THE ORDER CONJECTURE FAILS IN $\mathcal{S}$

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Abstract. We construct an entire function f with only three singular values whose order can change under a quasiconformal equivalence.

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1. Introduction

If f is entire, let $S(f)$ denote the singular set of f, that is, the closure of its critical values and finite asymptotic values. The class of entire functions for which $S(f)$ is a finite set was denoted $\mathcal{S}$ by Eremenko and Lyubich [6] in honor of Andreas Speiser. We let $\mathcal{S}_n$ denote the set of entire functions with exactly n singular values. The Speiser class is a subclass of the Eremenko-Lyubich class $\mathcal{B}$, consisting of those entire functions whose singular values are a bounded set in $\mathbb{C}$. The two classes are also called "finite-type" and "bounded-type" in holomorphic dynamics.

A natural measure of the growth of an entire function is its order:

$$\rho(f) = \limsup_{z\to\infty} \frac{\log_+ \log_+ |f(z)|}{\log_+ |z|},$$

where $\log_+ r = \max(0, \log r)$. The natural parameter spaces of entire functions (at least for dynamical considerations) are the quasiconformally equivalent functions: we say f, F are equivalent if there are quasiconformal maps ϕ, ψ of the plane so that

$$\psi \circ f = F \circ \phi.$$

Eremenko and Lyubich [6] proved that for $f \in \mathcal{S}_n$, the collection of functions equivalent to f forms a $n+2$ complex dimensional manifold, and it is natural to ask if the order is constant on each such manifold. This is true for $n = 2$ (e.g. Proposition 2.2 of [5] shows ψ, ϕ can be chosen affine in this case), but we will show:

Theorem 1.1. *There are equivalent functions in $\mathcal{S}_3$ with different orders.*

More generally, we say that f and F are topologically equivalent if $\psi \circ f = F \circ \phi$ for some pair of homeomorphisms ψ, ϕ of $\mathbb{R}^2$ to itself. The order conjecture asks if $\rho(f) = \rho(F)$ whenever f and F are topologically equivalent. For $f \in \mathcal{S}$, topological equivalence is the same as quasiconformal equivalence (e.g., Proposition 2.2 of [5]), but in general the two notions can differ. For meromorphic functions, the order can be defined using the Nevanlinna characteristic, and in this setting Kunzi [11] constructed a meromorphic function with finite singular set whose order can change under a topological equivalence. Mori's theorem (e.g., Theorem III.C in [1]) implies that a K-quasiconformal equivalence can change the order by at most a factor of K. Also, functions in $\mathcal{B}$ always have order $\geq 1/2$ ([2], [12], [13]).

The order conjecture was formulated by Adam Epstein (e.g., [4], [7]) on the basis of several examples where $\rho(f)$ can be computed from combinatorial data associated to f. For example, if f''/f' is polynomial of degree d then $\rho(f) = d$ and the order is the same for any entire function topologically equivalent to f. Another example comes from dynamics: if p is a polynomial of degree d with a repelling fixed point at 0 with multiplier λ then there is an entire function f (called the Poincaré function) so that $f(0) = 0, f'(0) = 1$ and $f(\lambda z) = p(f(z))$. This function has order $\rho(f) = 1/\log_d|\lambda|$ and the singular set of f is the closure of the critical orbits of p. Thus if p is post-critically finite, $f \in \mathcal{S}$. In [5], Epstein and Rempe prove that the order conjecture holds for such f. If the post-critical set of p is bounded (which is the same as saying its Julia set is connected), then $f \in \mathcal{B}$. By taking two quasiconformally conjugate polynomials with repelling fixed points at 0, but with multipliers of different absolute values, they show the order conjecture fails in $\mathcal{B}$.

We say a function $f \in \mathcal{B}$ has the area property if

$$\iint_{f^{-1}(K)\setminus\mathbb{D}} \frac{1}{|z|^2} dxdy < \infty,$$

whenever K is compact subset of $\mathbb{C} \setminus S(f)$. The area conjecture asks if every function in $\mathcal{B}$ has this property. This question was first raised by Eremenko and Lyubich [6] in the special case when f has no finite asymptotic values. Epstein and Rempe prove in [5] that the Poincaré functions associated to polynomials with Siegel disks do not have the area property; this gives counterexamples in $\mathcal{B}$. If $f \in \mathcal{S}$ has the area property then it also satisfies the order conjecture (e.g., Theorem 1.4 of [5]). Thus Theorem 1.1 gives a counterexample to the area conjecture in $\mathcal{S}$. Even stronger examples are given in [3], e.g., a function $f \in \mathcal{S}$ with $S(f) = \{-1, 0, 1\}$ and so that $\{z : |f(z)| > \epsilon\}$ has finite Lebesgue area for any $\epsilon > 0$.

During a visit to Stony Brook in April of 2011, Alex Eremenko asked me a question about the geometry of polynomials with only two critical values. This led to further discussions of the classes $\mathcal{B}$ and $\mathcal{S}$ with him and Lasse Rempe, and the current paper is among the consequences. I thank both of them for their lucid explanations of known results and generously sharing their ideas about open problems. I am grateful to David Drasin for his careful reading of an earlier draft of this paper. His comments prompted a re-write of several sections and improved the exposition and mathematics throughout the paper. Also thanks to the referee for mathematical corrections and suggestions to improve the exposition.

We use the notation $\mathbb{D}$ for the unit disk, $\mathbb{C}$ for the complex numbers, $\mathbb{R}$ for the real numbers and $\mathbb{H}_r$ for the right half-plane. When two quantities x, y depend on a common parameter, $x \lesssim y$ means that x is bounded by a multiple of y, independent of the parameter. We sometimes use the equivalent notation $x = O(y)$. If $x \lesssim y$ and $y \lesssim x$, we write $x \simeq y$. We let $|E|$ denote the diameter of a set E.

2. The basic idea

We first describe how to construct entire functions with exactly two critical values. At the end of the section we modify this idea to give a function with three critical values, and this is the type of function we will use to disprove the order conjecture.

Let $U = \mathbb{C} \setminus [-1, 1]$ and recall that the map

$$z \to \cosh(z) = \frac{e^z + e^{-z}}{2},$$

acts as a covering map from $\mathbb{H}_r$ to U, with the half-strip

$$S = \{x + iy : x > 0, |y| < \pi\}$$

being a fundamental domain. The role of $\cosh(z)$ in this paper is analogous to the role of $\exp(z)$ in [6] where Eremenko and Lyubich consider functions whose critical values are contained in a disk and $\exp(z)$ provided the covering map for the complement of this disk. We will also use the conformal map of S to $\mathbb{H}_r$ given by

$$z \to i\pi \sinh(z/2).$$

This is symmetric with respect to $\mathbb{R}$ and fixes the boundary points $-i\pi, 0, i\pi$.

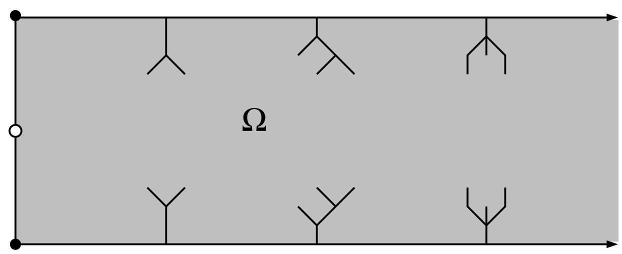
The idea of this paper is to build an entire function starting from a simply connected subdomain $\Omega \subset S$. We will assume that Ω is symmetric with respect to the real line and that it is obtained from S by removing finite trees rooted along the top and bottom sides of S. The edges of the trees will be line segments. See Figure 1.

The vertices of $\partial\Omega$ form a locally finite set that includes all vertices of the removed trees (including the points where the trees are attached to the sides of S), but may include other points on the tree edges or on the sides of S. We assume the origin is a vertex. Note that $\partial\Omega$ is an infinite tree and hence is bipartite, i.e., we can label the vertices with $+1$ and -1 so that no two adjacent vertices having the same label. Let V_+, V_- denote the vertices labeled $+1$ and -1 respectively.

Since cosh is 1-to-1 on S, it is a conformal map from Ω to $\Omega' = \cosh(\Omega)$. Let $\tau : \mathbb{H}_r \to \Omega$ be a conformal map that is also symmetric and fixes 0. Then

$$f(z) = \cosh(\tau^{-1}(\cosh^{-1}(z))), \tag{2.1}$$

is holomorphic from Ω' to U. The region Ω' is dense in the plane, so if we knew that f extended continuously to $\partial\Omega'$, we could deduce it is entire (since the boundary of Ω' is a union of smooth arcs, it is removable for continuous, holomorphic functions, e.g., [9], [10]).



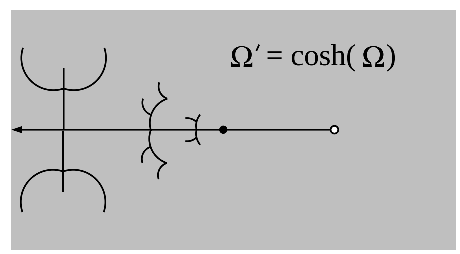


FIGURE 1. Ω is obtained by removing finite, straight edge trees from the half-strip S. $\Omega' = \cosh(\Omega)$ is dense in the plane. Our function f is holomorphic on Ω', but is it continuous across the boundary?

However, f is very unlikely to be continuous across $\partial\Omega'$. Suppose $z \in \partial\Omega$ is an interior point of one of the trees we have removed from S. Then τ will map at least two different points of $\partial\mathbb{H}_r$ to z. This will cause a discontinuity for f unless the final cosh in (2.1) maps all τ-preimages of z to the same point in $[-1, 1]$. In other words, for f to extend continuously to the whole plane, we need

$$\begin{aligned} \tau(ix) = \tau(iy) \quad &\Rightarrow \quad \cosh(ix) = \cosh(iy) \\ &\Leftrightarrow \quad \cos(x) = \cos(y) \\ &\Leftrightarrow \quad \mathrm{dist}(x, 2\pi\mathbb{Z}) = \mathrm{dist}(y, 2\pi\mathbb{Z}) \end{aligned}$$

This will not happen in general, but our goal is to construct examples where is does happen by replacing the conformal map τ by a quasiconformal map $\psi : \mathbb{H}_r \to \Omega$ with the essential property

$$\psi(ix) = \psi(iy) \quad \Rightarrow \quad \cos(x) = \cos(y). \tag{2.2}$$

When this holds we say ψ "correctly identifies" points. In this case, the function

$$g(z) = \cosh(\psi^{-1}(\cosh^{-1}(z)))$$

is quasiregular on Ω' and extends continuously across $\partial\Omega'$, and hence is quasiregular on the whole plane. The measurable Riemann mapping theorem then implies there is a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ so that $f = g \circ \phi$ is entire.

Where are the critical values of f? The function g is locally 1-to-1 on Ω' and ϕ is 1-to-1 everywhere, so $f = g \circ \phi$ has no critical points on Ω'. Thus all its critical points are in $\partial\Omega'$ and hence all critical values lie in $[-1, 1] = f(\partial\Omega')$. The only critical values due to cosh are ± 1; any others must correspond to critical points of ψ^{-1} on $\partial\Omega$ and these can occur only at vertices of $\partial\Omega$. Therefore we also assume

$$\psi(ix) \in V_+ \Leftrightarrow x \in 2\pi\mathbb{Z}, \qquad \psi(ix) \in V_- \Leftrightarrow x \in \pi + 2\pi\mathbb{Z} \tag{2.3}$$

If both and (2.2) and (2.3) hold, then f is entire and only has critical values ± 1. Moreover, (2.2) can be reduced to a much easier condition to check. Let $\mathcal{Z}$ be the partition of $i\mathbb{R} = \partial\mathbb{H}_r$ into segments with endpoints $\pi i\mathbb{Z}$.

Lemma 2.1. *If (2.3) holds and ψ is linear on each segment in $\mathcal{Z}$, then (2.2) holds.*

Proof. Suppose $v = \psi(ix) = \psi(iy)$ and $x \neq y$. If v is a vertex then by (2.3) either both x and y are in $2\pi\mathbb{Z}$ or both are in $\pi + 2\pi\mathbb{Z}$. In either case, (2.2) holds. If v is in interior point of an edge $e = [z, w] \subset \partial\Omega$, say $v = tz + (1-t)w$. If the two ψ-preimages of e containing ix and iy are $[ia, ib]$ and $[ic, id]$ respectively, with $\psi(ia) = \psi(ic) = z$, then $x = ta + (1-t)b$, $y = yc + (1-t)d$. Since a and c are either both in $2\pi\mathbb{Z}$ or both in $\pi + 2\pi\mathbb{Z}$, the distances of x and y to $2\pi\mathbb{Z}$ are the same, which implies (2.2). □

Our counterexample to the order conjecture will have three critical values instead of two, but the basic idea is the same as above. The only difference is that we will build the domain Ω so that its boundary vertices can be 3-colored with the labels $\{-1, 0, 1\}$ and so that the map

$$z \to \cosh(\psi^{-1}(z)),$$

is well defined at each vertex of $\partial\Omega$ and sends each vertex to the value of its label (each edge of $\partial\Omega$ is mapped to segment of length $\pi/2$ on $\partial\mathbb{H}_r$). To obtain such a ψ, not any 3-coloring will do; there are two conditions that must be met. First, as we traverse the boundary $\partial\Omega$, the labels must occur in the order

$$\dots, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0 \dots,$$

which is the same order we encounter them when traversing the boundary of $U = \mathbb{C} \setminus [-1, 1]$. Second, no leaf of the tree (a vertex of degree 1) can have label 0. Together, these conditions are necessary and sufficient. Then as before,

$$g(z) = \cosh(\psi^{-1}(\cosh^{-1}(z)))$$

extends to be quasiregular on the plane and the measurable Riemann mapping theorem gives a quasiconformal ϕ so that $f = g \circ \phi$ is entire with critical values $\{-1, 0, 1\}$. Note that the critical points with critical value 0 must correspond to vertices of $\partial\Omega$ with label 0 and degree ≥ 3.

3. Exponential partitions

We just described how our construction of an entire function f depends on the construction of a domain $\Omega \subset S$ and quasiconformal map $\psi : \mathbb{H}_r \to \Omega$ that correctly identifies points. The map ψ will be written as a composition $\psi = \psi_1 \circ \psi_0$ where $\psi_0 : \mathbb{H}_r \to S$ is quasiconformal and piecewise linear on the boundary, and ψ_1 will be piecewise linear from S to Ω. The map ψ_1 is the "interesting" part and contains the essential geometry; ψ_0 simply approximates the conformal map from $\mathbb{H}_r$ to S. In this section we describe ψ_0.

Since ψ_1 will be constructed to be piecewise linear, there is a partition of ∂S into a collection of segments $\mathcal{I}$ so that ψ_1 is linear on each $I \in \mathcal{I}$ (these will be the preimages under ψ_1 of the edges of $\partial\Omega$). The elements of $\mathcal{I}$ are taken to be closed line segments that cover ∂S and are pairwise disjoint except for endpoints. In addition, we will assume $\mathcal{I}$ satisfies

(1) $\mathcal{I}$ is symmetric with respect to the real line (i.e., the top and bottom edges of S are partitioned in exactly the same way).
(2) There is a $s > 0$ and a constant $C < \infty$ so that for all $I \in \mathcal{I}$,

$$|I| \simeq \exp(-s\,\text{dist}(I, I_0)).$$

We will call $\mathcal{I}$ an exponential partition of ∂S if both these conditions hold.

Assume the elements of $\mathcal{I}$ on the top edge of S are denoted $I_1, I_2, \dots$ in left to right order. The elements on the bottom edge are similarly denoted $I_{-1}, I_{-2}, \dots$.

Lemma 3.1. *If $\mathcal{I}$ is an exponential partition, then $|I_m| \simeq \frac{1}{m}$ for all $m \geq 1$.*

Proof. For $k \geq 1$, let $N_k > 0$ be the index of a partition element that contains $k + i\pi$ and let $n_k = N_{k+1} - N_k$ (this is approximately the number of elements that hit $[k + i\pi, k + 1 + i\pi]$). By assumption, $n_k \simeq \exp(sk)$ and hence (since a geometric sum is dominated by its last term) $N_k \simeq n_k \simeq \exp(sk)$. So for an integer $N_k \leq m \leq N_{k+1}$,

$$|I_m| \simeq \exp(-sk) \simeq \frac{1}{n_k} \simeq \frac{1}{N_k} \simeq \frac{1}{m}.$$

□

Recall that $\mathcal{Z}$ denotes the partition of $\partial\mathbb{H}_r = i\mathbb{R}$ with endpoints $i\pi\mathbb{Z}$. For $k \geq 1$, $Z_k = [(k-1)\pi, k\pi] \in \mathcal{Z}$ and $Z_{-k} = -Z_k$ (there is no interval labeled Z_0).

Lemma 3.2. *Suppose $\mathcal{I}$ is an exponential partition. Then there is a quasiconformal map $\psi_0 : \mathbb{H}_r \to S$ so that $\mathcal{I} = \psi_0(\mathcal{Z})$ and ψ is linear on each segment in $\mathcal{Z}$.*

Proof. Partition $\mathbb{H}_r$ as follows. Let W_0 be the region bounded by the vertical segment $[-2\pi i, 2\pi i]$ and an arc of the circle $|z| = 2\pi$. For $k = 1, 2, \dots$, let

$$W_k = \mathbb{H}_r \cap \{z : \pi k \leq |z| \leq \pi(k+1)\},$$

(this is a half-annulus). Partition S into rectangles by letting S_k, $k = 0, 1, \dots$ be the points in S that project vertically onto $I_{k+1} \in \mathcal{I}$. Then it is easy to check that we can map W_0 to S_0 by a map that is linear on each interval $Z_{-2}, Z_{-1}, Z_1, Z_2 \subset \partial W_0$ and that is linear from arclength on the circular side of W_0 to the right-hand side of S_0. See Figure 2.

Each half-annulus W_k, $k \geq 1$ can be quasiconformally mapped to S_k by a map that is linear on each component of $\partial W_k \cap \partial\mathbb{H}_r$ and is linear in the argument on the two circular sides. It is important to note that the quasiconstant can be chosen independent of k, but this is easy to see since both S_k and W_k are generalized quadrilaterals with modulus approximately k. □

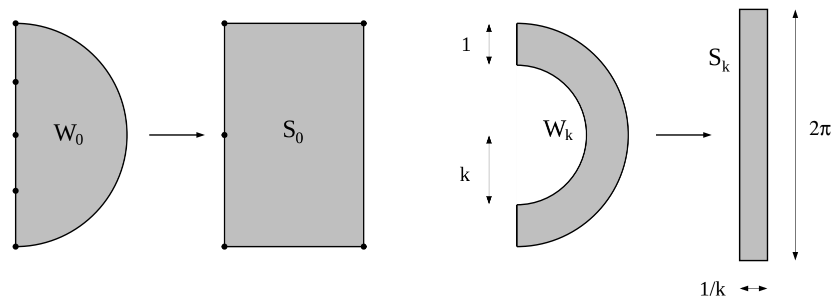


FIGURE 2. Define ψ_0 to be linear on boundary edges and uniformly quasiconformal.

4. A simple example

The main step in our construction is building the map $\psi_1 : S \to \Omega$. This map will be built using piecewise affine maps between combinatorially equivalent infinite triangulations of S and Ω. Although the triangulations are infinite, only a finite number of different shapes are used (up to Euclidean similarity), and so the quasiconstant is bounded by the maximum over a finite set of affine maps. In this section we will build such a map ψ_1 in a simple case, in order to introduce the basic idea before attacking the more complicated construction in the next section.

Triangulations of two polygonal domains Ω_1, Ω_2 are compatible if we have a 1-to-1 correspondence between the triangulations that preserves interior adjacencies (i.e., if two triangles share in edge in Ω_1 then the corresponding triangles share an edge in Ω_2). See Figure 3. We can then define a piecewise affine map from Ω_1 to Ω_2 by using the unique affine map between corresponding triangles that respects adjacency. If triangles share an edge on the boundary of Ω_2 the corresponding triangles in Ω_1 don't have to be adjacent. This will happen when Ω_1 has a Jordan boundary, but Ω_2 has slits.

Now for the example. Consider Figure 4. It shows the half-strip S and a subdomain $\Omega \subset S$ formed by removing certain vertical slits from S. Both S and Ω are partitioned into pieces that are numbered $1, 2, 3, \dots$ moving left to right. The pieces of S are all squares. The pieces of Ω have two shapes: rectangles and pieces that look like a letter "C" or its reflection. Figure 4 shows a triangulation for each piece and we can easily check that corresponding pieces for S and Ω have compatible triangulations. Using these building blocks we can find compatible triangulations of all of S and Ω

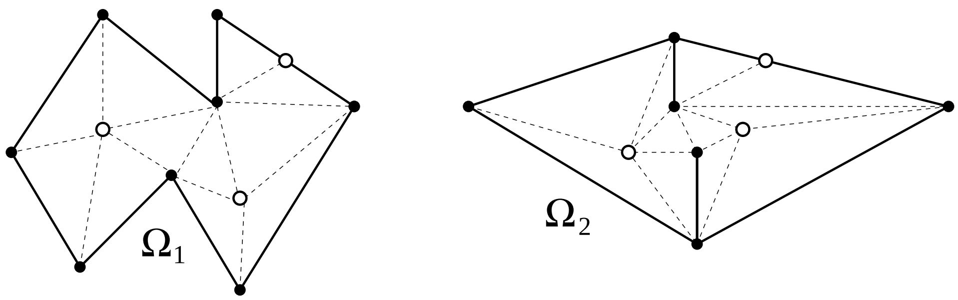


FIGURE 3. Compatible triangulations of two domains. Solid lines indicate the domain boundaries and dashed lines are the interior edges of the triangulations. We allow extra vertices either in the interior or on the boundary. Also note that adjacency only need be preserved when triangles share an edge inside the domain.

and hence a piecewise linear map from S to Ω. Since we are repeatedly using just two building blocks, the map is quasiconformal, with constant determined by the most distorted triangle from a finite set of possibilities. Note that the particular choice of triangulation is not important, and we have made no attempt to optimize the number of triangles used or the resulting QC constant.

The map ψ_1 is linear on a partition $\mathcal{J}$ of ∂S consisting of equal length segments. We define an exponential partition by dividing the segments on $\partial\Omega$ into sub-segments; about 2^n for segments that are between distance n and $n+1$ from the left side of Ω. Pulling back these edges on $\partial\Omega$ via ψ_1 gives an exponential partition of ∂S and then we apply the construction of the Section 3 to build ψ_0. If $\psi = \psi_1 \circ \psi_0 : \mathbb{H}_r \to \Omega$ then

$$g = \cosh \circ \psi^{-1} \circ \cosh^{-1}$$

is defined on $\Omega' = \cosh(\Omega)$ and extends to a quasiregular function on the plane. Thus there is a ϕ so that $f = g \circ \phi$ is entire with two critical values.

Also note that f has positive, finite order. It has order $\geq 1/2$ simply because all functions in class $\mathcal{B}$ do. To show f has finite order it suffices to show g does; the quasiconformal correction can only change the order by a multiple depending on the quasiconformal constant of the correction map ϕ (i.e., ϕ is Hölder). The main property of Ω that we need is that it contains a half-strip $\{x + iy : x > 0, |y| < w\}$ for some fixed $w > 0$.

Suppose $x \geq 2\pi$ and let $D = D(x, r)$ be the maximal disk centered at x contained in Ω. Since Ω is contained in the strip S, we have $r \leq \pi$ and ∂D hits $\partial\Omega$ at

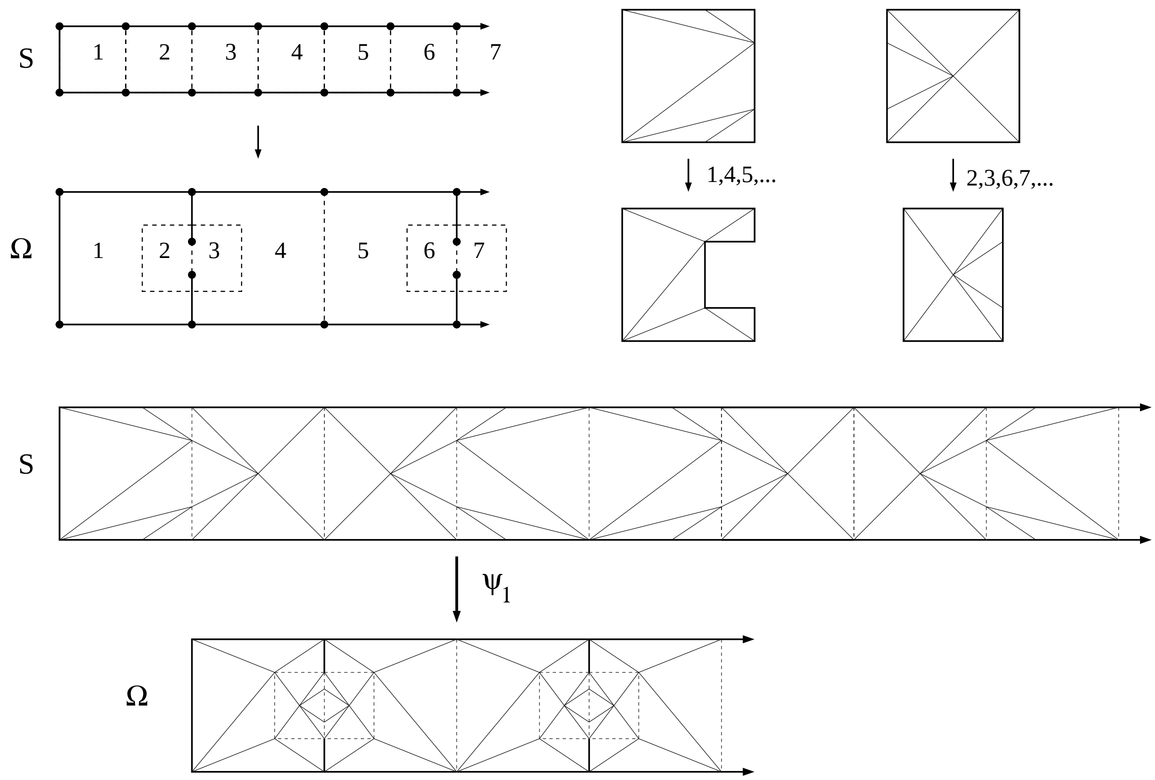


FIGURE 4. Constructing a piecewise affine map $\psi_1 : S \to \Omega$ using compatible triangulations. On the upper left we show partitions of S and Ω into corresponding subdomains and on the right we show that corresponding subdomains have compatible triangulations. On the bottom we insert the triangulations into the pieces to show the compatible triangulations of S and Ω. The same pattern is repeated forever.

symmetric points $t \pm iy$ with $|x - t| \le \pi$ and $y \ge w$. Let S_x be the hyperbolic geodesic in D connecting these two points. This arc cuts D into two subdomains that are quasicircles with uniformly bounded constant, and hence a result of Fernández, Heinonen and Martio [8] implies the image of each of these domains is a quasicircle with uniformly bounded constant in $\mathbb{H}_r$. This easily implies that $\gamma_x = \psi^{-1}(S_x)$ is a curve in $\mathbb{H}_r$ that is symmetric with respect to the real line and satisfies $R_x \simeq r_x \simeq \psi^{-1}(x)$ where

$$R_x = \max_{z \in \gamma_x} |z|, \qquad r_x = \min_{z \in \gamma_x} |z|.$$

(In other words, γ is contained in an annulus around the origin of fixed modulus, independent of x.) See Figure 5.

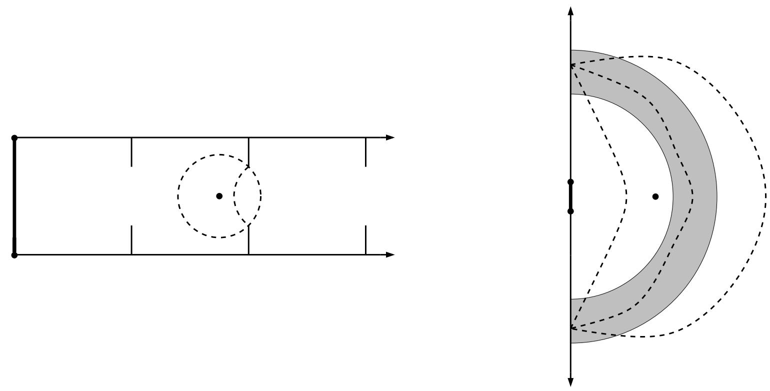


FIGURE 5. For each x we can find a nearby cross-cut of Ω that maps to an approximate half-circle in $\mathbb{H}_r$, centered at the origin, and an extremal length argument shows the logarithm of the diameter grows linearly with x if Ω contains an infinite strip. This implies f has finite order.

If I is the left side of Ω then the extremal distance from $\psi^{-1}(I)$ to γ_x in $\mathbb{H}_r$ is comparable to the extremal distance from I to S_x in Ω (since modulus is quasi-preserved by ψ). The first is easily seen to be comparable to $\log R_x$ and the second is comparable to x, since Ω is contained in the half-strip S and contains another half-strip of positive width. Finally,

$$\rho(g) \leq \limsup_{x\to\infty} \frac{\max_{u+iv\in\Omega, u<x} \log|\psi^{-1}(u+iv)|}{x} \leq \limsup_{x\to\infty} \frac{\log R_{x+\pi}}{x},$$

and our previous remarks show the rightmost term is bounded as $x \to \infty$.

More generally, this argument shows that g (and hence f) will have finite order whenever the domain Ω contains an infinite half-strip of positive width.

5. The main construction

The domain Ω used in the proof of Theorem 1.1 is illustrated in Figure 6. It consists of the half-strip S with finite trees removed along the integer points of the top and bottom edge. The domain is symmetric with respect to the real axis and in most of our pictures we will only show the lower half to simplify the illustrations. Note that there is a half-strip contained in Ω; this implies that the function we eventually obtain will have positive, finite order by our previous comments.

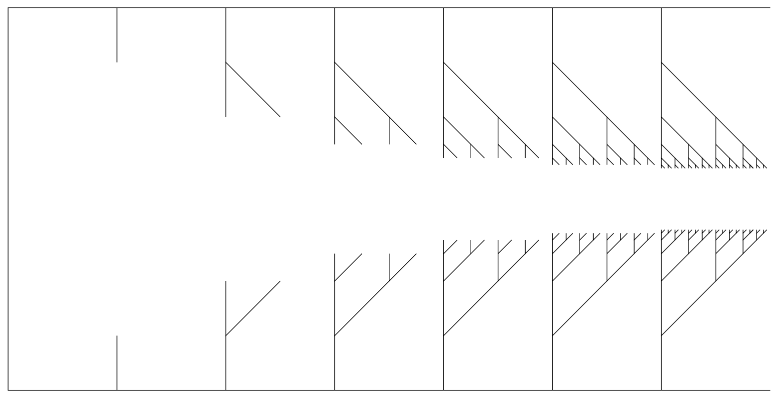


FIGURE 6. The domain in cosh coordinates. The domain is symmetric with respect to the real axis, and the trees are attached at the integer points. The line segments are further divided into edges by vertices of degree 2; see Figure 7.

Every removed tree contains a vertical unit line segment attached to ∂S. For $n = 1$ this segment is the entire tree. For $n = 2$ we add two segments: another vertical line segment of length 1 and a diagonal segment of slope 1 and length $\sqrt{2}$ (so the degree one vertices of the resulting tree are both distance 2 from ∂S). In general, the tree T_n attached at the nth point is the vertical segment plus a binary tree of depth $n-1$ as shown in Figure 6.

The edges of T_n are naturally divided into levels from 0 (attached to ∂S) to $n-1$ (adjacent to the leaves of the tree). We form a new tree T_n' by subdividing edges in the kth level of T_n into 4^k equal sub-edges. See Figure 7. Note that edges in the k-th level have length $\simeq 2^{-k} \cdot 4^{-k} = 8^{-k}$. We define the vertices of $\partial\Omega$ to correspond to the vertices of T_n'. It is these new, shorter edges that will eventually be mapped to segments of length $\pi/2$ on $\partial\mathbb{H}_r$. We also subdivide the edges of $\partial\Omega$ on ∂S by adding vertices where the real parts equal $n + \frac{1}{2}$, $n = 1, 2, 3 \dots$ along the top and bottom edges of T and adding the points $\pm i\pi/2$ to the left side of S.

This subdivision of T_n to obtain T_n' is important for two reasons. First, when we define the map $\psi_1 : S \to \Omega$, these new edges will pull back to an exponential partition of ∂S, and this allows us to define ψ_0 as before. Second, since every edge of T_n of level k is divided into 4^k sub-edges, we can label the vertices of T_n' so that the vertices of

T_n all get label 1, except for the root of T_n (the point where it is attached to ∂S) that gets label 0. Moreover, the roots of the trees are the only vertices labeled zero that are not degree 2 vertices. Hence these are the only preimages of 0 that are critical points. This fact will be important when we construct a quasiconformal deformation that changes the order of our function.

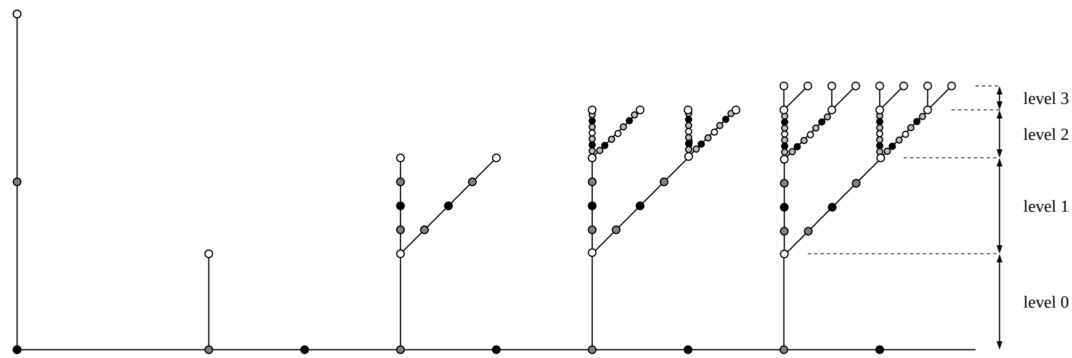


FIGURE 7. T_n' is T_n with extra vertices added. After subdividing, the new vertices of $\partial\Omega$ can be three-colored as shown (black = -1, gray = 0, white = 1). The labels are only shown for levels $0, 1, 2$ since edge lengths decrease exponentially with the level.

Next we have to describe the map $\psi_1 : S \to \Omega$. Consider the dashed curve γ in Figure 8. It is horizontal along the top of each tree and has slope 1/2 between trees (because the horizontal distance between T_n and T_{n+1} is 2^{-n+1} and T_{n+1} is 2^{-n} "taller" than T_n. The curve γ and its reflection across the real line bound a region $\Omega_1 \subset \Omega$ (half of Ω_1 is the shaded region in Figure 8). It is easy to map S to Ω_1 by a piecewise linear quasiconformal map that sends $S \cap L$ affinely to $\Omega_1 \cap L$ for every vertical line L.

The leaves of $\partial\Omega$ lie on $\partial\Omega_1$ and divide it into segments. We associate each such segment I to a region $Q_I \subset \Omega_1$. See Figure 9. If I is horizontal, then Q_I is the square in Ω_1 with I as one side. If I has slope 1/2 then Q_I is the right triangle in Ω_1 with hypotenuse I and vertical and horizontal legs. We also associate to I the component U_I of $\Omega \setminus \Omega_1$ that has I on its boundary.

We let $W_I = Q_I \cup U_I \cup I$ be the union of the regions above and below I. We will define a map $\Omega_1 \to \Omega$ by mapping each region Q_I to W_I. Our map will be the identity on $\partial Q_I \cap \Omega_1$, and hence extends continuously to all of Ω_1 by setting it to the identity outside all the Q_I's. This map is called a "filling" map for W_I.

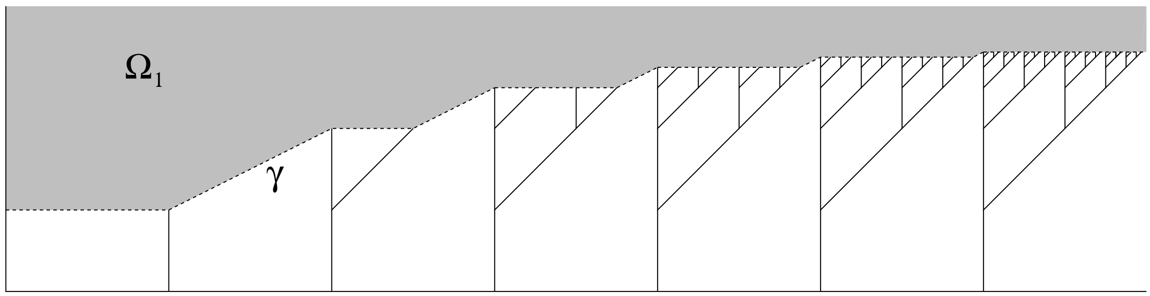


FIGURE 8. The first step is to map the half-strip S to the variable width strip Ω_1 by a piecewise linear quasiconformal map. This is easy.

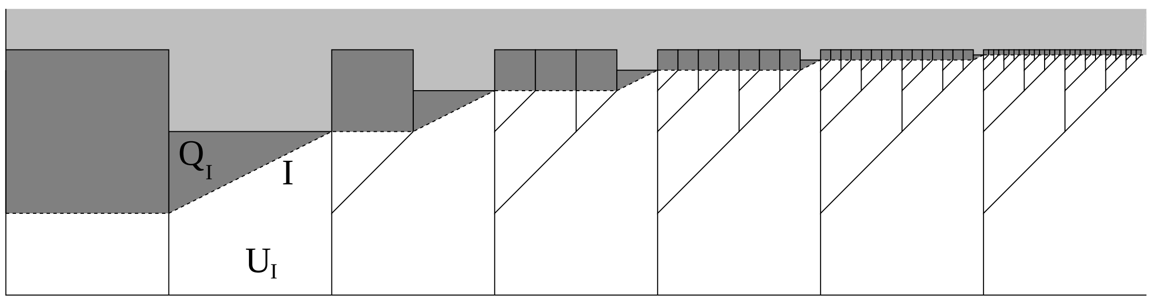


FIGURE 9. The leaves of $\partial\Omega$ partition $\partial\Omega_1$ and we associate to each segment I a region Q_I above it and a region U_I below it.

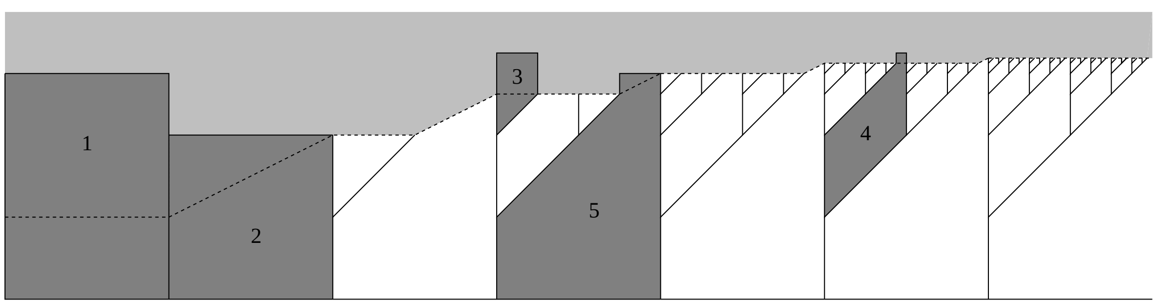


FIGURE 10. The regions Q_I are mapped to regions W_i that "fill in" $\Omega \setminus \Omega_1$. There are two special regions that are used just once each and three that are used repeatedly. The five types of regions are illustrated. Regions 1 and 2 are special cases that occur only once each and cases 3, 4 and 5 are called triangular, full and partial tubes respectively. There is only one type of triangular region, but the full and partial tubes come in infinitely many versions. However each of these versions is built from a finite number shapes.

There are five cases to consider: two special cases that occur once each, and three cases that occur infinitely often. The two special cases are the first two segments on γ; the ones that project vertically to $[0, 1]$ and $[1, 2]$. Figures 11 and 12 show how to define these maps using triangulations.

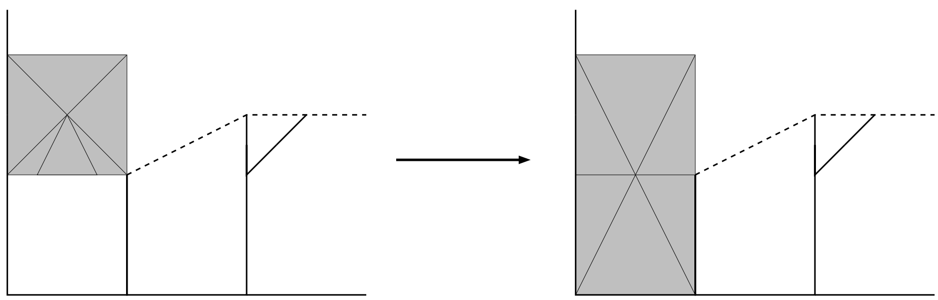


FIGURE 11. The first special case.

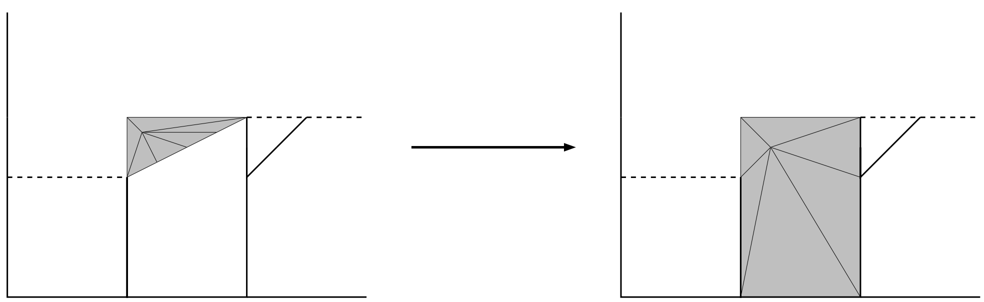


FIGURE 12. The second special case.

The remaining three cases are illustrated in Figures 13-17. We refer to these cases as triangles (U_I is a triangle), partial tubes (U_I is not a triangle and does not touch ∂S) and full tubes (U_I touches ∂S). See cases 3, 4 and 5 in Figure 10.

The filling map for full tubes is illustrated in Figures 14 and 15. In Figure 14, the left side shows the region Q_I divided into a number of subregions; a pentagon on top, a series of non-convex hexagons and a final triangle. The exact sizes of these regions is important and will be discussed in the next paragraph; for the moment, just assume that each of the middle regions (the non-convex hexagons) is similar to all the others. The regions are labeled with numbers indicating their levels; these are the same levels we used to subdivide the edges of the tree T_n to obtain the tree T'_n.

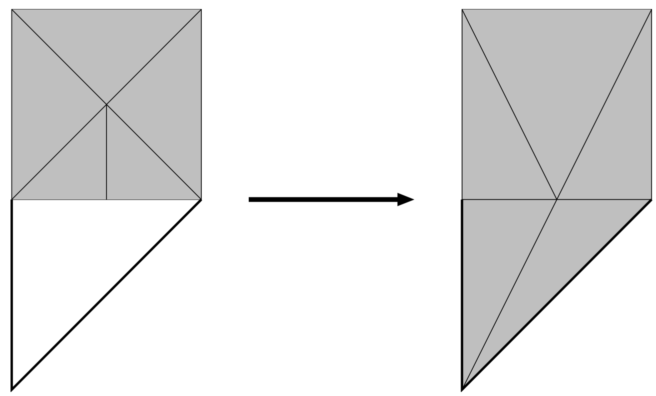


FIGURE 13. This shows the filling map for a triangular component.

The right side of Figure 14 shows the region W_I divided into subregions corresponding to the subregions of Q_I: a rectangle on top (this contains Q_I), a series of trapezoids, and a square on the bottom. The regions on left and right are numbered and each subregion on the left is mapped to the corresponding subregion on the right by a piecewise affine map. The triangulations that define this map are illustrated in Figure 15. Because we are only using three basic shapes (even though the middle shape may be used many times), each with a finite triangulation, the maps we build are clearly uniformly quasiconformal.

There is another property we need to check. The partition of the tree into edges is supposed to pull back to an exponential partition of the strip. This means that the vertices along the boundary of a full tube should pull back to a points on I that are approximately evenly spaced and grow exponentially with the distance of I from the left side of Ω. Consider Figure 14. The full tube is pictured on the right and divided into levels (starting with the square on the bottom, which is level 0). The sides of level k are divided into 4^k equal length edges of the tree (see Figure7). These points pull back to 4^k equally spaced points on a subsegment of I (on the left of Figure 14, the segment is where the boundary of a subregion hits I). In order for the collection of all preimage points to be equally spaced on I, we need each segment corresponding to a level k region to be 4 times longer than the segments corresponding to a level $k-1$ region, but this is easy to accomplish. This implies that when the edges of $\partial\Omega$ are pulled back, every preimage in $[n, n+1]$ has length $\simeq 4^{-n}$ and hence they form an exponential partition.

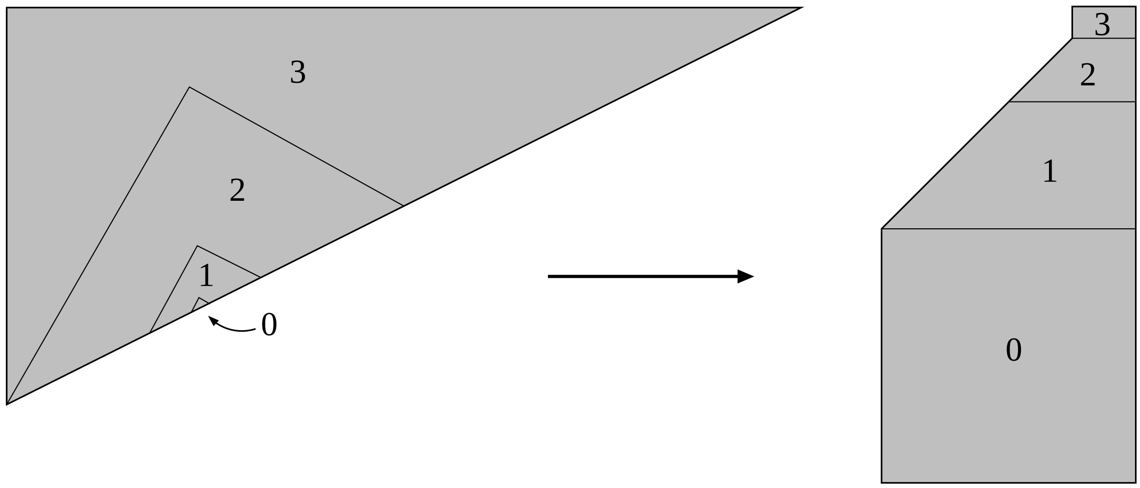


FIGURE 14. This shows the filling maps for the full tubes. We subdivide Q_I and W_I as shown and map the pieces as illustrated in Figure 15. All the filling maps for full tubes are of this form, differing only in the number of times the central trapezoid piece is repeated. The only critical feature is that components pieces in Q_I have diameters that decay like powers of 4 (this gives an exponential partition of ∂S).

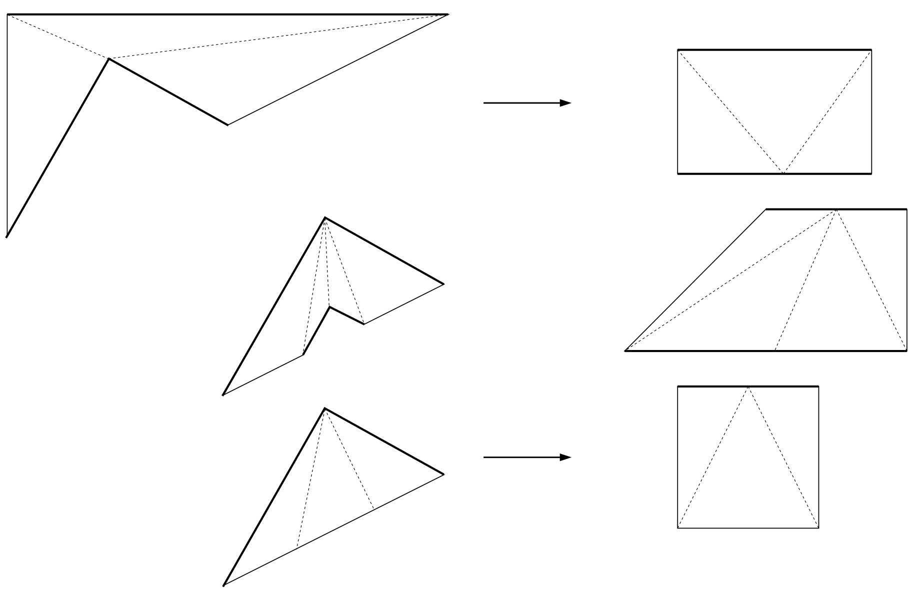


FIGURE 15. This shows the triangulations defining the filling maps for the full tubes. The reader should check that they are compatible.

The corresponding pictures for partial tubes are a little simpler. Figure 16 shows how we cut Q_i and W_I into corresponding pieces and Figure 17 shows how each piece is mapped via a triangulation. The rest of the argument is the same as for the full tubes (checking the map is uniformly quasiconformal and defines an exponential partition of the strip).

This completes our construction of the quasiregular map $g = \cosh \circ \psi^{-1} \circ \cosh^{-1}$ and hence of an entire function $f = g \circ \phi$ with three singular values. The next section will prove f is a counterexample to the order conjecture.

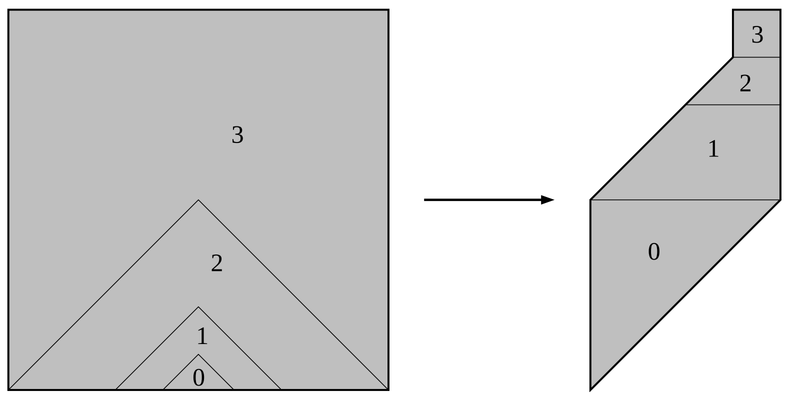


FIGURE 16. This shows how to decompose Q_I and W_I into corresponding pieces to construct the filling maps for partial tubes.

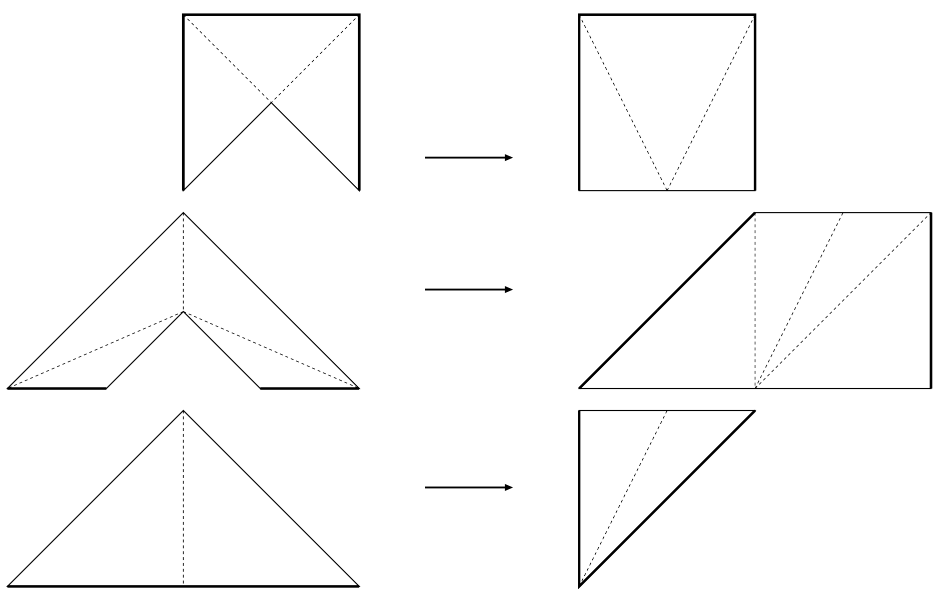


FIGURE 17. This shows how to map corresponding pieces of Q_I and W_I for partial tubes.

6. The order conjecture fails

Suppose f is the entire function constructed in the previous section. We claim we can choose quasiconformal maps τ, σ and an entire function F so that

$$F \circ \sigma = \tau \circ f,$$

where τ is the identity off a compact set and σ satisfies

$$|\sigma(z)| = O(\sqrt{|z|}), \tag{6.1}$$

for all sufficiently large z.

First we check that (6.1) gives the desired counterexample. Taking $z = \sigma(w)$,

$$\begin{aligned}
\lim_{|z|\to\infty} \frac{\log_+ \log_+ |F(z)|}{\log_+ |z|} &= \limsup_{|w|\to\infty} \frac{\log_+ \log_+ |F(\sigma(w))|}{\log_+ |\sigma(w)|} \\
&\geq \limsup_{|w|\to\infty} \frac{\log_+ \log_+ |\tau(f(w))|}{\log_+ |\sqrt{w}|} \\
&\geq 2 \limsup_{|w|\to\infty} \frac{\log_+ \log_+ |f(w)|}{\log_+ |w|},
\end{aligned}$$

since $f = \tau \circ f$ when $|f|$ is large enough. Thus the order of F is at least twice the order of f. Since the order of f is positive and finite, the two orders are different.

Now we prove the claims. First we define τ. Let $W = D(1,2) \setminus [0,1]$. W is a topological annulus and is conformally equivalent to the round annulus $A_t = \mathbb{D} \setminus t\mathbb{D}$ for some t. For $K > 1$, the map

$$r_K : z \to z|z|^{K-1},$$

is a K-quasiconformal from A_t to A_{t^K} that is the identity on the outer boundary of A_t and decreases the extremal length of the path family that separates the two boundary components of A by a factor of K. Let W_K be the domain of the form $D(1,2) \setminus [s,1]$ that is conformally equivalent to A^{t^K}. Thus r_K transfers to a K-quasiconformal map $\tau : W \to W_K$ that is the identity on the outer boundary of W decreases the extremal length of the path family that separates the two boundary components of A by a factor of K. It is easy to check that τ extends continuously across the slit boundary of W and can be extended by the identity outside $\overline{W}$ to give a K-quasiconformal map of the whole plane. This is the map τ we use in our quasiconformal equivalence.

By the measurable Riemann mapping theorem, there is a K-quasiconformal map σ of the plane so that $F = \tau \circ f \circ \sigma$ is entire. We can assume σ fixes $-1, 1, \infty$.

All that remains is to show that (6.1) holds. Let $V = D(1,2) \setminus [-1,1]$ and consider inverse images of V under f. By construction, every such preimage contains two preimages of 0 on its boundary and either

(I) both preimages are critical points,

(II) exactly one preimage is a critical point, or

(III) neither preimage is a critical point.

Case I occurs when the preimages of zero are the roots of two adjacent trees. Case II only occurs twice and corresponds to the corners of S in cosh coordinates. Case III occurs for preimages that do not touch ∂S in cosh coordinates. A cartoon of the different types of preimages is shown (in cosh coordinates) in Figure 18.

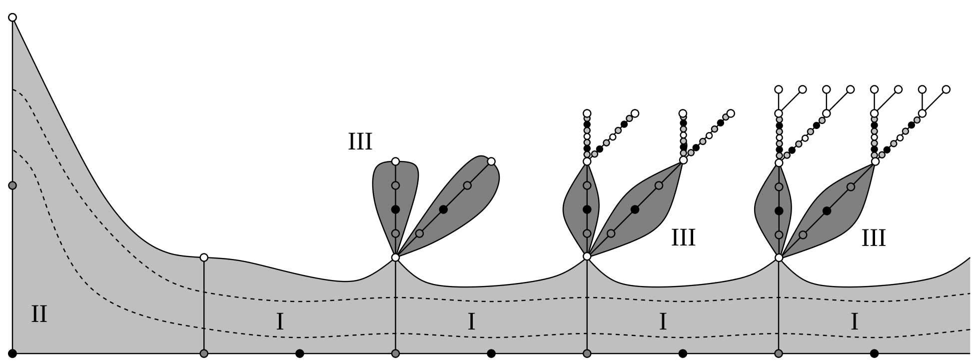


FIGURE 18. The support of our dilation lifted to Ω (its easier to see in Ω than in $\Omega' = \cosh(\Omega)$). The light gray is the union of type I and type II preimages and the darker gray shows a few type III preimages (this is a sketch, not a computation). The dashed lines represent the path family Γ_n whose extremal length is decreased by a factor of K by our dilatation. The grey dots correspond to preimages of 0.

Let $\gamma_0 = [-i\pi/2, i\pi/2]$ and for $n = 1, 2, \ldots$ let γ_n be the vertical crosscut of S that contains the point where the nth tree is attached. Let Γ_n be the path family connecting γ_0 to γ_n inside the closure of the type I and type II components. This family has extremal length $\leq Mn$ for some fixed M.

Applying cosh, γ_n, $n \geq 1$, maps to an ellipse E_n of bounded eccentricity and diameter $\simeq e^n$. Also, $\cosh(\Gamma_n)$ is a family of paths connecting $[0,1]$ to E_n. Let V_n be the region bounded by the ellipse E_n, minus $[-1,1]$ and let Σ_n be the path family in V_n connecting $[-1,1]$ to E_n. Then Σ_n contains $\cosh(\Gamma_n)$, and $V_n \setminus (-\infty,-1]$ is conformally equivalent to a $2\pi \times n$ rectangle, so if λ denotes extremal length we get

$$n \simeq \lambda(\Sigma_n) \leq \lambda(\cosh(\Gamma_n)) = \lambda(\Gamma_n) \simeq n.$$

When we apply the quasiconformal map σ, the extremal length of Γ_n is reduced by a factor of K; this is exactly why we choose τ as we did. Hence

$$\lambda(\sigma(\Sigma_n)) \leq \lambda(\sigma(\cosh(\Gamma_n))) \leq \frac{Mn}{K}, \tag{6.2}$$

if K is large enough. The eccentricity of the ellipse E_n tends to 1 as $n \nearrow \infty$ so $\sigma(E_n)$ is a $2K$-quasicircle when n is large enough. Let

$$R = \max_{\sigma(E_n)} |z|, \qquad r = \min_{\sigma(E_n)} |z|.$$

Since $\{1 \leq |z| \leq r\} \subset \sigma(V_n)$, we have $\frac{1}{2\pi} \log r \leq \lambda(\sigma(\Sigma_n)) \leq Mn/K$, and hence

$$r \leq \exp(2\pi Mn/K) \leq \exp(n/2)$$

if $K \geq 4\pi M$. Quasiconformal maps are quasisymmetric, so we have $R \leq C_K \cdot r$ for some constant depending only on K and hence $R \leq C_K \cdot e^{n/2}$. If $z \in V_n \setminus V_{n-1}$, then

$$|\sigma(z)| \leq R = O(e^{n/2}) = O(e^{(n-1)/2}) = O(\sqrt{|z|}),$$

which is (6.1). This proves Theorem 1.1, i.e., the order conjecture fails in $\mathcal{S}$.

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