overline{z}} \end{pmatrix} \begin{pmatrix} (g_w \circ f) \\ (g_{\bar{w}} \circ f) \end{pmatrix}$$

The determinate of the matrix is

$$f_z \bar{f}_{\overline{z}} - \bar{f}_z f_{\overline{z}} = f_z \overline{f_z} - \overline{f_{\overline{z}}} f_{\overline{z}} = |f_z|^2 - |f_{\overline{z}}|^2 = s_1 \cdot s_2 = J,$$

which is the Jacobian of f, so by Cramer's Rule,

$$(g_w \circ f) = \frac{1}{J} [(g \circ f)_z \bar{f}_{\overline{z}} - (g \circ f)_{\overline{z}} \bar{f}_z],$$

$$(g_{\bar{w}} \circ f) = \frac{1}{J} [(g \circ f)_{\overline{z}} f_z - (g \circ f)_z f_{\overline{z}}],$$

so

$$\begin{aligned} \mu_{g\circ f} &= \frac{(g\circ f)_{\bar z} f_z - (g\circ f)_z f_{\bar z}}{(g\circ f)_z \bar f_{\bar z} - (g\circ f)_{\bar z} \bar f_z} \\ &= \frac{\mu_{g\circ f} f_z - f_{\bar z}}{\bar f_{\bar z} - \mu_{g\circ f} \bar f_z} \\ &= \frac{f_z}{\bar f_z} \frac{\mu_{g\circ f} - \mu_f}{1 - \mu_{g\circ f}\overline{\mu_f}}. \end{aligned}$$

Now set $h = g\circ f$ or $g = h\circ f^{-1}$ to get

$$\mu_{h\circ f^{-1}} \circ f = \frac{f_z}{\bar f_z} \frac{\mu_h - \mu_f}{1 - \mu_h \overline{\mu_f}}. \tag{B.1}$$

Thus if h and f are differentiable and have the same dilatation μ, then $g = h\circ f^{-1}$ is conformal.

B.4. **Definition of quasiconformal maps.** There are two alternate definitions of quasiconformal maps that we will work with. It is well known that these are equivalent, but we will not need this fact, and we only prove one definition gives a subset of the other (this will cause a certain awkwardness in the presentation, but shortens the paper). The first definition is in terms of the dilatations described above.

The piecewise differentiable definition: h is K-quasiconformal on Ω if there are countable many analytic curves whose union is a closed set Γ of Ω such that h is continuously differentiable on each connected component of $\Omega' = \Omega\setminus\Gamma$ and $D_h \le K$ on Ω'.

The main motivating example is when $\Omega = \mathbb{C}$, Γ is a triangulation of the plane and μ_h is constant on the interior of each triangle. Such maps arise as piecewise linear maps between compatible triangulations. We will show the above definition implies the following one.

The geometric definition: A homeomorphism h, defined on a planar domain Ω, is K-quasiconformal if the

$$\frac{1}{K} M(Q) \le M(h(Q)) \le K M(Q),$$

for every quadrilateral $Q\subset\Omega$.

One can prove that this definition implies the map is absolutely continuous on almost all lines and differentiable almost everywhere (for Lebesgue are measure), so the partials exist almost everywhere, and the tangent maps have bounded dilatation almost everywhere. Thus the geometric definition implies an "almost everywhere" version of the differentiable definition, but one of our goals in this paper is to avoid using this fact.

Next we check that the first definition implies the second. Suppose $Q\subset\Omega$ and $h(Q)$ are respectively equivalent to $1\times a$ and $1\times b$ rectangles and h is has dilatation bounded by K. Since the dilatation is unchanged by composing with conformal maps, it suffices to show

Lemma B.3. *If we have a piecewise differentiable K-quasiconformal map between a $1 \times a$ and $1 \times b$ rectangle with dilatation $\leq K$, then $\frac{a}{K} \leq b \leq Ka$. Thus the piecewise differentiable definition implies the geometric definition.*

Proof. By integrating over horizontal lines in the first rectangle, we see

$$b \leq \int_0^a (|f_z| + |f_{\overline{z}}|) dx,$$

and integrating in the other variable,

$$b \leq \int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|) dx dy.$$

Thus by Cauchy-Schwarz

$$\begin{aligned} b^2 &\leq \left(\int_0^1 \int_0^a (|f_z| + |f_{\overline{z}}|)(|f_z| - |f_{\overline{z}}|) dx dy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dx dy\right) \\ &\leq \left(\int_0^1 \int_0^a (|f_z|^2 - |f_{\overline{z}}|^2) dx dy\right)\left(\int_0^1 \int_0^a \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dx dy\right) \\ &\leq \left(\int_0^1 \int_0^a J_f dx dy\right)\left(\int_0^1 \int_0^a D_f dx dy\right) \\ &\leq baK, \end{aligned}$$

so $b \leq Ka$. The other direction follows by considering the inverse map. □

In order for the above proof to work we need two things: (1) the area of the range to be bounded above by integrating the Jacobian over the domain and (2) each horizontal line segment S to have an image whose length is bounded above by the integral of $|f_z| + |f_{\overline{z}}|$ over S. This certainly holds if f_z and $f_{\overline{z}}$ are piecewise continuous on a partition of the plane given by countable many analytic curves, as we have assumed. Recall that μ_n tends to zero in measure if area$(\{z : |\mu(z)| > \epsilon) \to$ for any $\epsilon > 0$. We leave it to the reader to deduce the following results.

Corollary B.4. *If f is a piecewise differentiable K-quasiconformal on the whole rectangle and $(1+\epsilon)$-quasiconformal except on a set of area δ, then $b/a \leq 1 + \epsilon + K\delta$. In particular, a sequence of such maps whose dilatations satisfy $\sup_n \|\mu_n\|_\infty \leq k < 1$ and so that $\{\mu_n\}$ tends to 0 in measure, will tend to a 1-quasiconformal map.*

Corollary B.5. *A K quasiconformal map satisfying the piecewise differentiable definition on Ω changes the modulus of any path family in Ω by at most a multiplicative factor of K.*

B.5. **Compactness of K-quasiconformal maps.** The Arzela-Ascoli theorem states that a collection of continuous functions is relatively compact if and only if it is equicontinuous and pointwise bounded. Here we prove that K-quasiconformal maps of the plane, normalized to fix both 0 and 1, have both these properties, and are also closed under uniform convergence on compact sets. Thus normalized K-quasiconformal maps are compact. Some normalization is necessary; the maps $f_n(z) = nz$ are all 1-quasiconformal, but are not pointwise bounded or equicontinuous.

Lemma B.6. *If $\{f_n\}$ is a sequence of K-quasiconformal maps on Ω that converge uniformly on compact subsets to a homeomorphism f, then f is K-quasiconformal.*

Proof. Any quadrilateral $Q \subset \Omega$ has compact closure in Ω so $Q' = \lim_n f_n(Q)$ is a quadrilateral in $f(\Omega)$ and we need only check that if Q is a quadrilateral then $M(\lim_n f_n(Q)) = \lim_n M(f_n(Q))$. However, this follows from Lemma B.1. □

Lemma B.7. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a K-quasiconformal map that fixes both 0 and 1. Then $|f(x)|$ is bounded with an estimate depending on $|x|$ and K, but not on f.*

Proof. First suppose $\mathrm{Re}(x) \leq 1/2$ and consider the topological annulus with boundary component $[0, x]$ and $[1, \infty)$. The modulus of the path family separating the two boundary components is bounded below depending only on $|x|$. But if $R = |f(x)|$ then by using the metric $\rho(z) = 1/(|z| \log R)$, we see that the modulus of $f(\mathcal{F})$ is at most $1/\log R$. This is a contradiction if R is too large. □

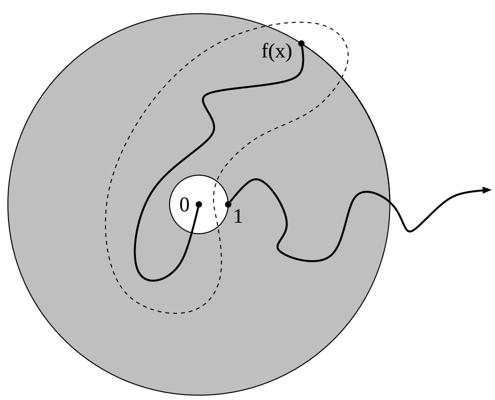

FIGURE 4. If $|f(x)| \gg |x|$ then the modulus of the path family separating $[0, x]$ and $[0, \infty)$ must change by more than a factor of K.

Theorem B.8. *A K-quasiconformal map of the plane that fixes both 0 and 1 is locally Hölder continuous.*

Proof. Suppose f is as in the lemma and $x, y \in D(0, r)$. By Lemma B.7, $D(0, 2r)$ is mapped into $D(0, R)$ for some $R = R(r, K)$. Surround $\{x, y\}$ by $N = \lfloor \log_2 \frac{r}{|x-y|} \rfloor$ annuli $\{A_j\}$ of modulus $\log 2$. See Figure 5. The image annuli $\{f(A_j)\}$ have moduli bounded away from zero, and hence $\mathrm{diam}(f(A_{j+1})) \leq (1-\epsilon)\mathrm{diam}(f(A_j))$ by Lemma A.16. Therefore

$$|f(x) - f(y)| \leq R(1-\epsilon)^N \leq R2^{\log_2(1-\epsilon)(1+\log_2 R - \log_2|x-y|)} \leq C(R)|x-y|^{\log_2(1-\epsilon)}.$$

□

One can prove the actual Hölder exponent is $\alpha = 1/K$.

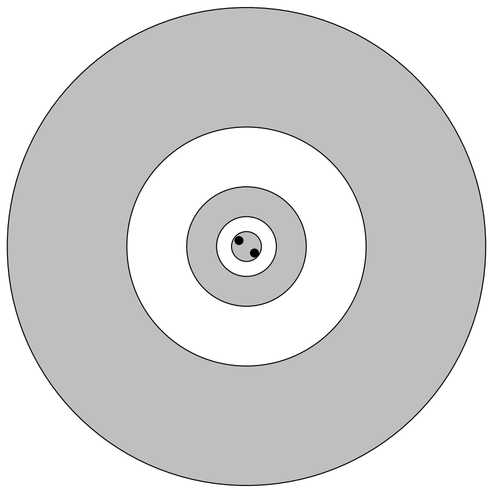

FIGURE 5. Annuli of fixed modulus map to annuli with modulus bounded below, and whose diameters shrink geometrically. Thus f is Hölder continuous.

Lemma B.9. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and onto, then φ extends continuously to a homeomorphism of $\mathbb{T} = \partial\mathbb{D}$ to itself.*

Proof. We may assume $f(0) = 0$; the general case follows after composing with a Möbius transformation.

Suppose $w, z \in \mathbb{D}$. We will show that

$$|f(z) - f(w)| \leq C|z-w|^\alpha,$$

for constants $C < \infty$, $\alpha > 0$ that depend only on the quasiconstant K of f. This implies f is uniformly continuous and hence has a continuous extension to the boundary of $\mathbb{D}$.

Let $d = |z-w|$ and $r = \min(1-|z|, 1-|w|)$. There are several cases depending on the positions of the points z, w and the relative sizes of d and r. See Figure 5.

To start, note that if $|z-w| \geq \frac{1}{10}$ we can just take $C = 20$ and $\alpha = 1$. So from here on, we assume $|z-w| < 1/10$.

Suppose $r > 1/4$, so $z, w \in \frac{3}{4}\mathbb{D}$. Surround the segment $[z, w]$ by $N \simeq \log d$ annuli with moduli $\simeq 1$. Then just as in the proof of Theorem B.8, the image annuli have moduli $\simeq 1$ (with a constant depending on K) and hence

$$|f(z) - f(w)| \leq (1 - \epsilon(K))^N = O(|z - w|^\alpha),$$

for some $\alpha > 0$ depending only on K.

Next suppose $|z| \geq 3/4$ and $d > r$. Then separate $[z, w]$ from 0 by $N \simeq \log d$ disjoint quadrilaterals with a pair of opposite sides being arcs of $\mathbb{T}$, and all with moduli $\simeq 1$. Since $f(0) = 0$ and the image quadrilaterals have moduli $\simeq 1$, there diameters shrink geometrically, so $|z - w| = (1 - \epsilon(K))^N = O(d^\alpha)$, as desired.

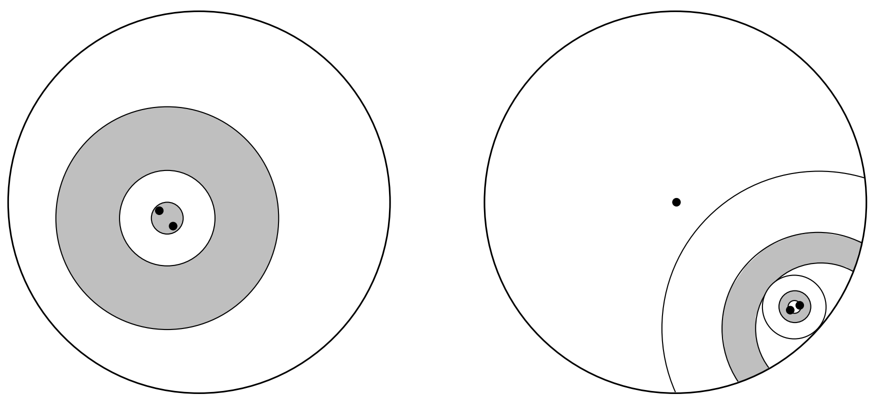

FIGURE 6. The proof of Hölder estimates in the disk is similar to the proof in the plane,except that we need to use quadrilaterals, as well as annuli, if the pair of points in near the boundary.

Finally, if $r \leq d$ we combine the two previous ideas: we start by separating $[z, w]$ from 0 by $\simeq \log d$ quadrilaterals with as above. The smallest quadrilateral then bounds a region of diameter approximately r containing $[z, w]$ and we then construct $\simeq \log r/d$ disjoint annuli with moduli $\simeq 1$ that each separate $[z, w]$ from this smallest quadrilateral. See Figure 6. The same arguments as before now show

$$|z - w| = (1 - \epsilon(K))^{-\log r}(1 - \epsilon(K))^{\log r/d} = O(d^\alpha) = O(|z - w|^\alpha).$$

□

B.6. **Continuous dependence on dilatation.** In this section we want to prove that if a quasiconformal map f on the plane has dilatation μ with small supremum norm, then f is close to linear $az + b$ in a precise way. This follows from compactness of K-quasiconformal maps, once we know that $\mu \equiv 0$ implies f is conformal on the plane, and hence linear. I follow Ahlfors presentation in [3], but add details where I found his argument hard to understand.

Corollary B.10. *There is an absolute $C < \infty$ so that the following holds. Suppose φ is a conformal map from a $\epsilon \times 1$ rectangle R to a Jordan domain that contains no disk larger than δ. Then for every $y \in [0, 1]$ there is a $t \in (0, 1)$ with $|t - y| < \epsilon$ and such that the horizontal cross-cut of R at height t maps to a arc of length $C\delta$.*

Proof. First assume $y \in (\frac{\epsilon}{2}, 1-\frac{\epsilon}{2})$ and choose a conformal map $\psi : \mathbb{D} \to R$ that sends 0 to $(\frac{\epsilon}{2}, y)$. By Lemmas A.23 and A.26, except for a set of small measure in $I = [y - \frac{\epsilon}{4}, y + \frac{\epsilon}{4}]$, all the horizontal cross-cuts corresponding to this interval have length bounded by $|\Phi'(0)| \leq C\delta$. □

Lemma B.11. *If f is a homeomorphism of $\Omega \subset \mathbb{C}$ that is K-quasiconformal in a neighborhood of each point of Ω, then f is K-quasiconformal on all of Ω.*

Proof. Suppose $Q \subset \Omega$ is a quadrilateral that is conformally equivalent via a map φ to a $1 \times m$ rectangle R and $Q' = f(Q)$ is conformally equivalent a $1 \times m'$ rectangle R'. Divide R into M equal vertical strips $\{S_j\}$ of dimension $1 \times m/M$. We have to choose M sufficiently large that two things happen.

First choose $\delta > 0$ so that f^{-1} is K-quasiconformal on any disk of radius δ centered at any point of Q' (we can do this since Q' has compact closure in Ω). Next, note that the closure of Q' is a union of Jordan arcs γ corresponding via $f \circ \varphi^{-1}$ to vertical line segments in R. By the continuity of $f \circ \phi^{-1}$ there is an $\eta > 0$ so that if $z \in R$ then $f(\phi^{-1}(D(z,\eta)))$ has diameter $\leq \delta$. By the continuity of the inverse map, there is an $\epsilon > 0$ so that $x, y \in Q'$ and $|x-y| < \epsilon$ implies $|\varphi(f^{-1}(x)) - \varphi(f^{-1}(y))| \leq \eta$. Thus for any $\delta > 0$ there is an $\epsilon > 0$ so that if $x, y \in \gamma \subset Q'$ are at most distance ϵ apart, then the arc of γ between then has diameter at most δ (and ϵ is independent of which γ we use).

Choose M so large that each region $Q'_j = f(\varphi^{-1}(S_j))$ contains a disk of radius at most ρ, where ρ will be chosen small depending on ϵ. Map Ω_j conformally to a $1 \times m'_j$ rectangle R'_j. By Lemma B.10 there is an absolute constant C so that every for every $y \in [0,1]$, there is a $t \in (0,1)$ with $|t-y| \leq Cm_j$ and so that the horizontal cross-cut of R'_j at height t maps via ϕ_j^{-1} to a Jordan arc of length $\leq C\rho$. Thus we can divide R'_j by horizontal cross-cuts into rectangles $\{R'_{ij}\}$ of modulus $m'_{ij} \simeq 1$ so that the preimages of these rectangles under ϕ_j are quadrilaterals with two opposite sides of length $\leq C\rho$ and which can be connected inside the quadrilateral by a curve of length $\leq C\rho$.

Taking δ as above, choose ϵ as above corresponding to $\delta/4$ and choose ρ so that $3C\rho < \min(\epsilon, \delta/4)$. Then all four sides of the quadrilateral Q'_{ij} have diameter $\leq \delta/4$ and hence Q'_{ij} has diameter less than δ and hence lies in a disk where f^{-1} is K-quasiconformal. Let m_{ij} be the modulus of corresponding preimage quadrilateral $Q_{ij} = f^{-1}(Q'_{ij})$. See Figure 7.

Then using the rules of extremal length

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m'_j} = \sum_i \frac{1}{m'_{ij}}, \qquad m' \geq \sum_j m'_j,$$

and by the definition of K-quasiconformal,

$$\frac{1}{K} \leq \frac{m_{ij}}{m'_{ij}} \leq K.$$

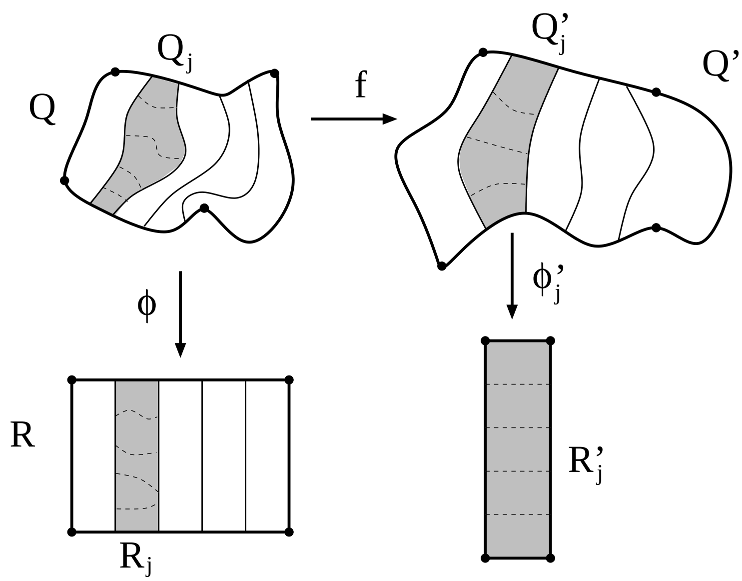

FIGURE 7. Notation in the proof of Theorem B.11.

Hence

$$\frac{M}{m} \geq \sum_i \frac{1}{m_{ij}} \geq \frac{1}{K} \sum_i \frac{1}{m'_{ij}} = \frac{1}{Km'_j}$$

or

$$\frac{m}{M} \leq Km'_j$$

for every j. Thus

$$m \leq \sum_{j=1}^{M} \frac{m}{M} \leq \sum_j Km'_j \leq Km'.$$

Applying the same result to the inverse map shows f is K-quasiconformal. □

If $K = 1$, then $m = m'$ the last line of the above proof becomes

$$m' = m \leq \sum_j \frac{m}{M} \leq \sum_j m'_j \leq m'.$$

so we deduce

$$\sum_j m'_j = m',$$

whereas in general, we only have $\sum_j m'_j \leq m'$. We want to use this to deduce that 1-quasiconformal map must be conformal. We start with

Lemma B.12. *Consider a $1 \times m$ rectangle R that is divided into two quadrilaterals Q_1, Q_2 of modulus m_1 and m_2 by a Jordan arc γ the connects the top and bottom edges of R. Then if $m = m_1 + m_2$, the curve γ is a vertical line segment.*

Proof. See Figure 8. Let φ_1, φ_2 be the conformal maps of Q_1, Q_2 onto $1 \times m_1$ and $1 \times m_2$ rectangles R_1, R_2 respectively. Set $\rho = |f_1'|$ on Q_1 and $\rho = |f_2'|$ in Q_2 and zero elsewhere. Then each horizontal line is cut by γ into pieces one of which connects the left vertical edge of R to γ, and another that connect γ to the right edge of R. The images of these connect the vertical edges of R_1 and R_2 respectively. Thus the images have lengths at least m_1 and m_2 respectively, there length of the image of the entire horizontal segment in Q is $\geq m_1 + m_2$. If we integrate over all horizontal segments in Q, we see

$$\int_Q (\rho - 1) dxdy \geq m_1 + m_1 - m = 0.$$

Similarly,

$$\int_Q (\rho^2 - 1) dxdy = \text{area}(f_1(Q_1) + \text{area}(f_2(Q_2)) - \text{area}(q) = (m_1 + m_2) - m = 0.$$

Thus

$$\int_Q (\rho - 1)^2 dxdy = \int_Q (\rho^2 - 1) - 2(\rho - 1) dxdy = 0.$$

Since $(\rho - 1)^2 \geq 0$, this implies $\rho = 1$ almost everywhere, i.e., f_1 and f_2 are both linear and the curve γ is a vertical line segment. □

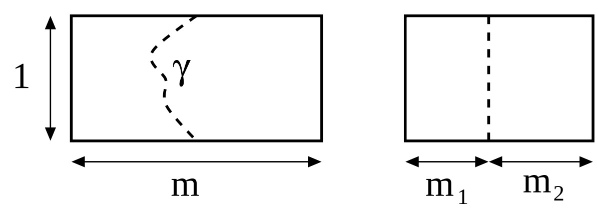

FIGURE 8. A partition of a rectangle as in the proof of Lemma B.13.

Lemma B.13. *If f is 1-quasiconformal on Ω, then it is conformal on Ω.*

Proof. If f is 1-quasiconformal in the proof of Theorem B.11, then as noted before Lemma B.12, we must have

$$\frac{M}{m} = \sum_i \frac{1}{m_{ij}}, \qquad \frac{1}{m_j'} = \sum_i \frac{1}{m_{ij}'}, \qquad m' = \sum_j m_j',$$

Thus the map $\psi = \varphi' \circ f \circ \varphi^{-1}$ between identical rectangles must be the identity map. Thus $f = (\varphi')^{-1} \circ \varphi$ is a composition of conformal maps, hence conformal. □

Corollary B.14. *For any $\delta > 0$ and and any $r > 0$ there is an $\epsilon > 0$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is $(1+\epsilon)$-quasiconformal and f fixes 0 and 1, then $|z - f(z)| \leq \delta$ for all $|z| < r$.*

Proof. If not, there is a sequence of $(1+\frac{1}{n})$-quasiconformal maps that all fix 0 and 1 and points $z_n \in D(0,r)$ so that $|z_n - f_n(z_n)| > \delta$. However, there is a subsequence that converges uniformly on compact subsets of the plane to a 1-quasiconformal map that fixes 0 and 1 and that moves some point by at least δ. However a 1-quasiconformal map is conformal on $\mathbb{C}$, hence of form $az+b$ and since it fixes both 0 and 1, it is the identity and hence doesn't move any points, a contradiction. □

Corollary B.15. *Suppose μ_t is a 1-parameter family of dilatations on $\mathbb{C}$ that move continuously in L^∞, and that F_t are quasiconformal maps with dilatation μ_t that all fix 0 and 1. Then $t \to F_t(z)$ is continuous in t for any fixed $z \in \mathbb{C}$.*

Lemma B.16. *If $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ is conformal and $\varphi : \Omega \to \Omega$ is a quasiconformal map (satisfying the piecewise differentiable definition) that extends continuously to the identity on $\partial\Omega$, then $\Phi = f^{-1} \circ \varphi \circ f$ is a quasiconformal map of the disk to itself that extends to the identity on $\partial\mathbb{D}$.*

Proof. Clearly $\Phi : \mathbb{D} \to \mathbb{D}$ is quasiconformal and hence extends continuously to a homeomorphism of the unit circle (see Theorem B.9). If the extension of Φ to $\partial\mathbb{D}$ is not the identity, then there is an arc $I \subset \mathbb{T}$ such that $I \cap \Phi(I) = \emptyset$. Choose a point $w \in I$ so that f has a finite radial limit at both z and $\Phi(z)$; we can do this because (1) conformal maps have finite radial limits except on a set of zero capacity (Corollary A.26), and (2) sets of zero capacity map to zero capacity under quasiconformal maps (Corollary B.5).

Take the union of the two radial line segments $[0,w]$ and $[0,\Phi(w)]$. Because φ extends as the identity to $\partial\Omega$, the images of these radial segments under f have the same endpoint on $\partial\Omega$ and hence their union is a a closed Jordan curve γ_w. Now, choose a distinct point $z \in I$ with the same properties and form the closed Jordan curve γ_z. Choose z so that the intersection of γ_z with $\partial\Omega$ is different that the intersection of γ_w with $\partial\Omega$; we can do this because only a set of logarithmic capacity zero on the circle can have the same radial limit. Then $\gamma_z \cap \gamma_w = f(0)$ and γ_z hits both sides of γ_w (since z and $\Phi(z)$ are in different components of $\mathbb{T} \setminus ([0,w] \cup [0,\Phi(w)])$. This contradicts the Jordan curve theorem (Theorem 12.9 in [44] or Theorem 13.4 in [51]), and thus Φ must extend to the identity on the boundary of $\mathbb{D}$. □

B.7. **A weak version of measurable Riemann mapping.** Recall that the uniformization theorem states that the only non-compact, simply connected Riemann surfaces are the plane and the disk. Liouville's theorem implies these surfaces are not conformally equivalent. To see they are not even quasiconformally equivalent, consider the path family connecting $\{|z| = \frac{1}{2}\}$ to $\mathbb{T}$ in the disk. It is easy to see this has positive modulus, but it is also an easy computation to show that the path family connecting any compact set in $\mathbb{C}$ to ∞ has zero modulus.

Theorem B.17. *Suppose* Γ *is a triangulation of the plane,* $0 \le k < 1$ *and* $\mu(z)$ *is constant on the interior of each triangle with* $|\mu| < k$*. Then there is a homeomorphism* f *of the plane with* $\mu_f = \mu$*.*

Proof. For each triangle T let A be the affine map with dilatation $\mu(T)$ and T_μ be the image of T under A. Form an Riemann surface by identifying the triangles T_μ along the same edges as in Γ. This defines a Riemann surface that is quasiconformally equivalent to the plane via the map $\Phi : R \to \mathbb{C}$ that is affine on each triangle. By the uniformization theorem, there is also a conformal map $\Psi : R \to \mathbb{C}$. Since R is simply connected and not-compact, it is conformally equivalent to either the disk or the plane and since it quasiconformally equivalent to the plane we know the extremal length of the path family connected an disk to ∞ on R is infinite, and hence it must be conformally equivalent to the plane. Then $\Psi \circ \Phi^{-1} : \mathbb{C} \to \mathbb{C}$ is quasiconformal with dilatation μ. □

Theorem B.18. *For any measurable* μ *on the plane with* $|\mu| \le k < 1$*, there is a quasiconformal map* f *with* $f = \lim_n f_n$ *and* $\mu_n = \mu_{f_n}$ *where* $\{\mu_n\}$ *satisfy the conditions of Theorem B.17 and* $\{f_n\}$ *are the corresponding maps.*

Proof. Take the standard equilateral triangulation of the plane and a series of refinements by recursively subdividing each triangle into four equilateral sub-triangles. Define a piecewise constant dilatation on the nth triangulation by taking the average of μ on each triangle and let $\{f_n\}$ be the corresponding sequence of quasiconformal maps, normalized to fix $0, 1, \infty$. Since these are all quasiconformal with the same bound, they form an equicontinuous family and we can extract a subsequence that converges uniformly on compact subsets of the plane. The limit function f is also K-quasiconformal by Lemma B.6. If μ is continuous on a disk D, then the dilatations μ_n converge uniformly to μ on compact subsets of the plane. □

Theorem B.19. *Suppose* f *is polynomial of degree* ≥ 2*, or a transcendental entire function and that* Ω *is a simply connected wandering domain whose forward orbit contains no critical values of* f*. Let* $\mu_{\mathbf{t}}$ *be the* f *invariant dilatation defined in the proof of Theorem 2.3. There is a family* $\{\Phi_{\mathbf{t}}\}$ *of quasiconformal maps so that*

1. *the dilatation of* $\Phi_{\mathbf{t}}$ *equals* $\mu_{\mathbf{t}}$ *on* Ω*,*
2. $\Phi_{\mathbf{t}}(z)$ *is continuous in* $\mathbf{t}$ *for each* z*,*
3. $\Phi_{\mathbf{t}} \circ f \circ \Phi_{\mathbf{t}}^{-1}$ *is entire.*

Proof. Suppose we are given an f-invariant dilatation μ that is non-zero only on the orbit (backwards and forwards) of a wandering component Ω. By conjugating by a Möbius transformation, we may assume that $\infty \in \Omega$, and that the entire support of μ is contained inside a bounded set. Choose nested, increasing compact sets $\{K_n\}$ inside the grand orbit of Ω, so that the union $E = \cup_n K_n$ is a bounded set containing the set $\{\mu \ne 0\}$. Choose continuous dilatations μ_n so that $\mu_n = \mu$ on K_n, and $\|\mu_n\|_\infty \le \|\mu\|_\infty$. Using our weak version of the measurable Riemann mapping theorem, we can find quasiconformal maps Φ_n with dilatation μ_n. The map $G_n = \Phi_n \circ f \circ \Phi_n^{-1}$ is holomorphic off $\Phi_n(E)$ and at points $z \in \Phi_n(K_n)$ that have a neighborhood whose image under f lies in $\Phi_n(K_{n+1})$. Thus G_n is holomorphic except on the closure of $\Phi_n(K_n) \setminus f^{-1}(\Phi_n(K_n)$.

Every point $z \in E$ satisfies this for large enough n, and hence G_n is a sequence of maps that holomorphic except on a sequence of bounded, nested, decreasing sets with empty intersection, and are locally K-quasiconformal (with a uniform K) except at the countably many critical points of f. The areas of these sets therefore tend to zero and Lemmas B.5 and B.13 imply G_n converges locally uniformly to a locally bounded holomorphic function G off the critical points of f. Since the critical points are isolated, each is a removable singularity, hence G is entire.

If we repeat this argument for each element of a 1-parameter family of f-dilatations μ_t, moving continuously in L^∞, and all with L^∞ norm $\leq k < 1$, we obtain a family of K-quasiconformal mappings Φ_t with $K = (k+1)/(k-1)$, and so that each Φ_t conjugates f to a holomorphic function G_t. If s, t are close, then the dilatations of $(\Phi_t)_n$ and $(\Phi_s)_n$ are close, and hence by Corollary B.15, these maps are close in the supremum norm on a large ball containing E, with an estimate that tends to zero as $|s-t|$ tends to zero and is independent of n. Thus the limiting maps Φ_s, Φ_t are also close on any compact set, so $\Phi_t(z)$ is continuous in t for a fixed z. In particular, for a fixed z, $G_t(z)$ moves continuously as a function of t. □

Appendix C. Topological dimension

C.1. **The definition.** In this appendix we prove that if $n > k$, and $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous, then there is a point y in the image so that $f^{-1}(y)$ contains a non-trivial continuum. The particular map f in this application is

$$\mu(t) \to \{f_{\mu(t)}(z_j)\}_{j=1}^k,$$

where $\mu(t)$ is a finite dimensional family of dilatations that varies continuously in the L^∞ norm, f_μ is the normalized quasiconformal map given by the measurable Riemann mapping theorem, and $\{z_j\}$ are a finite number of distinct points in the plane.

The usual proof of Sullivan's theorem is to use the fact that $f_{\mu(t)}(z_j)$ is a differentiable function of t and hence the desired result about f^{-1} follows from the rank theorem. This requires a significant amount of background on singular integrals and the almost everywhere differentiability of quasiconformal maps. It is somewhat easier to see that $f_{\mu(t)}(z_j)$ is a continuous function of t, and in this case the desired conclusion follows from the following purely topological result.

Theorem C.1. *If $n > k$ and $f : I_n = [0,1]^n \to \mathbb{R}^k$ is continuous, then there is a point $y \in f(I_n) \subset \mathbb{R}^{\subset}\mathbb{R}^k k$ so that $f^{-1}(y)$ contains a compact connected set with more than one point.*

This result is far from obvious and is closely related to Brouwer's "invariance of domain" theorem that states that $\mathbb{R}^n$ and $\mathbb{R}^k$ are not homeomorphic. We shall prove this along the way. The presentation here closely follows the proof in the classic book "Dimension Theory" by Hurewicz and Wallman, [38]; indeed, I will label results with the numbers from that book to make it easier to refer to them there. The book is written about separable metric spaces, although for our purposes, it suffices to consider subsets of Euclidean space and I will make this extra assumption to simplify the discussion.

The (topological) dimension of a set is defined inductively as follows:

(1) The empty set has dimension -1.
(2) A set X has dimension $\leq n$ if every point has arbitrarily small open neighborhoods whose boundaries have dimension $\leq n-1$.
(3) The set X has dimension $= n$ if it has dimension $\leq n$ but does not have dimension $\leq n-1$.

This says that X has dimension $\leq n$ if there is a basis for the topology of X made up of open sets whose boundaries have dimension $\leq n-1$. We shall let $\mathrm{Dim}(X)$ denote the topological dimension of X. In the course of this chapter we shall see that the topological dimension has several equivalent formulations, namely, $\mathrm{Dim}(X) \leq n$ if and only if

(1) X can be written as union of $n+1$ sets of dimension ≤ 0,
(2) any $n+1$ pairs of closed subsets of X can be separated by $(n+1)$ closed subsets that have empty intersection,
(3) Some continuous map of X into the n-cube has a stable value (a value that is attained by every continuous function sufficiently close to f in the supremum norm),
(4) every continuous function from any closed subset of X to the n-sphere can be continuously extended to all of X,
(5) X is homeomorphic to a zero $(n+1)$-measure subset of $\mathbb{R}^{2n+1}$.

In this language, the theorem we want follows immediately from two results in [38]:

Theorem C.2 (Proposition II.4.D)**.** *If X is compact, then X has dimension zero iff it is totally disconnected (i.e., contains no non-trivial connected components).*

Theorem C.3 (Theorem VI.7)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^k$ is a closed mapping (it is continuous and sends closed sets to closed sets), then there is an image point y so that $\mathrm{Dim}(f^{-1}(y)) \geq n-k$.*

The first result is fairly easy, but the second is quite involved and takes up most of this appendix. It interesting to note that the first result can fail if X is not compact. Knaster and Kuratowski [40] constructed a set $X \subset \mathbb{R}^2$ that is totally disconnected, but so that adding a single point $\{a\}$ makes it a connected set $Y = X \cup \{a\}$. Corollary II.3.2 of [38] states that adding a point to a set cannot change its dimension, so $\mathrm{Dim}(X) = \mathrm{Dim}(Y)$. Proposition II.2.D says that any zero dimensional set is totally disconnected, so $\mathrm{Dim}(X) = \mathrm{Dim}(Y) \geq 1$. A much harder result (Theorem IV.3 of [38]) says that a subset of $\mathbb{R}^n$ has dimension n if and only if it contains an open subset, The Knaster-Kuratowski example does not, so X is a totally disconnected set of topological dimension 1.

Suppose $\boldsymbol{C} \subset \mathbb{R}$ is the usual middle thirds Cantor set, let $E \subset \boldsymbol{C}$ be the countable set of endpoints of intervals in $\mathbb{R} \setminus \boldsymbol{C}$ and let $P = \boldsymbol{C} \setminus E$ be the remaining points. Let $a = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$ and for each $x \in \boldsymbol{C}$, let L_x be the line segment connecting x to a. For $x \in E$, let L_x^* be the points on L_x with rational y-coordinates, and for $x \in P$ let it denote the points on L_x with irrational y-coordinate. Then $X = \cup_{x \in \boldsymbol{C}} L_x^*$ is the desired set. See [40] or [57].

The point a is called an explosion point for the set X. This phenomenon is particularly interesting in transcendental dynamics, since similar sets arise naturally there: Mayer has shown that the set of landing points of dynamic rays for $\lambda(z)$ s totally disconnected, but becomes connected when we add $\{\infty\}$ [48], i.e., $\{\infty\}$ is an explosion point.

C.2. **Zero dimensional sets.** We say that two subsets $A_1, A_2 \subset X$ can be separated if there are disjoint open subsets U_1, U_2 that contain A_1 and A_2 respectively. Consider the four following properties:

(1) X is totally disconnected.
(2) Any two distinct points can be separated.
(3) Any point can be separated from any closed set not containing it.
(4) Any two disjoint closed sets can be separated.

For general X, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). We shall see that for compact X all four conditions are equivalent We say that A_1, A_2 are separated by a set $B \subset X$ if the open sets U_1, U_2 can be chosen to be in different connected components of $X \setminus B$.

Lemma C.4 (Definition II.1'). *A space X has dimension zero iff every point can be separated from any disjoint closed set $E \subset X$.*

Proof. If X is zero dimensional and $p \in X$ then p has an open and closed neighborhood U inside the open set $X \setminus E$. Thus p and E are separated by $B = X \setminus (E \cup U)$. The other direction is similar. □

Lemma C.5 (Proposition II.2.E). *If a space X is zero dimensional, then any two closed sets can be separated in X.*

Proof. Suppose K, L are disjoint closed sets in X. Every $p \in X$ has an open-closed neighborhood that is disjoint from either K or L (or maybe both) and a countable union $\{U_j\}$ of these cover X. Let $V_j = U_j \setminus \sum_{k=1}^{j-1} U_k$; this gives a disjoint open cover of X and each V_j is disjoint from either K or L. Taking the unions of V_j's that hit each of these sets gives disjoint open sets separating them. □

Theorem C.6 (Theorem II.2). *If $X = \cup X_j$ is a countable union of closed (in X), zero dimensional subsets, then X is also zero dimensional.*

Proof. It suffices to show that any two closed subsets K, L can be separated (contained in disjoint open sets). Since X_1 is zero dimensional, the sets can be separated in X_1 by Lemma C.5, so X_1 can be divide into two disjoint, closed subsets A_1, B_1 containing $K \cap X_1$ and $L \cap X_1$ respectively. Thus $K \cup A_1$, $L \cup B_1$ are disjoint closed sets in X and hence are contained in disjoint open subsets G_1, H_1 of X that have disjoint closures. Now repeat the argument replacing K and L by $\overline{G_1}$ and $\overline{H_1}$. By induction we obtain nested sequences of open sets so that

$$G_j \subset \overline{G_j} \subset G_{j+1}, \qquad H_j \subset \overline{H_j} \subset H_{j+1}.$$

Then $\cup G_j$, $\cup H_j$ are open, disjoint subsets of X that contain $K \cap X_j$ and $L \cap X_j$ respectively for every j and hence contain K and L respectively. □

Corollary C.7. *A union of two zero dimensional spaces, one of which is closed, is zero dimensional.*

This follows since if A, B are zero dimensional and B is closed, then $X = A \setminus B$ is open in $A \cup B$. But any open set in the separable metric space $A \cup B$ is a countable union of closed sets, and these sets have dimension zero, since they are subsets of A. Thus the corollary follows from the theorem. Since points are closed, we also get:

Corollary C.8. *Adding a point to a zero dimensional set does not increase its dimension.*

Lemma C.9. *Let $\mathcal{R}_n^m$ be the set of points in $\mathbb{R}^n$ that have exactly m rational coordinates. Then $\mathcal{R}_m^n$ has dimension zero.*

Proof. If $n = m$ then $\mathcal{R}_n^m$ is a countable union of points and hence has dimension zero (most small spheres around any point miss a countable set). If $m = 0$, then every point has small neighborhoods that are cubes whose faces have a rational coordinate, and again we get dimension 0.

For $0 < m < n$, fix a choice of m coordinates and fix m rational values and let H be the $k = n - m$ dimensional (in terms of linear algebra) subspace determined by these choices. Then $\mathcal{R}_n^m \cap H$ is a linear image of $\mathcal{R}_k^0$ and hence has dimension 0, and it is a closed subspace of $\mathcal{R}_n^m$ (although not closed in $\mathbb{R}^n$. Thus $\mathcal{R}_n^m$ is a countable union of closed, dimension zero, subspaces of itself, and hence has dimension zero. □

Lemma C.10 (Proposition II.4.B)**.** *Suppose X is compact and dimension zero, $p \in X$ and $K \subset X$ closed. If p can be separated from each point of K, it can be separated from K by open-closed sets.*

Proof. Fore each $q \in K$ there are disjoint neighborhoods U and $V$of p and q. Since K is compact, a finite union of V's cover K and the corresponding intersection of the U's is open and disjoint from the union. □

Lemma C.11 (Proposition II.4.C)**.** *If X is compact and dimension zero, and $p \in X$, then the set $M(p)$ of points that can't be separated from p is connected.*

Proof. Each point not in $M(p)$ has an open neighborhood disjoint from an neighborhood of p, so $X \setminus M(p)$ is open, so $M(p)$ is closed and contains p. If $M(p)$ were disconnected then $M(p) = K \cup L$ where K, L are open-closed in $M(p)$ hence closed in X. We may assume $p \in K$. There exists open U in X so $K \subset U$ and $\overline{U} \cap L = \emptyset$. Then $\partial U \cap M(p) = \emptyset$ (since it hits neither K nor L), and each point of ∂U is separated from p. Since ∂U is closed, Lemma C.10 says ∂U can be separated from p by disjoint open-closed neighborhood V of ∂U and $W = U \setminus V = U \setminus \overline{V}$ of p. But W is disjoint from L, so p is separated from points in L, contrary to the definition of $M(p)$. The contradiction shows $M(p)$ is indeed connected. □

Corollary C.12. *For compact sets X conditions (1)-(4) are equivalent. In particular, compact totally disconnected sets have dimension zero.*

Proof. Assume X is totally disconnected, i.e., no connected subset contains more than one point. Then by Lemma C.11 for each $p \in X$ $M(p)$ is connected, hence equals p. Thus (1) implies (2). Lemma C.10 gives (2) implies (3). The implication (3) implies (4) is Lemma C.5 and opposite directions are all trivial. □

C.3. **Subsets, unions and products.**

Lemma C.13. *A subset Y of a set X of dimension n has dimension $\leq n$.*

Proof. We use induction and note it is trivial for $n = -1$. Suppose $p \in Y \subset X$. By definition, for any $\delta > 0$, there is a neighborhood U of p in X with $U \subset B(p, \delta)$ and $\mathrm{Dim}(\partial U) \leq n - 1$. Let $V = U \cap Y$. Then V is a neighborhood of p in Y and $\partial V \subset \partial U \cap Y$ and this has dimension $\leq n - 1$ by induction. □

Lemma C.14 (Proposition III.2.A)**.** *A subset $Y \subset X$ has dimension $\leq n$, if and only if every point $p \in Y$ has arbitrarily small neighborhoods in X whose boundaries have intersections with Y of dimension $\leq n - 1$.*

Proof. Suppose the condition holds. For any $\delta > 0$ choose a neighborhood $U \subset B(p, \delta)$ of p in X so that $\mathrm{Dim}(\partial U \cap Y) \leq n - 1$. Then $V = U \cap$ has $\partial V \subset \partial U \cap Y$ so also has dimension $\leq n - 1$. This proves $\mathrm{Dim}(Y) \leq n$.

Conversely, suppose $\mathrm{Dim}(Y) \leq n$ and let $p \in Y$. For any δ we can choose a neighborhood $V \subset B(p, , \delta)$ of p and $\mathrm{Dim}(\partial V) \leq n - 1$. Since V and $Y \setminus \overline{V}$ are disjoint open subsets of Y, there is an open set W in X so that $V \subset W \subset B(p, \delta)$ and $\overline{W} \cap (Y \setminus \overline{V}) = \emptyset$. It follows that $\partial W \cap Y \subset \partial V$ and hence $\mathrm{Dim}(\partial W \cap Y) \leq n - 1$. □

Note that $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ writes a 1-dimensional set as a union of two 0-dimensional sets. We will show this is true much more generally.

Lemma C.15. *If $A, B \subset X$, then* $\mathrm{Dim}(A \cup B) \leq 1 + \mathrm{Dim}(A) + \mathrm{Dim}(B)$. *Thus a union of n zero dimensional sets has dimension at most $n - 1$.*

Proof. We use induction on both the dimension of A and B, noting that the cases $(m, -1)$ and $(-1, n)$ are all trivial. Assume it is true for the cases $(m, n - 1)$ and

$(m-1,n)$ and we will deduce it for (m,n); this suffices since we can then fill in the whole quadrant $(m,n), m \geq 0, n \geq 0$.

Suppose $p \in A \cup B$; we may assume $p \in A$. Let U be a neighborhood of p in X. By Lemma C.14 there is a neighborhood $V \subset U$ of p with $\dim(\partial V \cap A) \leq m-1$. Since $\partial V \cap B \subset B$ it also has dimension $\leq n$, so by the induction hypothesis,

$$\mathrm{Dim}(\partial V \cap (A \cup B)) \leq 1 + (m-1) + n = m+n,$$

and this proves $\mathrm{Dim}(A \cup B) \leq m+n+1$ by Lemma C.14. □

Lemma C.16. *Let $\mathcal{M}_n^m$ be the set of points in $\mathbb{R}^n$ that have at most m rational coordinates. Then $\mathcal{M}_m^n$ has dimension $\leq m$.*

Proof. Since $\mathcal{M}_n^m = \cup_{j=0}^m \mathcal{R}_n^j$, it is a union of $m+1$ sets of dimension 0. The result follows from the final conclusion of Lemma C.15. □

Lemma C.17 (Theorem III.2, Sum Theorem)**.** *A countable union of closed sets of dimension n has dimension $\leq n$.*

Before starting the proof of this, note that if we know the theorem for $n-1$, then we can deduce the following lemma.

Lemma C.18 (Δ_n)**.** *Any space X of dimension $\leq n$ is a union of a subset of dimension $\leq n-1$ and a space of dimension ≤ 0.*

Proof. By the definition of dimension, there is a basis of open sets whose boundaries have dimension $\leq n-1$ and since X is separable, this may be taken to be countable, $\{U_k\}$. Assuming Lemma C.17 for $n-1$, $B = \cup \partial U_k$ has dimension $\leq n-1$. It is easy to check that $\mathrm{Dim}(X \setminus B) \leq 0$. The lemma then follows from Lemma C.14. □

By induction we get a fact we will need later.

Corollary C.19 (Theorem III.3)**.** *A space has dimension $\leq n$ iff it can be written as a union of $n+1$ zero dimensional spaces.*

Proof of Lemma C.17. We use induction. The case $n=-1$ is trivial and the case $n=0$ was proven as Theorem C.6. By Lemma C.18 the case $n-1$ implies

We now resume the proof of Lemma C.17. Suppose $X = \cup K_j$ where each K_j is a closed set of dimension $\leq n$. Let $X_1 = K_1$ and

$$X_k = K_k \setminus \bigcup_{j=1}^{k-1} K_j.$$

Then these sets are disjoint, cover X, and have dimension $\leq n$ since $X_k \subset K_k$. Moreover each X_k is a F_σ, i.e., a countable union of closed sets. This holds since $X \setminus \bigcup_{j=1}^{k-1} K_j$ is open and hence F_σ; thus X_k is the intersection of a closed set and a F_σ and hence is F_σ.

By Lemma C.18, $X_k M_k \cup N_k$ where $\mathrm{Dim}(M_k) \leq n-1$ and $\mathrm{Dim}(N_k) \leq 0$. Thus $X = M \cup N = (\cup_k M_k) \cup (\cup_k N_k)$. Note that M_k is F_σ inside M since

$$M_k = M_k \cap X_k = (M_1 \cup \dots) \cap X_k = M \cap X_k,$$

is the intersection of a the F_σ set X_k and M (which is closed in itself). Thus by the induction hypothesis, $\mathrm{Dim}(M) = n-1$ and $\mathrm{Dim}(N) = 0$. Since $X = M \cup N$, we have

$$\mathrm{Dim}(X) \leq 1 + \mathrm{Dim}(M) + \mathrm{Dim}(N) \leq 1 + (n-1) + 0 = n.$$

□

Using the same arguments as with Theorem C.6 we obtain:

Corollary C.20. *The union of two sets of dimension $\leq n$, one of which is closed, has dimension $\leq n$*

Corollary C.21. *Adding a single point to a set does not increase its dimension.*

Lemma C.22 (Proposition II.2.F)**.** *If K, L are disjoint, closed subsets of X and $Y \subset X$ has dimension ≤ 0 then there is a separating set B for K and L so that $B \cap A = \emptyset$.*

Proof. There are open sets U, V with disjoint closures that contain K and L respectively. Since $\overline{U} \cap A$ and $\overline{V} \cap A$ are closed in A, they can be separated in A using Lemma C.5, since $\mathrm{Dim}(A) = 0$. Thus $A = Y \cup Z$ where Y, Z are disjoint open-closed sets in A and $\overline{U} \cap A \subset Y$. Then there is an open set W in X such that $K \cup Y \subset W$ and $\overline{W} \cap (L \cup Z) = \emptyset$. Thus $B = \partial W$ separated K from L and B is disjoint from both Y and Z and hence $B \cap A = \emptyset$. □

Lemma C.23 (Proposition III.5.B)**.** *If K, L are disjoint, closed subsets of X and $Y \subset X$ has dimension $\leq n$ then there is a separating set B for K and L so that $\mathrm{Dim}(B \cap A) \leq n-1$. If we take $A = X$, this says that disjoint, closed sets of a n-dimensional space X can always be separated by a $(n-1)$-dimensional set.*

Proof. We use induction. If $\mathrm{Dim}(A) = -1$, then $A = \emptyset$ and the result is obvious. If $\mathrm{Dim}(A) = 0$ then we proved this in Lemma C.22. Suppose $n > 0$. By Lemma C.18, We can write $A = D \cup E$ as a union of sets of dimension $\leq n-1$ and ≤ 0 respectively. By the case $n = 0$ of the induction, there is a separating set C for K and L that does not intersect E, so $A \cap B \subset D$ has dimension $\leq n-1$. □

Lemma C.24 (Proposition III.5.C)**.** *Suppose X is a set of dimension $\leq n-1$, and suppose $\{C_j, C'_j\}_{j=1}^n$ be n pairs of closed sets so that $C_j \cap C_j = \emptyset$ for $j = 1, \dots n$. Then there are n closed sets $\{B_j\}$ so that B_j separates C_j from C'_j and $\cap_{j=1}^n B_j = \emptyset$.*

Proof. By Lemma C.23 C_1, C'_1 can be separated by a set B_1 of dimension $\leq n-2$. By Lemma C.23 C_2, C'_2 can be separated by a set B_2 so that $\mathrm{Dim}(B_1 \cap B_2) \leq n-3$. Continuing in this way we get separating sets $\{B_k\}$ whose intersection has dimension $n-(n+1) = -1$, i.e., is empty. □

Theorem C.25 (Theorem III.4, Product Theorem)**.** $\mathrm{Dim}(A \times B) \leq \mathrm{Dim}(A) + \mathrm{Dim}(B)$.

Proof. We use induction. The result is trivial if either A or B is empty, i.e., for dimensions pairs $(m, -1)$ or $(-1, n)$, so we may assume it for both $(m, n-1)$ and $(m-1, n)$ and deduce it for (m, n).

Each point of $A \times B$ has a neighborhood of the form $U \times V$ where the boundaries of U and V have dimensions $\leq m-1$ and $\leq n-1$ respectively. Since

$$\partial(U \times V) \subset \overline{U} \times \partial V \cup \partial U \times \overline{V},$$

the induction hypothesis and Theorem C.17 imply

$$\mathrm{Dim}(\partial(U \times V)) \leq (m-1) + (n-1) + 1 = m + n - 1,$$

which proves the result. □

Equality holds in Theorem C.25 if $\mathrm{Dim}(B) = 0$, but not in general.

C.4. **The topological dimension of $\mathbb{R}^n$ is n.** The direction $\mathrm{Dim}(\mathbb{R}^n) \leq n$ is a rather obvious induction since points in $\mathbb{R}^k$ have small neighborhoods whose boundaries are $k-1$-spheres and one can show $\mathrm{Dim}(S_k) = \mathrm{Dim}(\mathbb{R}^k)$ since dimension is unchanged by homeomorphisms and adding a single point. The hard part is to show $\mathrm{Dim}(\mathbb{R}^n) \geq n$. We showed in Lemma C.24 that if X is a set of dimension $\leq n-1$, and $\{C_j, C'_j\}_{j=1}^n$ are n pairs of closed sets so that $C_j \cap C_j = \emptyset$ for $j = 1, \dots n$, then there are n closed sets $\{B_j\}$ so that B_j separates C_j from C'_j and $\cap_{j=1}^n B_j = \emptyset$. We will show that $\mathbb{R}^n$ does not have this property, and hence $\mathrm{Dim}(\mathbb{R}^n) \geq n$. We will make use of the following famous result.

Theorem C.26 (Brouwer's fixed point theorem)**.** *Every continuous map of I_n into itself has a fixed point.*

Proof. This is a "standard" result of algebraic topology: the idea is that given a map $f : I_n \to I_n$ with no fixed points, we can define a continuous retraction r from I_n to its boundary by following the ray from x to $f(x)$ until it hits the boundary at a point $r(x)$. Since the homology group H_{n-1} of I_n is trivial and H_{n-1} of its boundary is not, we get a contradiction. To avoid getting into homology groups, we give another well known proof, using Sperner's lemma in combinatorics.

If suffices to prove Brouwer's theorem for any homeomorphic space; it is convenient to consider the n-simplex

$$S_n = \{x \in \mathbb{R}^{n+1} : \sum_{k=1}^{n+1} x_k = 1, x_k \geq 0 \text{ for all } k\}.$$

Every point of the simplex has at least one non-zero coordinate, and a map f of the simplex into the discrete set $\{1, \dots, n+1\}$ is "proper" if it maps each point x to the index of a non-zero coordinate of x. Note that the kth vertex of the simplex must map to k. Sperner's lemma say that any simplicial subdivision must contain a cell whose $n+1$ vertices map to $n+1$ different values. In fact, there must be an odd number of such cells.

This is proven by induction on n. When $n = 1$, we are simply cutting $[0, 1]$ into finitely many subintervals. Since $f(0) = 0$ and $f(1) = 0$, there must be an odd number of subinterval whose endpoints have different values.

In general, suppose we have a proper map f on the n-simplex with a simplicial subdivision. Let Z be the number of $(n-1)$-faces whose vertices attain the n values $\{1, \dots, n\}$. We will compute Z in two different ways.

Let M be the number of n-cells with the maximal possible number of vertex values, namely $n+1$. These have one face counted by Z. Let N be the number of n-cells that have values in $\{1, \dots, n\}$. These have two faces counted by Z. Thus $Z = M + 2N$.

Let B be the number of $(n-1)$-faces on the boundary of the simplex that have values exactly $\{1, \dots, n\}$ and the I be the corresponding faces in the interior of the simplex. Then $Z = B + I$. Thus $M = Z - 2N = B + 2I - 2N$. By induction, B is odd, and hence M is odd, proving Sperner's lemma.

To deduce Brouwer's theorem, consider a sequence of simplicial subdivisions S_k with the cell sizes tending to zero. Assume f is a continuous selfmap of the n-simplex with no fixed points. For each vertex x in S_k assign a value $j = f(x)$ so that $f(x)_j < x_j$; there is such a value if $f(x) \neq x$. This implies $x_j > 0$, so the map is proper. By Sperner's lemma there is a n-cell with $n+1$ distinct labels. Taking the limit, we see there is a point y in the simplex with $f(y)_j \leq y_j$ for $j = 1, \dots n+1$. Since $\sum_j f(x)_j = \sum_j x_j = 1$ we have $f(y) = y$, so a fixed point exists. □

Lemma C.27 (Proposition IV.1.D). *Let $X = I_n = [-1, 1]^n \subset \mathbb{R}^n$ and let $\{C_j^-, C_j^+\}$ be the two components of $I_n \cap \{x = (x_1, \dots, x_n) : x_j = \pm 1\}$ (i.e., pairs of opposite faces of the cube). If $\{B_j\}$ are closed subsets of I_n so that B_j separates C_j and C_j', then $\cap_j B_j \neq \emptyset$. In particular, we must have* $\mathrm{Dim}(I_n) \geq n$.

Proof. To see how to deduce the lemma from Brouwer's theorem, let U_j^-, U_j^+ be open subsets of distinct components of $I_n \setminus B_j$ and define $v : I_n \to \mathbb{R}^n$ by setting the jth component to be

$$v(x) = \pm \mathrm{dist}(x, B_j),$$

with sign being chosen > 0 on the component of $I_n \setminus B_j$ containing U_j^+ and < 0 on the component containing U_j^- (and arbitrarily on any other components). Then $f(x) = x + v(x)$ is continuous and maps I_n into itself, because if x is not in U_j^+, then

$$1 - x_j = \mathrm{dist}(x, C_j^+) \geq \mathrm{dist}(x, B_j) = v_j(x),$$

so adding $v(x)$ to x can't make the jth coordinate larger than 1. Thus by Brouwer's theorem f has a fixed point y and so $v(y) = 0$, which means $\mathrm{dist}(y, B_j) = 0$ for $j = 1, \dots, n$, Since each B_j is closed, this means $y \in \cap_j B_j$, so the latter set in non-empty, as claimed. □

Now that we know $\mathrm{Dim}(\mathbb{R}^n) = n$, we have proven the following fundamental result.

Theorem C.28 (Invariance of domain)**.** *If $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.*

Thus in the setting of our goal, Theorem C.3, we now know that there must be an image point so that $f^{-1}(y)$ has more than one point. We need to improve this to a preimage of dimension ≥ 1.

C.5. **Embedding in $\mathbb{R}^{2n+1}$.** A cover of a set X is a collection of open sets whose unions contains X. We say it uses diameter δ if every set in the collection has diameter $\leq \delta$. The cover has order n is at most $n+1$ elements can contain a common point.

Lemma C.29. *If X is compact and* $\mathrm{Dim}(X) = 0$ *then X has a cover of order 0 using diameter $\leq \delta$ (i.e., a pairwise disjoint cover by small elements).*

Proof. By definition, each point has a neighborhood of diameter $\leq \delta$ and empty boundary (hence the set is both open and closed in X) and since X is compact, a finite number of these cover X. Replacing each open set by itself with the other removed gives a pairwise disjoint cover with even smaller diameters. □

Lemma C.30 (Corollary to Theorem V.1)**.** *If X is compact and* $\mathrm{Dim}(X) \leq n$*, then X has open covers using arbitrarily small diameters and order $\leq n$.*

Proof. By Theorem C.19 X is the union of $n+1$ dimension zero sets $X_1, \dots, X_{n+1}$, and each of these can be covered by collection of disjoint open sets using diameters $\leq \delta$. We claim the union of these $n+1$ collections has order $n+1$; if $n+2$ of the sets all contained the point p then by the pigeon hole principle, two come from the same collection and they can't both contain p since they are disjoint. □

Lemma C.31 (Theorem V.2)**.** *If X is compact and* $\mathrm{Dim}(X) \leq n$ *then the set of homeomorphisms from X into I_{2n+1} is a dense G_δ in the set of all continuous maps $X \to I_{2n+1}$. (the latter set is non-empty since it contains the constant maps.)*

Proof. We say g is an ϵ-mapping if $\mathrm{diam}(g^{-1}(y)) < \epsilon$ for every y (the empty set has diameter 0). It is easy to check that if X is compact and a continuous map g on X is an ϵ-mapping for every $\epsilon > 0$ then g is a homeomorphism. Similarly, compactness implies the ϵ-mappings form an open set: if h is close enough to g in the supremum norm and g is an ϵ-mapping, then so is h. We leave this as an exercise for the reader.

So by Baire's theorem suffices to show that ϵ-maps are dense. Given any continuous $f : X \to I_{2n+1}$ we must approximate it to within $\eta > 0$ in the supremum norm by an ϵ-map g. Since continuous on a compact set implies uniformly continuous, we may choose $\delta > 0$ so that $|x-y| < \delta$ implies $|f(x) - f(y)| \leq \eta/2$. Let $\{U_j\}$ be a cover of X of order n using diameters $\leq \delta$ and for each U_j choose a point $p_j \in I_{2n+2}$ so that $\mathrm{dist}(p_j, f(U_j)) \leq \eta/2$ and the p_j's are in general position, i.e., if we take two disjoint sets of the p_j's each with $\leq n+1$ points then the convex hulls in I_{2n+1} do not intersect.

For $x \in X$ let $w_j = \mathrm{dist}(x, X \setminus U_j)$ and define

$$g(x) = \frac{\sum w_j(x) p_j}{\sum w_j(x)}.$$

This is well defined and continuous since $w_j(x) > 0$ holds for at least one j for each x. Moreover, g approximates f at x since at most $n+1$ terms in the sum are non-zero, corresponding to the at most $n+1$ elements of the cover containing x. Since these all have diameter $\le \delta$, the values of f at these points differs from $f(x)$ by at most $\eta/2$ and hence the same is true for any weighted average.

Finally, associate to each $x \in X$ the linear space spanned by the points p_j where $w_j(x) > 0$. If $g(x) = g(y)$ then the convex hulls of the points p_j corresponding to x and y overlap, so the set of points themselves overlap by our general position condition. Thus x and y are in a common U_j and hence within $\delta < \epsilon$ of each other, as desired. □

C.6. **Stable values.** If $f : X \to Y$ is continuous and $y \in f(X)$, we call y a stable value of f if $y \in g(X)$ for every continuous $g : X \to Y$ that is sufficiently close to f in the supremum metric. Otherwise, we can make arbitrarily small perturbations of f that omit the value y. In this case, y is called an unstable value of f. For example, a constant map $f : [0,1] \to [0,1]$ has no stable values, whereas any non-constant continuous map from $[0,1]$ into itself has a stable value by the intermediate value theorem.

Lemma C.32 (Theorem VI.1)**.** *If X has dimension $< n$ and $f : X \to I_n$ is continuous, then f has no stable values.*

Proof. No value in ∂I_n can be stable since we can approximate f by $(1-\delta)f$. If y is an interior point we may apply a homeomorphism of I_n that maps y to 0 and so assume $y = 0$. Fix a small $\delta > 0$ and let $C_j^+ = \{x : f_j(x) \ge \delta\}$ where f_j is the jth coordinate of f, and similarly define $C_j^- = \{x : f_j(x) \le -\delta\}$. By Lemma C.24 there are separating sets B_j for these pairs so that $\cap B_j = \emptyset$. Define $g_j = f_j$ on $C_j^+ \cup C_j^-$, $g_j = 0$ on B_j and

$$g_j(x) = \frac{\mathrm{dist}(x, B_j)}{\mathrm{dist}(x, C_j^+) + \mathrm{dist}(x, B_j)},$$

on $U_j^+ \setminus C_j^+$ where U_j^+ is the component of $I_n \setminus B_j$ that contains C_j^+. Define g_j on $U_j^- \setminus C_j^-$ analogously. Then g is continuous, approximates f and never take the value 0, since the B_j's contain no common point. Thus 0 is an unstable value. □

Lemma C.33 (Remark VI.A)**.** *Suppose $f : X \to I_n$ and $y \in I_n \setminus f(X)$. Then for every $\delta < 0$ there is a map $g : X \to I_n$ so that $|f(x) - g(x)| < \delta$ for all $x \in X$ and $y \in I_n \setminus \overline{g(X)}$.*

Proof. If $y \in \partial I_n$ we can take $g = (1-\delta)f$. Otherwise let $d = \min(\delta, \text{dist}(y, \partial I_n))$, let B the ball of radius d around y and let φ be the identity outside B and the radial projection onto ∂B inside $B \setminus y$. then $g = \varphi \circ f$ is the desired map. □

We can now state and prove the converse of Lemma C.32. Note that this gives a characterization of n-dimensional sets in terms of the existence of stable values: X has dimension n if and only if there is a continuous map $f : X \to I_n$ that has a stable value.

Lemma C.34 (Theorem VI.2)**.** *If $X \subset I_n$ has dimension n, then there exists a continuous map $f : X \to I_n$ with a stable value.*

Proof. We prove the contrapositive: if no function from X to I_n has a stable value, then X is homeomorphic to a subset of $\mathcal{M}^{n-1}_{2n+1}$ (points with at most $n-1$ rational coordinates), and hence has dimension $\leq n-1$ by Lemma C.16.

Consider a continuous map $f : X \to I_{2n+1}$. Choose n coordinates $i_1, \dots i_n$ of $\mathbb{R}^{2n+1}$ and let $M \subset \mathbb{R}^{2n+1}$ be the affine space where these coordinates have fixed values $c_1, \dots c_n$. We claim the set of maps $g : X \to \mathbb{R}^{2n+1}$ can approximate any continuous $f : X \to \mathbb{R}^{2n+1}$. Given such an f, if we follow it by the orthogonal projection onto the chosen n coordinates, the resulting map has no stable values by assumption. In particular, $\mathbf{c} = (c_1, \dots c_n)$ is not a stable value, and (using Lemma C.33) we can approximate the composition by a map h so that $\mathbf{c} \notin \overline{h(X)}$. Replacing f in this coordinates by h gives the desired approximation g. Moreover, any small perturbation of g also approximates f and avoids $\mathbf{c}$, so the set of maps avoiding M is open and dense.

Now consider all possible M's taking all possible combinations of n coordinates (finitely many) and all possible values $\mathbf{c} \in \mathbb{Q}^n$ (countably many). For each of the countably many possible choices, the set of maps avoiding M is open and dense, so by Baire's theorem there is a dense set of maps that avoid every such M. By Lemma C.34 there is also a dense G_δ of homeomorphisms $X \to I_{2n+1}$ and applying Baire's theorem again gives a homeomorphism sending X into $\mathcal{M}^{n-1}_{2n+1}$, as desired. □

Lemma C.35 (Proposition VI.1.B)**.** *Suppose $f : X \to I_n$ is continuous and y is an interior point of I_n that is unstable. Fix $\delta > 0$. Then f can be approximated within δ by a map g that omits the value y but agrees with f whenever it takes values more than δ away from y. Thus stability is a local property.*

Proof. Without loss of generality we assume y is the origin and $U = B(0, \delta)$. Since y is unstable there is a $h : X \to I_n$ that approximates f and omits the value y. Define $g = h$ when $|f(x)| \leq \delta/2$, $g = f$ when $|f(x)| \geq \delta$ and

$$g(x) = (1 - \phi(t))h(x) + \phi(t)f(x),$$

otherwise where $t = |f(x)|$ and ϕ increases linearly from 0 to 1 as t goes from $\delta/2$ to δ. It is easy to check g has the desired properties. □

C.7. **Continuous extensions.** The following is standard, e.g., see Theorem 4.16 of [30] or Theorem 3.2 of [51].

Theorem C.36 (Tietze Extension Theorem)**.** *If K is a closed subset if a space X and $f : K \to I_1$ is continuous, then f can be extended to a continuous $F : X \to I_1$.*

Clearly I_1 can be replaced by I_n by extending the coordinates separately. Also since S_n is homeomorphic to ∂I_{n+1}, the Tietze theorem implies that a map $f : K \to S_n$ can be extended to an open neighborhood of K, by replacing S_n by ∂I_{n+1}, extending to I_{n+1}, restricting to the open subset where F avoids the origin and composing by radial projection back onto ∂I_{n+1}.

Lemma C.37 (Theorem VI.4)**.** *X has dimension $\leq n$ if and only if for each closed set $K \subset X$ and each continuous mapping $f : K \to S_n$, f has a continuous extension $X \to S_n$.*

Proof. **Sufficiency:** By Lemma C.34, it is enough to show that that no continuous mapping $f : X \to I_{n+1}$ has stable values. A stable value can't occur on the boundary of I_n, so assume there is a stable interior value y, and let U be a small ball around y. Let $K = f^{-1}(\partial U)$. This set is closed and by assumption there is a map $F : X \to S_n$ that extends $f : K \to \partial U = S_n$. Define g by setting $g = f$ on $f^{-1}(U)$ and $g = F$ otherwise. Then g approximates f uniformly and never equals y, so y is not a stable value of f.

Necessity: Suppose X has dimension $\leq n$, $K \subset X$ is closed and $f : K \to S_n$ is continuous. With loss of generality we may assume f maps into ∂I_{n+1} instead. By the Tietze extension theorem, f can be extended to a map $F : X \to I_{n+1}$. Lemma C.32 implies F has no stable values, so in particular, origin in not stable, so we can approximate F by a map G that never vanishes and agrees with F for values on ∂I_{n+1}. Hence G can be composed with radial projection to give a continuous map onto ∂I_{n+1} that extends f. □

Lemma C.38 (Corollary to Theorem VI.4)**.** *Suppose K is a closed subset of X. If $\mathrm{Dim}(X \setminus K) \leq n$, then every continuous map $f : K \to S_n$ has a continuous extension to X.*

Proof. Suppose $f : K \to S_n$ is continuous. Using Tietze's extension theorem as before, f has a continuous extension F to an open neighborhood U of K. Choose an open V with $K \subset V \subset \overline{V} \subset U$ and note that the restriction maps $\overline{V} \setminus K$ to S_n. Thus by the necessity part of Lemma C.37, this map can be extended to a continuous map G of $X \setminus K \to S_n$ and since this agrees with F on V, setting $G = f$ on K gives a continuous extension of f to all of X. □

We say $f, g : X \to S_n$ are homotopic if there is a continuous map $h : X \times [0, 1] \to S_n$ so that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

Theorem C.39 (Borsuk's theorem, Theorem VI.5)**.** *Suppose K is closed subset of X and $f, g : K \to S_n$ are homotopic. If there is an extension of f to X, then then there is an extension of g and the extensions are homotopic.*

Proof. We follow the proof in [38], which , in turn, follows a proof due to Dowker. We are assuming there is a map $F : X \to S_n$ that equals f on K. Let K' be the closed set $(X \times \{0\}) \cup (K \times [0, 1]) \subset X \times [0, 1]$. Extend F by setting $F : K' \to S_n$ by $F(x, 0) = F(x)$ and $F(x, t) = h(x, t)$. The Tietze's extension theorem implies F can be extended to some open neighborhood U of K'; the extension is still denoted F. There is an open set V in X so that $V \times [0, 1] \subset U$, so F makes sense on this cylinder set.

The closed sets K and $X \setminus V$ are disjoint, so there is a continuous $p : X \to [0, 1]$ that is 1 on K and 0 off V. Therefore $G(x, t) = F(x, tp(x))$ is continuous and defined on all of $X \times [0, 1]$. Clearly for $x \in K$ we have $G(x, 1) = F(x, p(x)) = h(x, 1) = g(x)$ and $G(x, 0) = F(x, 0) = f(x)$. Thus $G(x) = G(x, 1)$ is an extension of g that is homotopic to F, the extension of f. □

Lemma C.40 (Proposition VI.3.B)**.** *Suppose $f, g : X \to S_n$ are continuous and disagree on a set Y of dimension $\leq n - 1$. Then f and g are homotopic.*

Proof. Y is open. Define a closed set $Z \subset X \times [0, 1]$ by

$$Z = (X \times \{0\}) \cup (X \times \{1\} \cup (X \setminus Y) \times (0, 1).$$

Define the homotopy F by $F(x, t) = f(x) = g(x)$ for $x \notin Y$, $F(x, 0) = f(x)$, $F(x, 1) = g(x)$. The complement of Z is the product $Y \times (0, 1)$ which has dimension $\leq n$ by the product theorem Theorem C.25. By Lemma C.38 we can extend F to all of $X \times [0, 1]$, proving f and g are homotopic. □

Lemma C.41 (Proposition VI.3.C)**.** *Suppose X is the union of two closed subspaces K, L and $F : K \to S_n$ an $G : L \to S_n$ are continuous and they agree on $K \cap L$ except possibly on a set of dimension $\leq n - 1$. Then F can be extended to all of X.*

Proof. The mappings are homotopic by Lemma C.40, so it follows from Borsuk's theorem that F extends from $K \cap L$ to a function H on L. Taking F on K and H on L gives the extension to X. □

Lemma C.42 (Proposition VI.3.F)**.** *Suppose $K \subset X$ is closed and $\{V_\lambda\}$ is a collection of open sets that cover X and whose boundaries all have dimension $\leq n - 1$. If $f : K \to S_n$ can be extended continuously to each V_λ, then it can be extended continuously to all of X.*

Proof. Since X is separable we may assume $\{V_\lambda\}$ is a countable collection $\{V_j\}$. Assume we have already extended f to F_k on $X_k = K \cup \overline{V_1} \cup \cdots \cup \overline{V_k}$ and set $Y_k = (K \cup \overline{V_{k+1}}) \setminus (V_1 \cup \cdots \cup V_k)$. By hypothesis f has a continuous extension to both these sets and these extensions can only disagree in

$$(Y_k \cap Z_k) \setminus K \subset \bigcup_{j=1}^{k} \partial V_k$$

which has dimension $\leq n-1$ by Lemma C.17. Hence we may apply Lemma C.41. □

C.8. **Preimages with large dimension.**

Lemma C.43 (Proposition VI.3.G)**.** *Suppose a set X is a union of sets K_λ of dimension $\leq m$ and each K_λ has the property that any open neighborhood U of K_λ contains an open neighborhood V of K_λ whose boundary has dimension $\leq m-1$. Then X has dimension $\leq m$.*

Proof. Suppose K is compact and $f : K \to S_n$ is continuous. By Lemma C.38, f can be extended to $K \cup K_\lambda$ and hence to an open neighborhood U_λ of $K \cup K_\lambda$ (by the Tietze extension theorem each coordinate function can be extended to some f_j since it is real-valued and then we restrict to an open neighborhood where $\sum f_j^2 > 0$).

By hypothesis, each U_λ contains a sub-neighborhood V_λ whose boundary has dimension $\leq m-1$ and hence f extends to $K \cup \overline{V_\lambda} \subset U_\lambda$. By Lemma C.42, f extends to all of X, and by Lemma C.37 this proves Lemma C.43 □

Theorem C.3 clearly follows from

Lemma C.44. *Suppose X has dimension n, Y has dimension k and $f : X \to Y$ has the property that* $\mathrm{Dim}(f^{-1}(y)) \leq m$ *for all $y \in Y$. Then $n \leq k+m$.*

Proof. To prove this, we use induction on k, keeping m fixed. If $k = -1$, the set Y is empty and the result is trivially true. Next we assume the result for $k-1$ and deduce it for k.

Consider the family of all preimages $\{K_y\} = \{f^{-1}(y)\}$ for $y \in f(X)$. This is a decomposition of X into disjoint compact sets of dimension $\leq m$. We claim that these sets satisfy the hypotheses of Lemma C.43. To see this, take any neighborhood U of K_y and let $C = f(I_n \setminus U)$. There is a ball V around y that is disjoint from C and has boundary of dimension $k-1$. Then $f^{-1}(V)$ is an open neighborhood of K_y inside U and its boundary has dimension $\leq k-1+m$ by induction on k. Thus Lemma C.43 can be applied to deduce Lemma C.44. □

Appendix D. Entire functions

D.1. **The Speiser class and Eremenko-Lyubich class.** Suppose f is a transcendental entire function. The singular set, $S(f)$, is the closure of the union of all the critical values of f and a the finite asymptotic values (limits of f along curves tending to infinity). If $S(f)$ is finite, we say f is in the Speiser class, denoted $\mathcal{S}$. If $S(f)$ is bounded, we say f is in the Eremenko-Lyubich class, denoted $\mathcal{B}$. In this section we prove that any Fatou component of an Eremenko-Lyubich class function is simply connected, the transcendental analog of Lemma 2.1.

Lemma D.1. *If Ω is multiply connected Fatou component of an entire function f, and $\gamma \subset \Omega$ surrounds a point of the Julia set, then $f^n(\gamma)$ has positive index with respect to 0 for all sufficiently large n.*

Proof. Suppose the index is zero for an infinite subsequence of γ, and tends to ∞ uniformly on γ. By the argument principle, $f^n(\gamma)$ having index zero implies implies f^n has no zeros inside γ. Thus the infimum of $|f^n|$ over the interior of γ is attained on γ. Hence f^n tends to ∞ inside γ. But γ surrounds a point of the Julia set and hence it surrounds a pre-periodic point (Theorem A.9), which necessarily has a bounded orbit. This gives a contradiction. □

Corollary D.2. *If f is a transcendental entire function that is bounded along a curve σ tending to ∞, then all Fatou components are simply connected.*

Proof. If U is a multiply connected component, then by Lemma D.1, it contains a curve γ whose iterates $f^n(\gamma)$ intersect σ for all sufficiently large n. This contradicts the assumption that f is bounded on σ since $f(f^n(U)\cap\sigma) \subset f^{n+1}(U)$ is as far from the origin as we wish. Thus f can't have any multiply connected Fatou components. □

Lemma D.3. *Suppose f is entire and U contain no critical values. Then f is a smooth covering map from $V = f^{-1}(U)$ to U.*

Proof. If $z \in V$ then $f(p) \neq S(f)$, so $f'(z)$ exists and is non-zero. Thus a small enough disk around z maps homeomorphically to a neighborhood of $f(z)$. □

The map is called a regular covering map if given any $y \in Y$ and any $x \in X$ such that $f(x) = y$, then any arc in Y starting at y can be lifted to an arc in X starting at x. It is a standard result (e.g., Theorem 14C of [4]) that any two liftings of the same arc with the same initial point must agree, but the existence of a lifting is not always true. The monodromy theorem say that if two arcs in Y have the same endpoints and are homotopic by a homotopy that keeps the endpoints fixed, then any lifts of these arcs that have the same initial point, must also have the same terminal point. This is proved by noting that the homotopy lifts to a homotopy whose terminal point must always lie in $f^{-1}(b)$; since this is a discrete set, any continuous motion within it must be constant.

Lemma D.4. *Suppose f is entire and U contain no singular points. Then f is a regular covering map from $V = f^{-1}(U)$*

Proof. From the previous lemma we know f is smooth covering map on V. Choose points $z \in V$, $w \in U$ such that $f(z) = w$ and let $D = D(w, \epsilon)$ be so small that $D \cap S(f) = \emptyset$. Define a branch g of f^{-1} so that $g(w) = z$ and extend it along a radius of D as far as possible. Because f is a smooth covering map, this extension is possible along some maximal open interval $[0, t)$. If $t < \epsilon$, consider the lifted arc corresponding to this radial segment. We claim it leaves every compact set, for if it stayed within some compact set then we could take a sequence of points on the lifted path that converged to a point that, by continuity of f, must map to $w + te^{i\theta}$. This contradicts the maximality of t. Thus the lift leaves every compact set, but f has a limit along the lift, showing f has an asymptotic value in D, a contradiction. Thus g can be defined on all of D. Thus the connected component W of $f^{-1}(D)$ containing z is mapped onto D by f. If two points of this component map to the same point of D, then an arc connecting these points maps to a closed loop in D. Since D is simply connected, this loop is homotopic to constant path, hence the arc in W must have been constant, hence the two points were actually a single point. Thus f is a bijection from W to D. This proves that f is a regular covering map over U. □

Corollary D.5. *Suppose f is entire and $S(f) \subset \mathbb{D}_R = \{z : |a| < R\}$. Let $\mathbb{D}_R^* = \{z : |z| > R\}$. Then f is covering map from $\Omega = f^{-1}(\mathbb{D}_R^*) = \{z : |f(z)| > R\}$ to $\mathbb{D}_R^*$. Each connected component of Ω (called a tract of f) is an unbounded, simply connected domain whose boundary is an analytic Jordan curve that tends to ∞ in both directions.*

Lemma D.6. *If $f \in \mathcal{B}$, then every component of $\mathcal{F}(f)$ is simply connected.*

Proof. Suppose $f \in \mathcal{B}$ and choose $R > 0$ so that $S(f) \subset \mathbb{D}_R$. Let $\Omega = f^{-1}(\mathbb{D}_R^*)$. By Lemma D.4, f is a regular covering map from each component of Ω to $\mathbb{D}_R^*$. Since $\mathbb{D}_R^*$ is unbounded, each component of Ω is unbounded, but $|f| = R$ on the boundary. Thus Corollary D.2 applies. □

D.2. **Multiply connected Fatou components wander.**

Theorem D.7 (Baker [7])**.** *If f is a transcendental entire function, then every multiply connected component of the Fatou set is bounded.*

Proof. Suppose not, i.e., suppose Ω is an unbounded multiply connected Fatou component and let $\gamma \subset \Omega$ is a closed curve surrounding a point of the Julia set. Then by Lemma D.1 the iterates of $\gamma_n = f^n(\gamma)$ hit Ω (and hence are contained in Ω for all large enough n. Thus Ω is forwards invariant.

Choose a compact, connected set $K \subset \Omega$ that contains both γ and $f(\gamma)$ and choose a domain V so that $K \subset V \subset \overline{V} \subset \Omega$. Since $|f^n| \to \infty$ uniformly on $\overline{V}$, $\log|f^n|$ is a sequence of well defined, positive harmonic functions on V and so by Harnack's inequality there is a constant $C = C(K)$ so that

$$\log|f^n(w)| \leq C \log|f^n(z)|,$$

for all $z, w \in K$, independent of n. Thus

$$|f^n(w)| \leq |f^n(z)|^C.$$

Since $\gamma_{n-1} \cup \gamma_n \supset f^{n-1}(K)$, we have

$$\sup_{\gamma_n} |f(z)| \leq \inf_{\gamma_{n-1}} |f(z)|^C = \inf_{\gamma_n} |z|^C.$$

In particular, $|f(z)| \leq |z|^C$ for every $z \in \gamma_n$. Since the curves $\{\gamma_n\}$ eventually surround every point and we can easily deduce f is a polynomial. This contradiction proves the theorem. $\square$

Corollary D.8. *The Julia set of transcendental entire function can't be a Cantor set. Thus it always contains a non-trivial connected component.*

For a survey of Baker's many contributions to transcendental dynamics, see [53]. As noted in Section 2, the following is due to Matthew Herring. It was proved independently by Andreas Bolsch, who also showed that in the unbounded case, at most one point it omitted.

Lemma D.9. *If Ω is a bounded Fatou component of f, then $f(\Omega)$ is contained in a bounded Fatou component and equals the whole component. The map is a branched covering.*

Proof. Suppose W is the Fatou component containing $f(\Omega)$. Since Ω is bounded, $f(\Omega) \subset W$ is bounded. Suppose $f(\Omega) \neq W$. Then there is $w \in W \cap \partial f(\Omega)$, and hence there are points $\{z_k\} \subset \Omega$ so that $f(z_k) \to w$. Since Ω is bounded, we can pass to a subsequence so that $z_k \to z \in \overline{\Omega}$. If $z \in \partial\Omega \subset \mathcal{J}(f)$, then $w = f(z) \in \mathcal{J}(f)$, a contradiction. If $z \in \Omega$, then $w = f(z) \in f(\Omega)$, also a contradiction. Therefore $f(\Omega) = W$ $\square$

Corollary D.10. *If f is a transcendental entire function then every multiply connected component of the Fatou set is a wandering domain.*

Proof. We already know that multiply connected components are bounded and iterate to infinity uniformly on compact sets, so they can't be periodic. If they were preperiodic they would have to land on a periodic domain where every point iterates to infinity (such a Fatou component is called a Baker domain). However, such a domain must be unbounded, whereas $f(U)$ must be bounded, contradicting Lemma D.9. Thus there are no pre-periodic, multiply connected Fatou components. $\square$

D.3. **Baker's example.** Lemma D.10 suggests how to build an entire function with a wandering domain: build a function with a multiply connected Fatou component. Here we give such an example due, again, to Baker.

Theorem D.11 (Baker)**.** *There exists an entire function with a multiply connected Fatou component, hence with a wandering domain.*

Proof. The function will be

$$f(z) = z^2 \prod_{k=1}^{\infty} (1 + \frac{z}{R_k}),$$

where $R_k \nearrow \infty$ is a sequence of positive real numbers that we define inductively. Suppose $R_0 > 0$ is large and and set $f_0(z) = F_0(z) = z^2$. In general, let $R_n =$

$\max_{|z|=R_{n-1}} |f_{n-1}(z)|$ and let

$$F_n(z) = (1 + \frac{z}{R_n}).$$

Set

$$f_n(z) = \prod_{k=0}^{n} F_k(z), \qquad \text{and} \qquad f(z) = \lim_{n\to\infty} f_n(z) = z^2 \prod_{k\in S}(1 + \frac{z}{R_k}).$$

The first step is to check that the product defining f converges and for this we need to know that $R_k \nearrow \infty$ fast enough. However, each F_k (and hence each f_k) takes its maximum modulus on $\{|z| = r\}$ where this circle intersects $(0, \infty)$, so

$$R_n \;=\; \max_{|z|=R_{n-1}} |f_{n-1}(z)| \geq {R_{n-1}}^2 \prod_{k\in S, k<n} (1 + \frac{R_{n-1}}{R_k}) \geq {R_{n-1}}^2,$$

since every term in the product is ≥ 1. Thus $R_n \geq R_0^{2^n}$ and, more generally, $R_n \geq R_k^{2^{n-k}}$ for $1 \leq k \leq n$. From this it easily follows that the product defining f converges uniformly on compact sets.

Next, for $n \in \mathbb{N}$, define the annulus

$$A_n = \{z : \frac{1}{4} R_n \leq |z| \leq 4R_n\},$$

and let B_n be the annulus separating A_n and A_{n+1}, i.e.,

$$B_n = \{z : 4R_n < |z| \leq \frac{1}{4} R_{n+1}\}.$$

We claim that $f(B_n) \subset B_{n+1}$. If this is true, then the iterates of B_n clearly converge uniformly to ∞, so that $B_n \subset \mathcal{F}(f)$. On the other hand, if $n \in \mathbb{N}$, then A_n contains a zero of f and 0 is a super-attracting fixed point of f. Thus A_n contains a Fatou component that does not iterate to ∞ and hence must contain some point of the Julia set (in fact a continuum of such points). Thus B_n surrounds a point of $\mathcal{J}(f)$ and the Fatou component containing it must be multiply connected.

Thus we must prove $f(B_n) \subset B_{n+1}$. The idea is that A_n is bounded by two circles and that after applying f these two circles are further apart; enough so that the region between them contains A_{n+1}. We break the product for f into three pieces

$$f(z) \;=\; (z^2 \prod_{k\in S, k<n} (1 + \frac{z}{R_k})) \cdot F_n(z) \cdot (\prod_{k\in S, k>n} (1 + \frac{z}{R_k})) \tag{D.1}$$

$$\;=\; I(z) \cdot II(z) \cdot III(z) \tag{D.2}$$

For $z \in A_n$, the third term is bounded between

$$\prod_{k\in S, k>n} (1 - \frac{R_n}{R_k})) \leq III \leq \prod_{k\in S, k>n} (1 + \frac{R_n}{R_k}))$$

Now use the estimate $R_k \geq R_n^{2^{k-n}}$ for $k > n$,

$$\prod_{k \in S, k>n} (1 - {R_n}^{1-2^{n-k}}) \leq III \leq \prod_{k \in S, k>n} (1 + {R_n}^{1-2^{k-n}})$$

$$1 - O(R_n^{-1}) \leq III \leq 1 + O(R_n^{-1})$$

and this gives

$$\frac{9}{10} \leq III \leq \frac{10}{9},$$

if R_0 is large enough.

The second term in (D.1) satisfies

$$|II(z)| \leq 3, \quad |z| = 2R_n,$$

$$|II(z)| \geq \frac{1}{2}, \quad |z| = R_n/2,$$

and

$$|II(z)| \leq 2, \quad |z| = R_n.$$

Define $C_n = \prod_{k<n} R_k^{-1}$. Then the first term in (D.1) satisfies

$$\begin{aligned} z^2 \prod_{k<n} (1 + \frac{z}{R_k}) &= z^2 \prod_{k<n} \frac{z}{R_k}(1 + \frac{R_k}{z}) \\ &= C_n z^{2+n} \prod_{k<n} (1 + \frac{R_k}{z}) \\ &= C_n z^{2+n} \prod_{k<n} (1 + O(\frac{R_k}{R_n}) \\ &= C_n z^{2+n}(1 + O({R_n}^{-1/2})). \end{aligned}$$

Thus if R_0 is large enough,

$$I = (1 + o(1)) C_n z^{2+n}.$$

Thus we can deduce that

$$R_{n+1} = (1 + o(1)) 2 C_n R_n^{2+n}$$

$$|f(z)| \leq 2(1 + o(1)) 2^{-2-n} < \frac{1}{4} R_{n+1}, \quad |z| = R_n/2,$$

$$|f(z)| \geq 2(1 + o(1)) 2^{2+n} > 4R_{n+1}, \quad |z| = 2R_n.$$

Thus the two boundaries of B_n both land inside B_{n+1}, and since f has no zeros in B_n (they all lie in the A_n's) the minimum and maximum principles imply $f(B_n) \subset B_{n+1}$. $\square$

It is not immediately clear whether the wandering domains constructed above are finitely or infinitely connected, but it is easy to make a small change which forces infinite connectivity. With the same inductive definition of $\{R_n\}$, place the zeros slightly outside the circles of radius R_n, i.e.,

$$f(z) = \prod_{k \in S} (1 - \frac{z}{3R_k}).$$

Everything goes through as above to show that $f(B_n) \subset B_{n+1}$ and hence f has multiply connected wandering domains, but now we also can show that for $z \in \gamma$, $\gamma = \{z : |z + 3R_n| = R_n\}$,

$$|f(z)| \geq 4R_{n+1},$$

Hence γ iterates into B_{n+1}, so is in the Fatou set. Moreover, γ is clearly not homotopic to $\{|z| = 4R_n\}$ in the Fatou set since $\mathbb{D}_{R_n}$ contains points of the Julia set. Thus the Fatou component containing B_n always has connectivity at least 3. By a result of Kisaka and Shishikura [39], the eventual connectivity must be 2 or ∞, so in this case the wandering component has infinite connectivity.

D.4. **Other examples.**

Theorem D.12 (Herman)**.** *$f(z) = z - 1 + e^{-z} + 2\pi i$ has a wandering domain.*

Proof. This was published by Baker [9] with credit to Herman. See also the survey by Dierk Schleicher, [55]. The map $N(z) = z - 1 + e^{-z}$ is the Newton's method map for $g(z) = e^z - 1$. The basin of attraction for $z = 0$ is invariant under N and the basins for $z = 2\pi i n$ are each translates of this basin (and are disjoint since they iterate to different points). Note that $N(z + 2\pi i) = N(z) + 2\pi i$ so the Julia and Fatou sets of N are $2\pi i$ periodic. Since $f(z) = N(z) + 2\pi i = N(z + 2\pi i)$, if z is a repelling fixed point of N of period k and multiplier λ, then $f^{nk}(z) = z + 2\pi i n k$ and

$$D_E^S f^{nk}(z) \geq \frac{|\lambda|^n}{1 + (|z| + |2\pi n k|)^2} \to \infty,$$

so $\{f^n\}$ is not normal at z by Marty's theorem (e.g. see Ahlfors' book [2]):

Theorem D.13 (Marty's theorem)**.** *A family $\mathcal{F}$ of meromorphic functions on a hyperbolic planar domain Ω is normal iff*

$$\sup_{f \in \mathcal{F}} \sup_{z \in K} D_E^S f(z) < \infty,$$

for every compact $K \subset \Omega$. Here $D_E^S f(z)$ denotes the norm of the gradient of f from the Euclidean metric on the domain to the spherical metric on the image.

As noted earlier, it is known that the repelling fixed points of N are dense in $\mathcal{J}(N)$, so we can deduce $\mathcal{J}(N) \subset \mathcal{J}(f)$. On the other hand, since f preserves the Fatou components of N, it is normal on these components, hence $\mathcal{F}(N) \subset \mathcal{F}(f)$. Thus equality holds. Thus each basin for $2\pi i n$ moves by up by $2\pi i$ under f and hence are wandering domains. □

The following example appears in Baker's paper [9], but may have been known earlier.

Theorem D.14. *$f(z) = z + \sin z + 2\pi$ has a bounded, simply connected wandering domain.*

Proof. For $g(z) = z + \sin z$, all points $(2n+1)\pi$ are super-attracting fixed points, hence in different Fatou components. Since $g(z+2\pi) = g(z) + 2\pi$, the Julia set is 2π-periodic and arguing as in the previous proof, $\mathcal{J}(f) = \mathcal{J}(g)$. Thus f maps the g-basin for $(2n+1)\pi$ to the g-basin for $(2n+3)\pi$ and so these are wandering domains for f. All the critical points of g are super-attracting fixed points, so their basins of attraction are simply connected (otherwise they are in the escaping set, but a fixed point of f can't escape).

To see that these components are bounded, note that the imaginary axis is preserved by g, as are its translates by $2\pi\mathbb{Z}$ and that all points on these lines iterate to ∞, except for those on the real line. Thus these vertical lines cannot be in the basins of attraction of $\{(2n+1)\pi\}$, so these basins are separated by these lines. We claim these basins are bounded. Suppose Ω_n is the basin of attraction of $(2n+1)\pi$. We know it is trapped between the vertical lines $L_0 = \{x = 2n\pi\}$ and $L_1 = \{x = (2n+2)\pi\}$. Suppose Ω intersects the horizontal segment $S = \{2n\pi < x < (2n+2)\pi, y = \frac{\pi}{2}\}$ and let γ be the shortest hyperbolic curve connecting the fixed point $(2n+1)\pi$ to S. Suppose the endpoint on S is $x + iy$ $(y = \pi/2)$. By the Schwarz lemma, $f(\gamma)$ has at most the hyperbolic length of γ. Since $y = \pi/2$,

$$\mathrm{Im}(g(x+iy)) = y + \mathrm{Im}(\sin(x+iy)) = y + \frac{1}{2}(e^x \sin(y) - e^{-x}\sin(-y)) = y + \frac{1}{2}(e^x + e^{-x}) > y,$$

so $g(\gamma)$ connects the fixed point to a point above S. By the Schwarz lemma the hyperbolic length of $g(\gamma)$ is less or equal the hyperbolic length of γ. Thus a subset of $g(\gamma)$ connects the fixed point to S and has strictly shorter hyperbolic length than γ, a contradiction. Thus the attracting basins do not intersect the lines $|y| = \pi/2$, and hence the basins are bounded sets. □

In [26] Eremenko and Lyubich use Runge's approximation theorem to build entire functions with wandering domains. They build one example where every orbit $f^n(z)$ for z in the wandering component has an infinite accumulation set, and another in for which the maps f^n are univalent on the wandering component. In a third example, every wandering orbit tends to ∞, and f^n are all univalent on the wandering component (the latter does not occur for multiply connected wandering components).

The approximation methods used by Eremenko and Lyubich do not give good control of the singular values of the constructed function, so it remained open whether an Eremenko-Lyubich function could have a wandering component (their paper [27] shows such a component cannot be escaping). In [22], the author used quasiconformal methods to construct such an example, and several variations were given by Kirill Lazebnik [42], Yanhua Zhang and Gaofei Zhang [64], Núria Fagella, Xavier Jarque, and Lazebnik [28]. A simpler method of obtaining examples with bounded singular set is given in [46] by David Martí-Pete and Mitsuhiro Shishikura. Recent (and as yet unpublished) work of Anna Miriam Benini, Vasiliki Evdoridou, Núria Fagella, Phil Rippon, and Gwyneth Stallard describes the internal dynamics of simply connected wandering domains. See [13]. A detailed description of the internal dynamics and geometry of multiply connected wandering domains is given in [19] by Bergweiler, Rippon and Stallard.

Very recently, Luka Boc Thaler [59] has shown that every bounded connected regular open set, whose closure has a connected complement, is a wandering domain of some entire function. In particular, every Jordan curve is the boundary of a wandering Fatou component of some entire function. Even more exotic examples have been produced Martí-Pete, Lasse Rempe and James Waterman [45]. They show that a wandering Fatou components can form "lakes of Wada", i.e., three or more simply connected regions that all have a common boundary.

Finally, we should at least mention that holomorphic polynomials in several variables can have wandering domains: see the construction in [5] by Matthieu Astorg, Xavier Buff, Romain Dujardin, Han Peters and Jasmin Raissy, based on an idea of Misha Lyubich.

D.5. **No wandering in the Speiser class.** In this section we sketch the proof that functions in the Speiser class do not have wandering domains. The proof follows the case of polynomials. Theorem A.9 stated that the Julia set of a transcendental entire function is contained in the closure of the pre-periodic points, and that the Fatou components of a Speiser class function are simply connected (we proved this for the larger Eremenko-Lyubich class). The only non-trivial new step is to prove that the collection of entire functions with a given finite singular set is finite dimensional. This is due to Eremenko and Lyubich in [27] and independently to Goldberg and Keen in [33].

By Lemma D.6, Ω is simply connected. Let M_g denote the collection of all entire functions f that are topologically equivalent to g. To entire functions are called quasiconformally equivalent if there are quasiconformal maps φ, ψ of the plane to itself so that $\psi \circ f = g \circ \varphi$. Eremenko and Lyubich proved that for $g \in \mathcal{S}$, the collection M_g of all f that are quasiconformally equivalent to g form a finite dimensional, complex analytic manifold. We shall just prove a part of this, showing that M_g is finite dimensional in the following sense.

Lemma D.15. *If $f, g \in \mathcal{S}$ have the same singular values then there is an $\epsilon > 0$ so that the following holds. If*

$$\psi \circ g = f \circ \varphi,$$

where ψ, φ are $(1+\epsilon)$-quasiconformal, then $g(z) = f(az+b)$ for some $a, b \in \mathbb{C}, a \neq 0$.

Proof. The proof is essentially an exercise about covering spaces, and we will need the following lifting lemma that is Theorem 14.3 of Munkres' book [51]:

Theorem D.16 (The general lifting lemma)**.** *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $F : Y \to E$ such that $F(y_0) = e_0$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

Here π_1 denotes the fundamental group and f_* is the map between fundamental groups induced by the continuous map f.

In our application, we let $X = \mathbb{C} \setminus S(f) = \mathbb{C} \setminus S(g)$ and let $Y_f = \mathbb{C} \setminus f^{-1}(S(f)), Y_g = \mathbb{C} \setminus g^{-1}(S(g))$. Choose some point $z_0 \in Y_g$. One can prove that $f : Y_f \to X$ and $g : Y_g \to X$ are covering maps. Since $S(g)$ is a finite set, there is a positive lower bound $\delta > 0$ for the distance between any two points in $S(g)$. Since $S(g)$ is bounded, there is an $\epsilon > 0$ so that any $(1+\epsilon)$-quasiconformal map fixing $0, 1, \infty$ moves each point of $S(f)$ by less than $\delta/10$. Thus if φ is $(1+\epsilon)$-quasiconformal, it is isotopic to the identity via a path of quasiconformal maps that fix each point of $S(g)$. Thus for any closed loop γ in Y_g, the image loop $g(\gamma) = \psi^{-1} \circ f \circ \varphi(\gamma)$ is homotopic to $f \circ \varphi(\gamma)$. Thus $g_*(\pi_1(Y_g, z_0)) \subset f_*(\pi_1(Y_f, \varphi(z_0)))$. In fact, we have equality, since $\pi_1(Y_f)$ is isometric to $\pi_1(Y_g)$ via the homeomorphism φ. By the general lifting lemma we get a homeomorphism $h : F : Y_g \to Y_f$ and this map is locally a composition of g and a branch of f^{-1} and hence is holomorphic. Thus it must be conformal linear, i.e., $h(z) = az + b, a \neq 0$, as claimed. □

This completes the proof that there are no wandering domains for Speiser class functions. Some other classes with no wandering domains are functions of the form:
- $f(z) = z + r(z)e^{p(z)}$,
- f has finite order and $f'(z) = r(z)e^{p(z)}(f(z) - z)$,
- f so that $f'(z) = r(z)(f(z) - z)^2$ or $f'(z) = r(z)(f(z) - z)(f(z) - c)$,

all where r is rational, p is a polynomial and $c \in \mathbb{C}$. These are described given in Bergweiler's survey [14]. Bergweiler's paper [15] shows non-existence of wandering domains for certain maps arising from Newton's method. Another collection of entire functions without wandering domains (based on the behavior of singular orbits) is given in [49] by Helena Mihaljević-Brandt and Lasse Rempe. Suppose f is an Eremenko-Lyubich function for which the singular values escape to ∞ uniformly, let A be a forward invariant closed set in the plane containing the singular set so that all the connected components of A are unbounded and suppose there exist $\epsilon > 0$ and $0 < c < 1$ so that when $z \in A$ s sufficiently large then $\text{dist}(f(\{w : |w - z| < c|z|\})S(f)) > \epsilon$. Then f has no wandering domain. Similarly if f is Eremenko-Lyubich class and $S(f) \cup f(\mathbb{R}) \subset \mathbb{R}$, and there are constants r, K such that $|f'(x)| \cdot |x| \leq K|f(x)| \cdot \log|f(x)|$ for $|x| > r$ and $|f(x)| > r$, then f has no wandering domain. For example, this result applies to $\frac{\lambda}{z} \sinh z + a$ when $a, \lambda \in \mathbb{R}$ and $\lambda \neq 0$. A conjecture from [49] is disproven by Lazebnik [43] who constructs an Eremenko-Lyubich functions with wandering domains even though each singular value escapes to ∞. The question of whether wandering domains can occur if the singular values escape *uniformly* to ∞ remains open (as of this writing).

Bergweiler, Haruta, Kriete, Meier, and Terglane [18] give another criteria for f not to have wandering domains: they show that, if Ω is a wandering domain of f, then all the limit functions of iterates of f on Ω are contained in the set of limit points of $P(f)$, the the orbit of the singular set (plus the point ∞).

Barański, Fagella, Jarque and Karpińska [11] prove that if $\{\Omega_n\}$ is the orbit of a wandering domain Ω then for for every $z \in \Omega$, there is a sequence $p_n \in P(f)$ such that $\text{dist}(p_n, \Omega_n) = o(\text{dist}(f^n(z), \partial\Omega_n)$. If we assume the map f is topologically hyperbolic, i.e., $\text{dist}(P(f), \mathcal{J}(f)$, and if $\Omega_n \cap P(f) = \emptyset$ for all n, then for every compact set $K \subset \Omega$ and every $r > 0$, we have $\{w : |w - f^n(z)| < r\} \subset \Omega_n$ for every $z \in K$ and all sufficiently large n. In particular, $\text{diam}(\Omega_n) \to \infty$ and $\text{dist}(f^n(z), \partial\Omega_n) \to \infty$. This is used to prove the non-existence of wandering domains for Newton maps of entire functions of the form $ae^z + bz + c$ for some values of a, b and c.

In [52], Nicks, Rippon, and Stallard investigates Baker's conjecture: the Fatou components of a transcendental entire function with order of growth $< 1/2$ must be bounded. Among other interesting results, the authors prove that if f is a real (i.e., $f(\mathbb{R}) \subset \mathbb{R}$) transcendental entire function of order less than 1 with only real zeros, then f has no unbounded wandering domains. Bounded multiply connected wandering domains of this type can be created by modifying the construction in Section D.3. In general, simply connected wandering domains can be either bounded or unbounded open sets.

Is there an entire function with a wandering domain whose orbit is bounded? Here we mean either a bounded wandering domain whose forward orbit lies within a bounded set, or an unbounded wandering component for which the forward orbit of each point is bounded. Since a wandering domain (if any existed) of a polynomial would have to have a bounded orbit in order to avoid the attracting basin at infinity, proving there is no transcendental example is, perhaps, the most natural extension of Sullivan's theorem to entire functions. As of this writing, it remains one of the most important open problems in transcendental dynamics.

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