CONFORMAL REMOVABILITY IS HARD

CHRISTOPHER J. BISHOP

Abstract. A planar compact set E is called conformally removable if every homeomorphism of the plane to itself that is conformal off E is conformal everywhere and hence linear. We show that the collection of removable subsets of $[0,1]^2$ is not a Borel subset of the space of compact subsets of $[0,1]^2$ with its Hausdorff metric.

1. Introduction

A planar compact set E is called removable for a property P if every function with property P on $\Omega = E^c = \mathbb{C} \setminus E$ is the restriction of a function on $\mathbb{C}$ with this property. For example, if P is the property of being a bounded holomorphic function, then E is removable iff every bounded holomorphic function on its complement extends to be bounded and holomorphic on the whole plane (and hence is constant by Liouville's theorem). A standard result in many complex variable classes is the Riemann removable singularity theorem, that says single points are removable in this sense. While there are a wide variety of properties that could be considered, most attention has been devoted to the following cases:

- H^∞-removable: $P =$ bounded and holomorphic,
- A-removable: $P = H^\infty$ and uniformly continuous,
- S-removable: $P =$ holomorphic and 1-to-1 (also known as conformal or schlicht),
- CH-removable: $P =$ conformal and extends to a homeomorphism of $\mathbb{C}$.

Recall that uniformly continuous on Ω is equivalent to extending continuously from Ω to the Riemann sphere $\mathbb{S}$. For any excellent survey of what is currently known about each of these classes, see Malik Younsi's 2015 survey paper [48].

Date: June 16, 2020.

1991 *Mathematics Subject Classification.* Primary: 30C35, Secondary: 28A05, 20H05, 30C85 .

Key words and phrases. conformally removable, analytic capacity, Borel sets, co-analytic complete, descriptive set theory, wellfounded trees, countable sets, conformal welding.

The author is partially supported by NSF Grant DMS 1906259.

The basic problem is to find "geometric" characterizations of these sets. For example, Xavier Tolsa has given a characterization of H^∞-removable sets in terms of the types of positive measures supported on the set (see Section 2). Alhfors and Beurling [1] gave a characterization of S-removable sets in terms of a quantity called "absolute area zero" (the complement of every conformal image of Ω has zero area). On the other hand, although there are various known sufficient conditions and necessary conditions, e.g., [27], [29], [30], there is no simple characterization of A-removable or CH-removable sets. Thus it seems that the characterizing these sets is "harder" than characterizing H^∞-removable or S-removable sets (or perhaps simply that smarter people have worked on the latter problems). The following is a precise formulation of the idea that A-removability and CH-removability are actually harder.

Theorem 1.1. *Let $S = [0, 1]^2$ be the unit square in $\mathbb{C}$ and let 2^S denote the hyperspace of S, i.e., the compact metric space consisting of all compact subsets of S with the Hausdorff metric. Within this metric space, the collection of*

1. *H^∞-removable subsets is a G_δ,*
2. *S-removable subsets is a G_δ,*
3. *A-removable sets is not Borel,*
4. *CH-removable sets is not Borel,*
5. *A-removable closed Jordan curves is not Borel.*

Thus, in some sense, removability for conformal homeomorphisms is infinitely more complicated than for bounded holomorphic functions. It turns out the proof of parts (1) and (2) are relatively elementary, parts (3) and (4) follow from well known results in descriptive set theory and complex analysis (the most recent result we cite dates to 1960), and (5) requires a new construction. The obvious question of whether CH-removable Jordan curves are Borel or not remains open, and seems closely connected to other difficult problems, such as deciding whether the collection of conformal weldings is a Borel subset of the space of circle homeomorphisms. We shall prove that the set of CH-non-removable curves is residual in the space of all closed Jordan curves; see Section 9.

Below I shall sketch a quick proof of (1) and (4) for experts familiar with both areas; the remainder of the paper provides the full details for readers who are familiar with only one (or perhaps neither) side of the story. Although I have not found this theorem stated in the literature, I would not be surprised if it were known to some experts, e.g., various papers of Robert Kaufman contain all the relevant facts for proving (1)-(4).

We start with some definitions. Given a compact set K, we define the Hausdorff distance between compact subsets K_1, K_2 as

$$d_H(K_1, K_2) = \inf\{\epsilon : K_2 \subset K_1(\epsilon), K_1 \subset K_2(\epsilon)\},$$

where $K_j(\epsilon) = \{z : \text{dist}(z, K_j) < \epsilon\}$ is an ϵ-neighborhood of K_j, $j = 1, 2$. This defines a compact metric space on the set of all compact subsets of K, called the Hausdorff hyperspace of K and denoted 2^K (e.g., Theorem A.2.2 of [11]). In this note we mainly deal with the examples of the unit interval $I = [0, 1] \subset \mathbb{R}$, the unit square $S = [0, 1]^2 \subset \mathbb{R}^2 = \mathbb{C}$ or the Riemann sphere $\mathbb{S}$. The collection of Borel sets is the smallest σ-algebra containing the open sets (a σ-algebra is closed under countable unions, countable intersections and complements). An F_σ set is a countable union of closed sets; a G_δ is a countable intersection of open sets (this terminology originates with Hausdorff in 1914). Analytic sets (also known as Suslin sets) are continuous images of Borel sets and need not be Borel (more about this later); the complement of an analytic set is called co-analytic and need not be analytic itself (if it is, then it is also Borel). Cases (3)-(5) in Theorem 1.1 turn out to be co-analytic complete, a condition we will define in Section 5 and that implies non-Borel.

The removable sets in the first three cases of Theorem 1.1 all form σ-ideals of compact sets, i.e., they are closed under taking compact subsets and compact countable unions. The subset property is obvious, and the fact that a compact set that is a countable union of removable sets is also removable is proven in [48] for each of these three classes. The dichotomy theorem for co-analytic σ-ideals (e.g., Theorem IV.33.3 in [32]) then says these collections must be either G_δ or co-analytic complete in 2^S. Theorem 1.1 indicates which possibility occurs in each case. It is not known whether the CH-removable sets form a σ-ideal; indeed, it is not even known if the union of two overlapping CH-removable sets is CH-removable. If the sets are disjoint, then this is true, but is remains open even if both sets are Jordan arcs sharing a single endpoint.

The fact that H^∞-removable sets are G_δ is an easy normal families argument. We sketch it here for complex analysis experts, giving the details in Section 2. Suppose

K_n denotes the set of H^∞-non-removable sets in $S = [0,1]^2$ whose analytic capacity (a non-negative number measuring of how non-removable the set is) is at least $1/n$. If $\{E_k\}$ is a sequence of compact sets in K_n that converge in the Hausdorff metric to a set E, then, by normal families, we can prove E is also non-removable with analytic capacity $\geq 1/n$. Thus K_n is closed collection of sets of 2^S. The set of all H^∞-non-removable sets is the union of the K_n, hence is F_σ. The H^∞-removable sets are the complement of this F_σ in 2^S, and therefore are a G_δ subset. The proof for S-removable sets is similar but the lower bound on analytic capacity is replaced by lower bounds on Laurent series coefficients.

To show that the CH-removable sets are not Borel, it suffices to show that they are the image of a known non-Borel subset K of some Polish space X under a continuous map from X into 2^S. A Polish space is a separable topological space X that has a compatible metric making it complete. Since continuous preimages of Borel sets are Borel, this implies the CH-removable sets can't be Borel in 2^S. For us, a convenient choice is to take X to be the hyperspace of $I = [0,1]$ and K to be the collection of countable, compact subsets of $[0,1]$. A famous result of Hurewicz [28] says that K is co-analytic but not Borel in 2^I. The continuous map $2^I \to 2^S$ is simply $E \mapsto E \times [0,1]$. We then quote the 1960 result of Fred Gehring [22] that for a compact set E, the product set $E \times [0,1]$ is CH-removable if and only if E is countable. This proves CH-removable sets are not Borel. The proof for A-removable sets is similar, but using a result of Carleson in place of Gehring's theorem. See Section 6.

The remainder of the paper is devoted filling in the details for the interested non-expert. (The bored expert may jump to the last three sections where some related open problems are discussed.)

Although it is a basic theorem of descriptive set theory that every uncountable Polish space X contains analytic and co-analytic sets that are not Borel (see Section 4), it is very interesting to obtain "natural" examples arising in various areas of mathematics. For example, if $X = C([0,1])$ (continuous functions on $[0,1]$ with the supremum norm) the following subsets of functions are all known to be co-analytic complete, and hence non-Borel:

- everywhere differentiable [39],
- differentiable except on a finite set [43] or countable set [24],

• nowhere differentiable [38],
• everywhere convergent Fourier series [3].

For the space $C([0,1])^{\mathbb{N}}$ of sequences of continuous functions on $[0,1]$ the space CN of everywhere convergent sequences is co-analytic complete, as is the space CN_0 of sequences converging to zero everywhere. See Theorem IV.33.11 of Kechris's book [32]. For further examples from analysis and topology, see Howard Becker's 1992 survey [5].

When X is the hyperspace of the unit circle $\mathbb{T}$ (compact subsets with the Hausdorff metric), we have already mentioned the countable compact sets are co-analytic non-Borel. Other known examples of non-Borel subsets of 2^I are:
• sets of uniqueness [33],
• sets of strict multiplicity [31].

A closed set $E \subset \mathbb{T}$ is a set of uniqueness if any trigonometric series that converges to zero everywhere off E must be the all zeros series. E is a set of strict multiplicity if it supports a measure whose Fourier coefficients tend to zero; the Fourier series of such a measure shows that its support is not a set of uniqueness in a strong way. These particular examples have an intimate connection to the foundations of modern mathematics: Cantor showed that finite sets are sets of uniqueness and the problem of extending this to infinite sets led him to the creation of set theory. For more about this fascinating episode in the history of mathematics, see e.g., [15], [16], [37], [44].

We should also mention that many non-Borel sets arise naturally in Banach space theory. For example, Bossard, Godefroy and Kaufman [25] show that set of rotund norms equivalent to any infinite dimensional Banach space norm is co-analytic but not Borel in the space of all norms (a norm is rotund if $1 = \|x\| = \|x_n\| = \lim_n \|(x + x_n)/2\|$ implies $\|x - x_n\| \to 0$). After choosing a (standard) embedding of separable Banach spaces as a closed subsets of the Cantor set, Bossard [12] shows the isomorphism relation (as a subset of the product of Cantor sets) is analytic non-Borel, the class of spaces which do not contain an isomorphic copy of a given space is co-analytic non-Borel, as is the set of reflexive spaces and the spaces with separable dual.

This note was prompted by email discussions with Guillaume Baverez, in which he proposed a possible characterization of CH-removable Jordan curves in terms of their conformal weldings (see Section 9). I doubted such a concise criterion could be given, and eventually I found a counterexample to his conjecture, but the interchange raised the question of quantifying the difficulty of the problem. This note was written in the hope that gathering the basic facts needed from descriptive set theory might be of interest to fellow complex analysts, and perhaps motivate some of them to attack harder variants of these problems, e.g., those discussed in Sections 8, 9 and 10.

2. H^∞-removability is "easy"

As we shall explain below, identifying removable sets isn't exactly easy in the usual sense, but in terms of descriptive set theory the collection of such sets is pretty simple:

Lemma 2.1. *The collection of H^∞-non-removable subsets of $S = [0,1]^2$ is an F_σ subset of 2^S. The H^∞-removable sets are therefore a G_δ subset.*

Proof. Suppose $E \subset [0,1]^2$ is non-removable for H^∞. Then there is a non-constant, bounded holomorphic function f defined on the complement of E. Near infinity f has a Laurent expansion

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots$$

and has at least one non-zero coefficient c_k for some ≥ 1. If $c_1 = 0$, the function

$$f_1(z) = z(f(z) - c_0) = \frac{c_2}{z} + \frac{c_3}{z_2} + \dots$$

is also bounded, non constant and holomorphic off E. Continuing in this way, we see that we eventually obtain a bounded holomorphic function on $\Omega = \mathbb{C} \setminus E$ that has non-zero coefficient c_1 in its Laurent expansion.

Let X_n be the collection of non-removable sets in $[0,1]^2$ whose complements support a holomorphic function whose absolute value is bounded by 1 and whose Laurent coefficient satisfies $|c_1| \geq 1/n$. We claim X_n is a closed set in 2^S. Suppose $\{K_n\} \subset X_n$ are compact sets converging to K in the Hausdorff metric. Assume f_n is the holomorphic function on K_n^c attesting to its membership in X_n. Each compact disk D in the complement of K is eventually contained in the complements of the K_n for n large enough, and by a normal families argument, we may extract a subsequence that converges to a holomorphic function f_D on D. Covering K^c by a countable union of such disks and applying a diagonalization argument, we may extract a subsequence converging to a holomorphic function f bounded by 1. Applying the Cauchy integral formula to a fixed circle surrounding $[0,1]^2$ we see that the Laurent coefficients of f_n converge to the Laurent coefficients of f and hence $|c_1(f)| \geq 1/n$. Thus $K \in X_n$. Since every non-removable set is in some X_n, the collection of all non-removable sets is an F_σ in 2^S. □

The proof that S-removable sets form a G_δ is very similar, except that the trick of replacing $f(z)$ by $z(f(z)-c_0)$ to get $|c_1| > 0$ might not give a 1-to-1 map. Instead, we may assume the map conformal off E has an expansion $f(z) = z + c_1/z + c_2/z^2 + \dots$ and that $c_k \neq 0$ for some k. Thus it suffices to prove each member of the countable family $K_{n,m}$ where $|c_k| \geq 1/n$ is closed. This proof proceeds just as above.

Of course, just because H^∞-non-removable sets are Borel in 2^S does not mean that it is an easy task to find an elegant characterization of them. Indeed, it is a deep result of Xavier Tolsa that E is non-removable for bounded holomorphic functions if and only if it supports a positive measure μ of linear growth, i.e.,

$$\mu(D(x,r)) \leq Mr, \tag{2.1}$$

(for some $M < \infty$ and all $x \in \mathbb{R}^2$ and $r > 0$) and has finite Menger curvature in the sense that

$$c^2(\mu) = \int\int\int c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z) < \infty, \tag{2.2}$$

where $c(x,y,z)$ is the reciprocal of the radius of the unique circle passing thorough (x,y,z) (linear growth implies μ^3 gives zero measure to the set were two or more of x,y,z agree).

We know the proof of Lemma 2.1 must break down for A-removable and CH-removable sets. In both cases we can find non-removable sets contained in the strip $[0,1]\times[0,\frac{1}{n}]$ and with corresponding functions that converge to non-constant functions on $\mathbb{C} \setminus [0,1]$ as $n \to \infty$. For A-removability this is done in [8]; one simply has to take an Jordan arc which has tangents on at most a set of zero linear measure. Thus any "flat enough" fractal arc will work. For CH-removability, see the construction of "flexible curves" in [9] or [10].

3. Analytic sets

A topological space X is called Polish if it is separable (has a countable dense set) and has a compatible metric that makes it complete (Cauchy sequences converge).

Standard examples include Euclidean space $\mathbb{R}^n$, the continuous functions on $[0,1]$ with the supremum norm $C([0,1])$, and the collection of compact subsets of a compact set $K \subset \mathbb{R}^n$ with the Hausdorff metric. Another important example is the Baire space $\mathbb{N}^\mathbb{N}$ of infinite sequences of non-negative integers equipped with the metric given by $d((a_n),(b_n)) = e^{-m}$, where $m = \max\{n \geq 0 \colon a_k = b_k \text{ for all } 1 \leq k \leq n\}$. One can show $\mathbb{N}^\mathbb{N}$ is homeomorphic to the irrational numbers (with the usual topology) although they are different as metric spaces (one is complete and the other is not). In fact, every Polish space is the continuous image of the Baire space. (e.g., Theorem B.1.2 in [11]).

If X is a Polish space, then $A \subset X$ is called analytic if there is another Polish space Y and a Borel set $E \subset X \times Y$ so that A is the projection on E onto A, i.e.,

$$A = \{x \in X : \exists y \in Y \text{ such that } (x,y) \in E\}.$$

There are several equivalent characteriations of analytic sets, including

(1) A is the projection of a closed set in $X \times \mathbb{N}^\mathbb{N}$,
(2) A is the continuous image of $\mathbb{N}^\mathbb{N}$,
(3) A is a continuous image of a Polish space,
(4) A is the continuous image of a Borel subset of a Polish space,
(5) A is the Borel image of a Borel subset of a Polish space.

In comparison, Borel subsets of a Polish space are characterized by being

(1) a continuous 1-to-1 image of $\mathbb{N}^\mathbb{N}$,
(2) a continuous 1-to-1 image of a Borel subset of a Polish space,
(3) a 1-to-1 projection of a closed set in $X \times \mathbb{N}^\mathbb{N}$,
(4) both a co-analytic and analytic set (see below).

If $A \subset X$ is analytic, then $A^c = X \setminus A$ is called co-analytic. Analytic sets are denoted $\boldsymbol{\Sigma}_\mathbf{1}^\mathbf{1}$ and co-analytic sets $\boldsymbol{\Pi}_\mathbf{1}^\mathbf{1}$ (using light-faced characters refers to something else). These form the simplest elements of the projective hierarchy of sets, much as closed and open sets are the simplest sets of the Borel hierarchy. Such sets can be quite complicated, e.g., although every uncountable analytic set contains a perfect subset, Gödel [26] showed that this question for co-analytic sets is undecidable (similar to his results for the Axiom of Choice and the Continuum Hypothesis). Similarly, all analytic sets are Lebesgue measurable, but proving general projective sets are measurable requires additional axioms, e.g., the assumption that certain "large cardinals" exist. See [45].

In a Polish space every open set and every closed set is analytic, and the analytic sets are closed under countable unions and intersections. See [32] or Appendix B of [11]. From this it follows that every Borel set is analytic. However, it is known that any uncountable Polish space contains an analytic set that is not Borel (see Lemma 4.1), and several explicit examples were already mentioned in Section 1.

Analytic sets are also known as Suslin sets in honor of Mikhail Yakovlevich Suslin, who proved that a set is Borel if and only it is both analytic and co-analytic. While a research student of Lusin in 1917, Suslin constructed a Borel set in the plane whose projection on the real axis is not Borel, contradicting a claim in a 1905 paper of Lebesgue, (Cooke [15] refers to this as "one of the most fruitful mistakes in all the history of analysis"). Suslin died of typhus in 1919 at the age of 24, having published just one 4 page paper while alive, and one posthumously one with Sierpinski. His work was further developed by Lusin[1], Sierpinski[2] and others, and Suslin's legacy remains very active a century later.

To prove that the conformally non-removable subsets of $S = [0,1]^2$ form an analytic subset of the hyperspace of S, we first record a few simple facts.

Lemma 3.1. *For any Borel map $f : X \to Y$ between Polish spaces, the graph of f is a Borel set in $X \times Y$.*

Proof. It suffices to prove the complement of the graph is Borel. Since Y is separable, there is a countable basis $\{B_k\}$ for the topology. Thus given any $x \in X$ and $y \in Y$ so that $y \neq f(x)$ there is a basis element B_k so that $f(x) \in B_k$ and $y \neq B_k$. In other words, (x, y) is contained in the Borel product set $f^{-1}(B_k) \times (Y \setminus B_k) \subset X \times Y$ and this set is disjoint from the whole graph of f. Thus the complement of the graph of f is a countable union of Borel sets, and hence is Borel itself. □

[1]In 1936 Lusin was the victim of a political attack that included charges of taking credit for Suslin's work and publishing too much in Western journals. Lusin survived the incident and was officially rehabilitated in 2012. See [19]. However, Lusin's thesis advisor, Egorov, died in 1931 following a hunger strike in prison after similar attacks.

[2]According to [15] although Sierpinski was technically under arrest in Moscow during World War I as an Austrian citizen, he was allowed to participate in the academic life of Moscow University.

Lemma 3.2. *If A is an analytic subset of 2^K, then the collection of supersets of A is also analytic.*

Proof. Since A is analytic it is the continuous image of some Polish space, say $A = f(X)$. Define a map $X \times 2^K \to 2^K$ by $(x, E) \mapsto f(x) \cup E$. It is easy to check that taking unions is a continuous map from $2^K \times 2^K \to 2^K$. Since products of Polish spaces are also Polish, we see the union of supersets is a continuous image of a Polish space, hence is analytic. □

Note that adding all the supersets to a non-Borel collection of sets can sometimes create a Borel collection. For example, the collection of supersets of the countable compact sets is all compact sets.

Lemma 3.3. *Suppose X is a Polish space. Suppose $K \subset \mathbb{C}$ is compact and that each open $U \subset \mathbb{C}$ is associated to a closed set $X(U) \subset X$ so that $\cap_\alpha X(U_\alpha) = X(\cup_\alpha U_\alpha)$ for any collections of open sets $\{U_\alpha\}$. Then the map from points of X to compact subsets of $K = [0,1]^2$ defined by*

$$x \to K_x = K \setminus \cup\{U : x \in X(U)\},$$

is Borel from X to 2^K.

Proof. Note that if $V \subset W$ are open sets, then $V \cup W = W$ so $X(V) \supset X(V) \cap X(W) = X(V \cup W) = X(W)$, so our map has a "reverse monotone" property.

For each closed set $E \subset K$ and $\epsilon > 0$ the collection $\{F \subset K : d_H(F, E) < \epsilon\}$ is an open ball in the hyperspace of K and sets of this form are a basis of the topology of the hyperspace. Thus if suffices to show preimages of such sets are Borel. Each such set is a countable union of sets of the form $\{F \subset K : d_H(F, E) \le \delta\}$, say with $\delta = \epsilon(1 - 1/n)$, so it suffices to show preimages of these sets are Borel.

Clearly $d_H(K_x, E) \le \delta$ is equivalent to having x in both $Y_1 = \{x \in X : K_x \subset \overline{E(\delta)}\}$ and $Y_2 = \{x \in X : E \subset \overline{K_x(\delta)}\}$.

We claim that $x \in Y_1$ iff $x \in X(U)$ where $U = \{z : \text{dist}(z, E) > \delta\}$. If the latter condition holds, then the former holds by the definition of K_x (U is one of the open sets we subtract from K). If the former condition holds, then for any $y \notin \overline{E(\delta)}$ we must have $x \in X(U_y)$ for some open U_y containing y. Thus $x \in \cap_y X(U_y) = X(\cup_y U_u) \subset X(U)$; where the union is over all $y \notin \overline{E(\delta)}$ and the last inclusion is implied by $U \subset \cup_y U_y$ and the reverse monotone property. By assumption $X(U)$ is a closed subset of X, so Y_1 is closed.

Next we consider Y_2. Its complement $X \setminus Y_2$ consists of points x so that K_x is more than distance δ from E, i.e., K_x misses some closed disk $\overline{D} = \overline{D(y,r)}$ with $r > \delta$ that contains a closed disk of radius δ centered at some point of E. The disk D can chosen from the countable collection of disks with rational centers and radii. For each point $z \in \overline{D}$, $z \notin K_x$ implies $x \in X(U_z)$ for some open set U_z containing z, hence $x \in \cap_z X(U_z) = X(\cup_z U_z) = X(V)$ where V is an open set containing $\overline{D}$ but disjoint from K_x. This is a closed set of points in X. Thus $Y_2^c = X \setminus Y_2$ is F_σ, so Y_2 is a G_δ. Thus $Y_1 \cap Y_2$ is G_δ. Recalling that the inverse image of an open ball was a countable union of such sets, we deduce that the inverse image of any open set is a $G_{\delta\sigma}$ set. □

Lemma 3.4. *The CH-non-removable subsets of $[0,1]^2$ form an analytic subset of the hyperspace of $[0,1]^2$. Thus the removable sets are co-analytic.*

Proof. Let X be the space of homeomorphisms of the plane to itself that are holomorphic off $K = [0,1]^2$ and normalized to be $h(z) = z + O(1/|z|))$ at infinity. This is a Polish space with the supremum metric.

For each open set $U \subset \mathbb{C}$ let $X(U)$ be the elements of U that are holomorphic on U. Since uniform limits of holomorphic functions are holomorphic, this is a closed subset of X. Moreover, if h is holomorphic on each sets in a collection $\{U_\alpha\}$ it is holomorphic on the union so $X(\cup_\alpha U_\alpha) = \cap_\alpha X(U_\alpha)$. (All functions in this set may be homomorphic on a strictly larger set, e.g., if the union has removable complement, but this equality still holds, and simply gives an example where $X(V) = X(W)$ even if V is strictly contained in W.)

For each $h \in X$, and let $U_h = \mathbb{C} \setminus K_h$ be the largest open set so that h is holomorphic on some neighborhood of every $z \in U_h$. Lemma 3.3 says that $h \mapsto K_h$ from X to Y is a Borel map, Lemma 3.1 says its graph $\{(h, K_h)\}$ is a Borel set in $X \times Y$, and the projection onto the second coordinate gives an analytic set $A = \cup_{h \in X} E_h$ (projections of Borel sets are analytic). By definition, a compact subset of K is conformally non-removable if and only if it contains a set in A. Thus conformally non-removable sets are analytic in 2^K by Lemma 3.2. □

Corollary 3.5. *The A-removable subsets of $S = [0,1]^2$ are co-analytic in 2^S.*

Proof. This is exactly the same as the proof of Lemma 3.4, except that now we work in the Polish space of all continuous functions on the Riemann sphere that are holomorphic off S. As before, the map sending each such function to the complement of the set where it is holomorphic is a Borel mapping of this Polish space into 2^S, and the projection of its graph onto the second coordinate gives an analytic subset of 2^S. Taking all supersets gives all A-non-removable sets. □

4. Analytic non-Borel sets exist

This is another standard result, but we include the simple proof for completeness. We follow the argument in [13].

Lemma 4.1. $\mathbb{N}^\mathbb{N}$ *contains an analytic set that is not Borel. The complement of this set is co-analytic and not Borel.*

Proof. This is a diagonalization argument. We claim it suffices to show there is an analytic subset $X \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ so that every analytic subset $A \subset \mathbb{N}^\mathbb{N}$ occurs as a slice $A = X_y = \{x \in \mathbb{N}^\mathbb{N} : (x, y) \in X\}$ for some y. Given such a set X, then

$$B = \{x \in \mathbb{N}^\mathbb{N} : (x, x) \in X\}$$

is the projection of the intersection of X with the (closed) diagonal of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ and hence is the continuous image of an analytic set, and therefore is itself analytic. The complementary set

$$B^c = \{x \in \mathbb{N}^\mathbb{N} : (x, x) \notin X\}$$

is automatically co-analytic, and if B^c were also analytic, then it would be equal to a slice X_y of X for some y. In this case,

$$X_y = \{x : (x, y) \in X\} = \{x : (x, x) \notin X\} = B^c,$$

and $y \in B$ and $y \in B^c$ both lead to contradictions. Thus B^c can't be analytic and hence neither B nor B^c is Borel (since Borel sets are closed under complements and all Borel sets are analytic). Thus we have reduced finding a non-Borel analytic set to finding an analytic set $X \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ which has every analytic subset of $\mathbb{N}^\mathbb{N}$ as a slice.

First we show this is possible for closed slices. If Y is a Polish with a countable basis $\{B_k\}$ for the topology and $y \in Y$ let $S(y) \subset \mathbb{N}$ be all the k's with $y \notin B_k$ and

$T(y) \subset \mathbb{N}^{\mathbb{N}}$ all the sequences with elements in $S(y)$. Then $\{(y, T(y)) \in Y \times \mathbb{N}^{\mathbb{N}}\}$ is a closed set: if $y_n \to y$ and $y_n \notin B_k$ for large n, then $y \notin B_k$, since B_k^c is closed. The second coordinates also converge, since the topology on $\mathbb{N}^{\mathbb{N}}$ agrees with the product topology. If we fix a sequence $(a_k) \in N^{\mathbb{N}}$ as the second coordinate, the first coordinate ranges over $Y \setminus \cup_k B_{a_k}$, so any closed subset of Y can occur as a slice.

Next, to obtain every analytic subset of $\mathbb{N}^{\mathbb{N}}$ as a slice, we apply the previous argument to $Y = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ to get a closed set $X \subset (\mathbb{N}^{\mathbb{N}})^3 = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^n$ so that every closed subset of $(\mathbb{N}^{\mathbb{N}})^2$ occurs as a slice of X. Hence every analytic subset of $\mathbb{N}^{\mathbb{N}}$ occurs when we project X onto the first coordinate. Since projections of analytic sets are analytic, projecting X onto the first and third coordinates gives an analytic subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ where the first coordinate ranges over all analytic subsets of $\mathbb{N}^{\mathbb{N}}$, as desired. □

Note that this implies the cardinality of the analytic subsets of a Polish space is at most the cardinality of $\mathbb{N}^{\mathbb{N}}$, which is same as $\mathbb{R}$, the continuum c. Since points are analytic sets, the analytic subsets of $\mathbb{R}$ have cardinality c. Thus the Borel subsets of $\mathbb{R}$ have the same cardinality.

5. Co-analytic complete sets

A co-analytic subset $A \subset X$ of Polish space is called complete co-analytic if for any co-analytic set B of $\mathbb{N}^{\mathbb{N}}$ there is a Borel map $f : \mathbb{N}^{\mathbb{N}} \to X$ so that $f(y) \in A$ iff $y \in B$. Thus membership in any such B can be reduced to checking membership in A. Since Borel pre-images of Borel sets are Borel, and we know that $\mathbb{N}^{\mathbb{N}}$ contains a non-Borel co-analytic set, we can deduce that any complete co-analytic set A must be non-Borel. Thus a common strategy for proving a co-analytic set $A \subset X$ is non-Borel is to find a Borel map $f : \mathbb{N}^{\mathbb{N}} \to X$ so that its inverse image $f^{-1}(A)$ is a known non-Borel set. A common example to use for this purpose is the collection of wellfounded trees.

Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers (including the empty sequence). A tree T is a subset of $\mathbb{N}^*$ that is closed under removing the final element, i.e., if a finite sequence is in T, so is every initial segment, including the empty one. An infinite branch of T is an element of $\mathbb{N}^{\mathbb{N}}$, all of whose finite initial segments belong to T. The set of all branches is denoted $[T]$.

A tree is wellfounded if it has no infinite branches. Finite trees are obviously wellfounded, and the infinite set of finite sequences $(n, n-1, n-2, \dots, 1)$ with $n \in \mathbb{N}$, together with all initial segments of these sequences, form an infinite wellfounded tree. Since $\mathbb{N}^*$ is countable and a subset can be identified with its indicator function, any tree can be identified with a point of $2^{\mathbb{N}}$, i.e., the Cantor set of infinite binary sequence. In fact, the set of all trees corresponds to a closed subset of $2^{\mathbb{N}}$ with the usual metric, making it a Cantor set itself. However, the collection of wellfounded trees is co-analytic complete, and hence non-Borel, in this space. To prove this we will use the following.

Lemma 5.1. *Every closed set in $\mathbb{N}^{\mathbb{N}}$ is of the form $[T]$ for some tree T. For every analytic set $A \subset \mathbb{N}^{\mathbb{N}}$ there is a tree T so that $a = (a_1, a_2, \dots) \in A$ if and only if there is some $b = (b_1, b_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ so that*

$$W(a, b) = (a_1, b_1, a_2, b_2, \dots) \in [T].$$

Proof. The first part is straightforward. Let T be the tree of all finite initial segments of all elements in K. Using the definitions and the fact $\mathbb{N}^{\mathbb{N}}$ is complete, we see that K is closed iff the limit x of any Cauchy sequence in the set is also in the set iff for every n the sequence of initial n-segment eventually equals $(x_1, \dots, x_n)$ iff x is a branch of T.

To prove the second part, note that $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ by the 1-1, continuous map that interweaves sequences:

$$W : (a_1, a_2, \dots) \times (b_1, b_2, \dots) \mapsto (a_1, b_1, a_2, b_2, \dots).$$

Thus, A is the projection onto the first coordinate of the closed set $W^{-1}([T]) \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and hence A is analytic. Conversely, if A is analytic, then it is a continuous image $A = f(\mathbb{N}^{\mathbb{N}})$ and hence the projection of the closed set $(x, f(x)) \in \mathbb{N}^{\mathbb{N}} \times X$ (recall that graphs of continuous functions are closed sets). Taking T to correspond to this closed set gives the condition in the lemma. □

Lemma 5.2. *The wellfounded trees are co-analytic complete in $2^{\mathbb{N}}$.*

Proof. Suppose A is analytic in $\mathbb{N}^{\mathbb{N}}$. Then there is a tree T so that $a = (a_1, a_2, \dots) \in A$ iff $W(a, b) \in [T]$ for some $b = (b_1, b_2, \dots) \in \mathbb{N}^{\mathbb{N}}$. If we fix a, then the map $\mathbb{N}^{\mathbb{N}}$ to itself given by $b \mapsto W(a, b)$ is continuous. Since the inverse image of a closed set is closed, we see that $T(a) = \{b \in \mathbb{N}^{\mathbb{N}} : W(a,b) \in T\}$ is a closed set in $\mathbb{N}^{\mathbb{N}}$, and this corresponds to a tree by the previous lemma. We claim the mapping $\mathbb{N}^{\mathbb{N}} \to 2^{N^*}$ given by $a \mapsto T(a)$ is Borel. If this is true, then it suffices to show that the image of A^c under this map is the set of well founded trees (since A^C can be any co-analytic set in $\mathbb{N}^{\mathbb{N}}$). Note that a sequence $x \in A^c$ iff $W(a,b) \notin [T] \forall b \in \mathbb{N}^{\mathbb{N}}$. Thus $x \in A^c$ iff $T(a)$ is a wellfounded tree.

To check that the map $a \mapsto T(a)$ is Borel, we recall $T(a)$ is a closed set of sequences in $\mathbb{N}^{\mathbb{N}}$ and that a neighborhood of such a set is a countable union of basis elements where we specify a finite initial segment and allow the remaining elements to be free. The inverse image of one such basis element is the collection of all sequences a, so that (1) interweaving the initial elements of a with the specified elements of the basis gives a finite string in T and (2) so that there is some continuation of the specified elements to an infinite sequence so that interweaving is a branch of T. Thus a is simply the sequence of odd coordinates of all branches of T that pass through the specified vertex, and this is a closed set. Thus inverse images of open sets are countable unions of closed sets, so the mapping is Borel, as desired. □

Theorem 5.3 (Hurewicz, [28])**.** *The compact countable subsets of $I = [0,1]$ are co-analytic complete in 2^I.*

Proof. We have to construct a continuous map from the space of trees into 2^I, so that the image of T is countable if and only if T is wellfounded. For each $n = 1, 2, \dots,$ let $A_n = \{x \in [0,1] : \frac{1}{2n+1} \leq |x - \frac{1}{2}| \leq \frac{1}{2n}\}$. Then the A_n are all disjoint and each consist of two compact intervals. For any $S \subset \mathbb{N}$ define

$$A_S = \{\frac{1}{2}\} \bigcup \cup_{n \in S} A_n$$

This is a compact subset of $[0,1]$, and equals $\{1/2\}$ if and only if S is empty.

Suppose we are given a tree T. The root vertex (labeled by the empty string) is associated to $E_0 = I_\emptyset = [0,1]$. In general, Suppose E_n is a compact subset of $[0,1]$ whose connected components are a countable number of points labeled by strings of length $< n$, and a countable number of non-trivial closed intervals I_s labeled by strings of length n. All strings that occur as labels correspond to labels of vertices in level n of T, and for each such label, 2^n intervals in E_n will have that label. To construct E_{n+1} from E_n, we keep every point component from E_n (and leave the label the same) and replace each interval component J_s labeled by a string s of length n by $L_S(A_S)$, where S is the set of integers that can be appended to S to give a length $n+1$ string in T (i.e., these correspond to the edges leading out of vertex s), A_S is as above, and L_S is a linear map from J to J_s. Since each A_n consists of two intervals, each nth generation interval with a given label gives rise to two intervals in the next generation with identical labels. Let $E_T = \cap E_n$. Since the E_n are nested, this is a non-empty compact subset of $[0, 1]$.

If T as an infinite branch, then following this branch through the construction gives a Cantor subset of E, hence E is uncountable. Conversely, if E is uncountable, then $E \cap J_1$ must be uncountable for one of the countably many connected components of E_1. Then $E \cap J_2$ must be uncountable for one of the countably many components of E_2 contained in J_1. Continuing in this way, we obtain nested, non-degenerate components $J_1 \supset J_2 \supset J_3 \supset \dots$ whose labels form an infinite branch of T, so T is not wellfounded. □

The endpoints of all the components of E_n in the previous proof are rational numbers. Thus the sets E that arise from wellfounded trees are subsets of $\mathbb{Q}$, and we could reformulate the result to say that compact subsets of $\mathbb{Q} \cap I$ are co-analytic complete in 2^I. Theorem 5.3 also gives a rather concrete example of a non-Borel set in $[0, 1]$. Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ and for $K \in 2^I$ define

$$f(K) = \sum_{r_n \notin K} 3^{-n}.$$

Clearly f is 1-to-1 (since distinct sums of powers of 3 are distinct). The sets $\{K : f(K) > \alpha\}$ are easily checked be open in 2^I, so f is Borel. Thus

$$X = \{f(K) : K \subset \mathbb{Q} \cap [0, 1] \text{ and is compact } \} \subset [0, 1],$$

cannot be Borel.

6. A-removable sets are co-analytic complete

We start with a well known fact from complex analysis.

Lemma 6.1. *If $E \subset [0, 1]$ has positive length, then it is H^∞-non-removable.*

Proof. If E is an interval, then we take $E_2 = [0, 1]$ and apply the Riemann mapping theorem to get a non-constant bounded holomorphic function on the complement.

The general case was proven by Ahlfors and Beurling in [1] (or see Section I.6 of Garnett's book [20]):

$$F(z) = \int_E \frac{dz}{z-w} = \int_E \frac{t-x}{(t-x)^2+y^2} + i\int_E \frac{y}{(t-x)^2+y^2}$$

is holomorphic on $\Omega = E^c$, has imaginary part bounded by π and Laurent expansion $c_1 = \ell(E)$. Thus $G = \exp(F)$ takes values in the right half-plane, and $(G-1)/(G+1)$ maps Ω holomorphically into the disk and one can compute its Laurent coefficient $c_1 = \ell(E)/4$. □

Extending this result from subsets of $\mathbb{R}$ to subsets of graphs of real Lipschitz functions was a major breakthrough by Alberto Calderon which led to many important developments in in harmonic analysis and geometric measure theory over the last fifty years, including Tolsa's result, discussed in Section 2. For some of the related history, see [18], [40], [46], [47].

The following is stated and proved on page 117 of Carleson's 1951 paper [14]:

Theorem 6.2. *If $E_1, E_2 \subset [0,1]$ are compact and E_2 has positive Lebesgue measure then $E = E_1 \times E_2$ is A-removable iff E_1 is countable.*

Proof. For completeness, we recreate Carleson's proof, although we will only need the direction that E_1 countable implies E is removable, which we prove first.

Suppose f is continuous on the sphere and holomorphic off $E = E_1 \times E_2$. Then f is uniformly continuous on the whole sphere; thus its modulus of continuity

$$\omega(f,\delta) = \max_{|z-w|\le\delta} \max_{z,w\in Q} |f(z)-f(w)|$$

tends to zero uniformly with the diameter of Q. Near infinity $f(z) = c_0 + c_1/z + c_2/z^2 + \dots$.

Fix $\epsilon > 0$. If E_1 is countable, enumerate it as $\{x_n\}$ and for each n choose a dyadic interval I_n that contains x_n and so that the variation of f over I_n is less than $\epsilon 2^{-n}$. A countable number of such intervals cover E_1 and for each I_n used in this cover, $O(1/||I_n|)$, squares of diameter $|I_n|$ suffice to cover $(E_1 \cap I_n) \times E_2$ (in fact, this suffices to cover $E_1 \cap I_n \times [0,1]$). Doing this for each I_n gives a covering of E by squares $\{Q_k\}$

Near infinity $f(z) = c_0 + c_1/z + c_2/z^2 + \dots$. Let z_Q be the center of the square Q. Using the Cauchy integral formula

$$\begin{aligned} |c_1| = \frac{1}{2\pi}\sum_Q \int_{\partial Q} |f(z) - f(z_Q)||dz| &= O\left(\sum_Q \text{diam}(Q)\omega(f, \text{diam}(Q))\right) \\ &= O\left(\sum_{n>0} \omega(f, 2\text{diam}(I_n))\right) \\ &= O\left(\epsilon \sum_{n>0} 2^{-n}\right). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, c_1=0. But the same argument applies to $f_1 = z(f(z)-c_0) = c_1 + c_2/z + \dots$ to show $c_2 = 0$. Continuing in this way, we see f is constant, and hence E is removable.

Conversely if E_2 has positive length, then there is a non-constant bounded analytic function f on the complement of iE_2 (see remark following this proof). If E_1 is uncountable, then it supports a non-atomic, positive, finite measure μ. Then $F(z) = \int f(z+x)d\mu(x)$ is continuous on the sphere and holomorphic off $E = E_1 \times E_2$. We may assume $c_1 \neq 0$ (otherwise recursively replace f by $z(f(z)-c_0)$ until this happens) and the fact

$$\frac{1}{z-x} = \frac{1}{z} + (\frac{1}{z+x} - \frac{1}{z}) = \frac{1}{z} + \frac{x}{z^2},$$

implies F also has non-zero Laurent coefficient and hence non-constant. Hence E is non-removable. □

Corollary 6.3. *The A-removable compact subsets of $S = [0,1]^2$ are co-analytic complete in 2^S, hence not Borel.*

Proof. We already know this set is co-analytic by Corollary 3.5. To prove completeness, note that the mapping $E \mapsto E \times [0,1]$ is continuous between the respective Hausdorff metrics and hence reduces the set of countable compact subsets of $[0,1]$ to the set of A-removable sets. Since the former is co-analytic complete, so is the latter. □

7. A-removable Jordan curves are co-analytic complete

A case of particular interest among compact planar sets are the closed Jordan curves. Let $\mathrm{Homeo}(X,Y) \subset C(X,Y)$ denote the 1-to-1 continuous maps of X into Y. It is easy to see that this subset is neither open nor closed in $C(X,Y)$. However, a map $f : \mathbb{T} \to \mathbb{C}$ is 1-to-1 if and only if any two disjoint closed dyadic intervals have disjoint images (an open condition) and hence $\mathrm{Homeo}(\mathbb{T},\mathbb{C})$ is a G_δ set in $C(\mathbb{T},\mathbb{C})$.

We can think of closed Jordan curves elements of $\mathrm{Homeo}(\mathbb{T},\mathbb{C})/\mathrm{Homeo}(\mathbb{T},\mathbb{T})$, i.e., modulo re-parameterizations. Thus $f, g \in \mathrm{Homeo}(\mathbb{T},\mathbb{C})$ are equivalent if $f = g \circ \rho$ for some $\rho \in \mathrm{Homeo}(\mathbb{T},\mathbb{T})$. We can define a metric between equivalence classes as

$$d([f],[g]) = \inf\{\|f - g \circ \rho\|_\infty : \rho \in \mathrm{Homeo}(\mathbb{T},\mathbb{T})\},$$

although Jordan curves are not complete in this metric. A complete metric on Jordan curves separating 0 and ∞ is described by Pugh and Wu in [41], where they attribute the idea to Thurston (one takes conformal maps of $\mathbb{S} \setminus \mathbb{T}$ to $\mathbb{S} \setminus \Gamma$ normalized to fix 0 and ∞ respectively and have positive derivative at these points, and then use the supremum metrics between conformal maps).

Theorem 7.1. *The collection of A-removable Jordan curves contained in $S = [0,1]^2$ is co-analytic complete in 2^S.*

Proof. We will construct a continuous map from trees into Jordan curves. As in earlier arguments, it suffices to show that the preimage of the removable curves is precisely the set of wellfounded trees.

To simplify some formulas, we work in $[-1,1]^2$ instead of $[0,1]^2$. We start with a map from trees to compact subsets of $[-1,1]$ that maps wellfounded trees into countable sets, using a slightly different map than we did in the proof of Theorem 5.3. For $S \subset \mathbb{N}$, we define

$$A_n = \{x : \frac{1}{4} + \frac{1}{2n+1} \le |x| \le \frac{1}{4} + \frac{1}{2n}\},$$

and

$$A_S = \{\pm\frac{1}{4}\} \bigcup \bigcup_{n \in S} A_n \subset [-1,1].$$

This is similar to what we did before, except that now the pairs of intervals A_n converge to two different points $\pm 1/4$, instead of a single point. However, the rest of the construction is the same and associates to each tree T a compact set E_T that is countable if and only if T is wellfounded. Recall that each string s of length n is associated to 2^n intervals which we label I_s^j, $j = 1, \dots 2^n$. We assume these are numbered left to right.

Next we construct a Cantor set $K = \cap_n K_n \subset [-1, 1]$ of positive Lebesgue measure where each K_n is a union of 2^n equal length closed intervals which we denote K_n^k, $k = 1, \dots, 2^n$. We assume the components are numbered left to right.

For a string $s = s_1, \dots s_n$ let $N(s) = s_1 + \cdots + s_n$. Each such string can be associated to $2^{n+N(s)}$ closed rectangles $R_s^{j,k} = I_s^j \times K_{N(s)}^k$ with $1 \leq j \leq 2^n$ and $1 \leq k \leq 2^{N(s)}$. For an non-empty string $s = s_1, \dots s_n$ we will often drop the j, k and let R_s denote one of the rectangles $R_s^{j,k}$. The empty string is associated to a single rectangle $R_\emptyset = [-1, 1]^2$.

Each rectangle R_s has four marked points on its boundary: two on the left vertical side and two on the right vertical side. The exact placement is unimportant, but in Figure 1 I have drawn these points to be symmetric with respect to the midpoints of these sides and the same height on both sides. As in Figure 1 we connect the upper left point for $R_\emptyset$ to the upper left of $R_1^{1,2}$, the lower left point for $R_\emptyset$ to the lower left point for $R_1^{1,1}$. The points on the right side of $R_\emptyset$ are connected to the corresponding points on the right sides of $R_1^{2,1}$ and $R_1^{2,2}$, as shown in the figure. The lower left point of $R_1^{1,2}$ is connected to the lower right point point of $R_1^{2,2}$, and similarly the upper point of the left side of $R_1^{1,1}$ is connected to the upper point of the right side of $R_1^{2,1}$.

In general, each rectangle R of the form $R_s^{j,k}$ is immediately to the left or right of two rectangles R', R'' of the form $R_{s'}^{p,q}$, where the last element of s' is one larger than the last element of s. Suppose R is to the left of R' and R'' and R' is above R''. Then we connect the upper right point of R to the upper left point of R', and connect the lower right point of R to the lower left point of R''. Then the lower left point of R' is connected to the upper right point of R''.

We continue inductively over all n and add the set $\{-\frac{1}{4}, \frac{1}{4}\} \times K$. We obtain the picture in Figure 1: the union of the rectangles, Jordan arcs and two copies of K form two connected sets. We can connect these two sets by adding the vertical segments on the left and right sides of $R_\emptyset$ that connect the upper and lower points on these sides. This picture is the basic template for the construction.

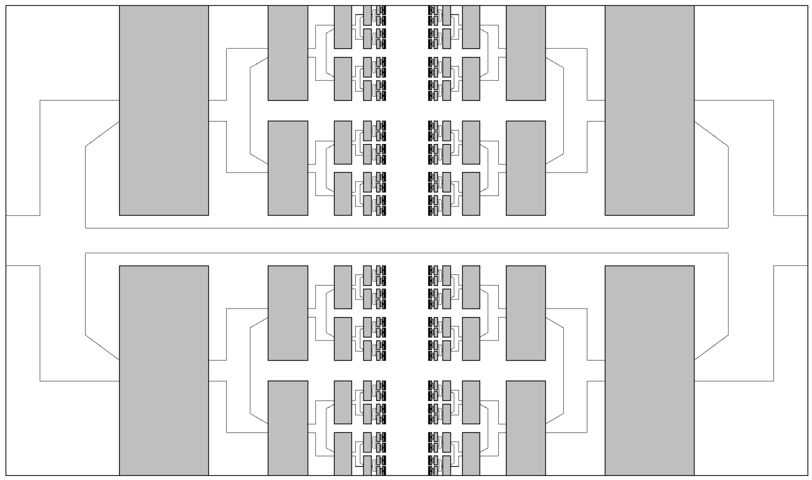

FIGURE 1. Each vertex of T labeled by a string s of length n is associated to $2^{n+N(s)}$ rectangles. Here we show all the rectangles for strings of length 1, with $R_1^{j,k}$ being the four "outermost" ones, $R_2^{j,k}$ the eight rectangles "adjacent" to these, $R_3^{j,k}$ the next sixteen, and so on. As $n \nearrow \infty$, these rectangles accumulate on two vertical copies of K, which will each lie on the curve we are constructing.

In the general case, suppose that we have a compact set that is the union of string indexed rectangles, Jordan arcs connecting rectangles and Cantor sets. Suppose each rectangle is indexed by a finite sequence of the form $s = s_1, \dots, s_n, m$, where $s' = s_1 \dots, s_n$ labels a vertex in the tree T. Then the longer sequence s is either in the tree or it is not. If it is not, then the rectangle R_s is replaced by two horizontal line segments that connect the two upper points of R_s and the two lower points. The curve in this rectangle is now finished and will not be altered at later stages. For example, if the tree consists of just the one vertex labeled by the emptyset, then we add these horizontal segment to every rectangle of the form $R_s^{j,k}$ were $s = 1, 2 \dots$ is a string of length 1. The result is the curve in Figure 2, the simplest curve in our family.

If the vertex s is in the tree, then we replace each rectangle $R_s^{j,k}$ by a template as in Figure 1. The horizontal direction of the template is scaled linearly, but the vertical direction is not quite because when we split a component interval of K_n into

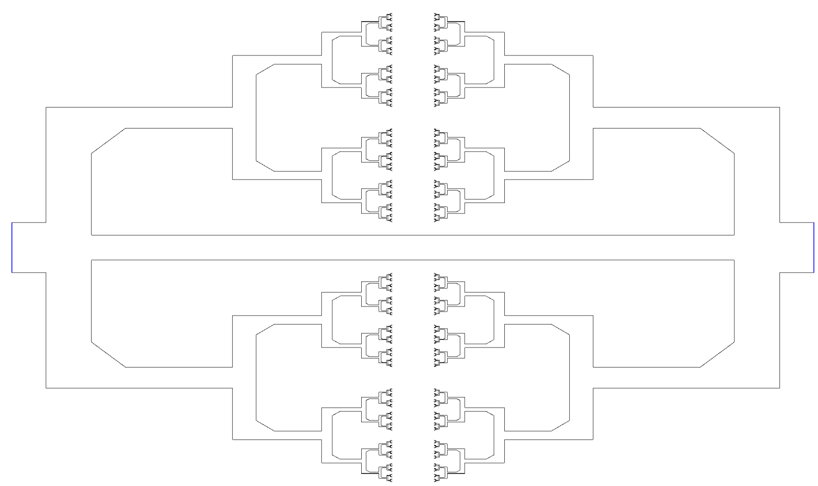

FIGURE 2. The curve corresponding to the 1-vertex tree. This is a countable union of line segments and two linear Cantor sets and hence is A-removable. It is the "simplest" curve in our collection.

two components of K_{n+1}, the gap between them is not a fixed fraction of the interval (this would lead to a zero length Cantor sets), but is a smaller fraction at each stage, so the template has to be adjusted to accordingly. Figure 3 shows what happens when we replace each of the four rectangles of the form $R_1^{j,k}$ with the appropriate template, and Figure 4 shows the curve assuming this is the only time the template is used (this corresponds to the tree with two vertices $\{\emptyset, 1\}$. Figure 5 shows the tree for a tree with three vertices $\{\emptyset, 1, 2\}$ (curves for larger trees get harder to draw and to see).

If T is wellfounded, then the final curve is countable union of line segments and linear Cantor sets and hence is A-removable by one direction of Carleson's theorem. If T has an infinite branch then the curve contains a copy of $E \times K$, where E is a Cantor set depending on the branch, and thus it is non-A-removable by other direction of Carleson's theorem.

The map from trees to curves is continuous from the product topology to the Hausdorff metric because if two trees have the same set of vertices in $[1, \dots, N]^N$ then the two curves will agree except on a union of rectangles with small diameter (tending uniformly to zero with N) and each contains at least one point inside each

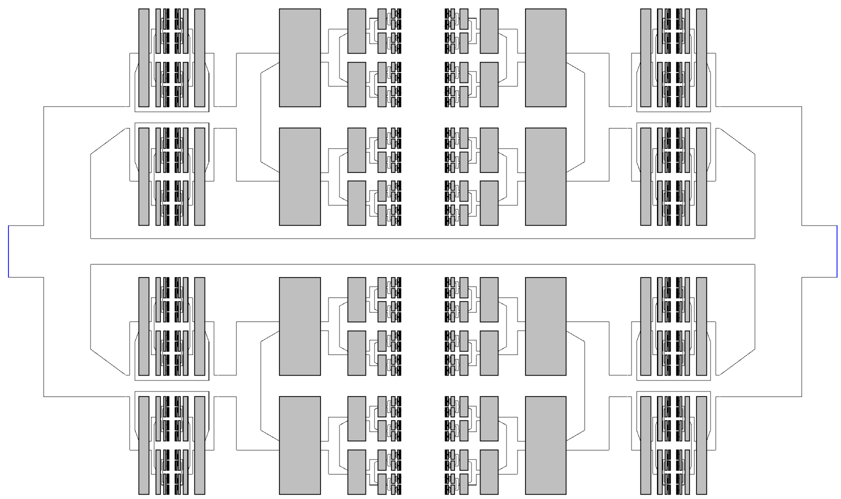

FIGURE 3. The four rectangles of the form $R_1^{j,k}$ have been replaced by a second stage template. Any curve containing the vertices $\{\emptyset, 1\}$ will contain these arcs. The minimal such tree is shown in Figure 4.

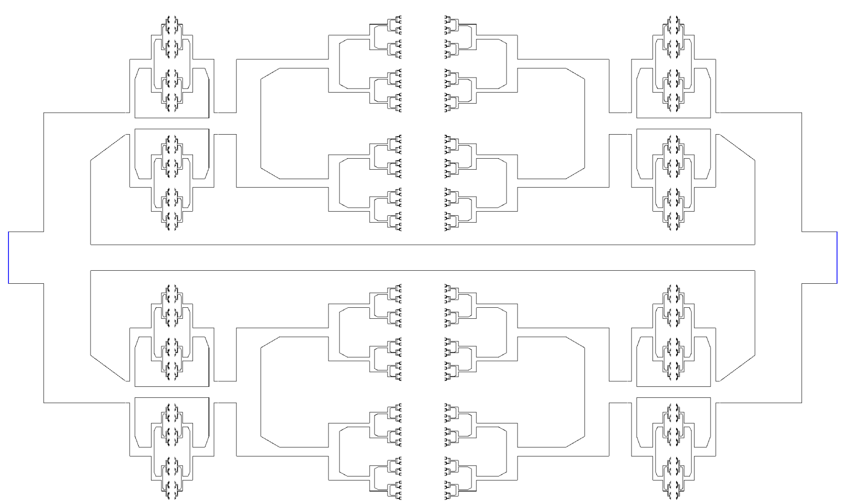

FIGURE 4. The curve corresponding to the tree with vertices $\{\emptyset, 1\}$.

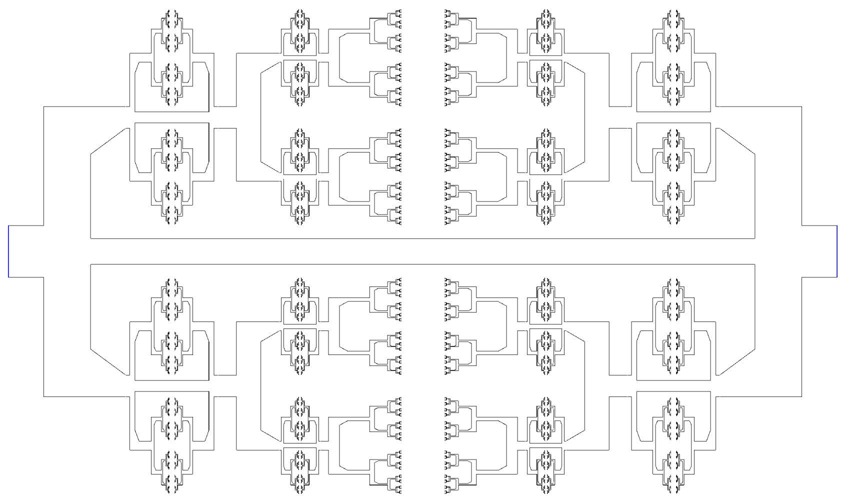

FIGURE 5. The curve corresponding to the tree with vertices $\{\emptyset, 1, 2\}$. There are countably many segment and 10 copies of the linear Cantor set K.

of these rectangles; thus the curves are close in the Hausdorff metric. Thus the set of wellfounded trees is the exact preimage of the set of A-removable curves under a continuous map from trees into the hyperspace of $[-1, 1]$. Thus this collection of A-removable curves is co-analytic complete, hence non-Borel. □

8. CH-removable sets are co-analytic complete

The following is due to Fred Gehring [22] in 1960. We include a proof for the reader's convenience.

Lemma 8.1. *For compact sets $E \subset [0, 1]$, $E \times [0, 1]$ is CH-non-removable if and only if E is uncountable.*

Proof. If E is compact and uncountable then it supports positive, finite, non-atomic measure μ. By restricting μ to an appropriate subset set E_0 of zero Lebesgue measure and multiplying by an appropriate constant we may assume μ is singular to Lebesgue measure, is supported in an interval $J = [a, b] \subset [0, 1]$, has total mass equal to half the length of J. Fix a constant $c \in [0,1]$ and define $h_c(x) = x$ outside J and

$$h_c(x) = x + c\left(\int_0^x d\mu(t) - \frac{x-a}{2}\right),$$

inside J. It is easy to check this is a homeomorphism that is linear with slope $1 - \frac{c}{2}$ on each component of $J \setminus E_0$. On the other hand, it maps E_0 to a set of length $c\ell(J)/2 > 0$.

Let $g(y) = \max(0, \frac{1}{2} - |x - \frac{1}{2}|)$ and define

$$F(x,y) = (h_{g(y)}(x), y).$$

See Figure 6. This is a homeomorphism of the plane that is the identity off $J \times [0,1]$, and for any component K of $J \setminus E_0$ F is a skew linear map on $J \times [0, \frac{1}{2}]$ and $J \times [\frac{1}{2}, 1]$ with uniformly bounded dilatation. Thus F is quasiconformal off $E_0 \times [0,1]$. It is not quasiconformal on the whole plane because the zero length set $E_0 \times \{y\}$ is mapped to a set of positive length for each $0 < y < 1$, and thus $E_0 \times [0,1]$ is a set of zero area that is mapped to positive area; this is impossible for quasiconformal maps, see e.g., [2]. Using the measurable Riemann mapping theorem we can find a quasiconformal mapping φ of the whole plane so that $\varphi \circ F$ is conformal off $E \times [0,1]$ but not quasiconformal everywhere, hence not conformal everywhere. Thus $E \times [0,1]$ is CH-non-removable.

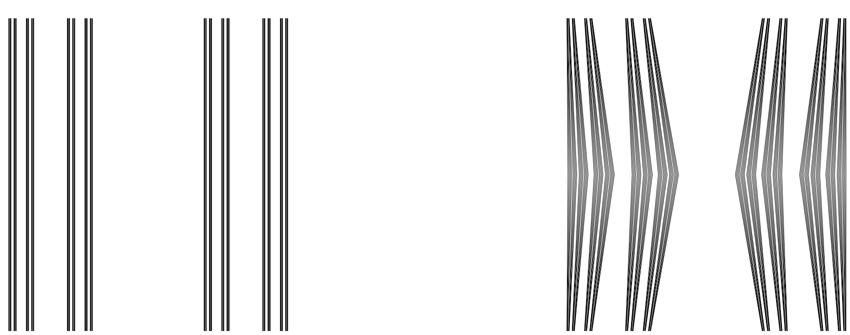

FIGURE 6. If E is a Cantor set there is homeomorphism h of $\mathbb{C}$ that is quasiconformal off $E \times [0,1]$ and maps $E \times [0,1]$ to a set of positive area. This can't happen if E has zero length and h is quasiconformal on the whole plane.

If E is C-non-removable with witness f and $z_0 \notin E$, then $g(z) = f(z) - f(z_0)/(z - z_0)$ continuous and bounded on the plane and holomorphic off E, so E is also A-non-removable. If E is compact and countable, then Carleson's Theorem 6.2 show that $E \times [0,1]$ is A-removable, and the previous sentence implies E is CH-removable. □

Corollary 8.2. *CH-removable sets in $S = [0,1]^2$, are co-analytic complete in 2^S.*

The proof is the same as for A-removable sets, except using Gehring's result in place of Carleson's. On the other hand, I have been unable to give an analogous construction to Theorem 7.1:

Question 1. *Are the CH-removable curves co-analytic complete?*

One approach to this question would be to use a theorem of Robert Kaufman [31], who proved that whenever $E \subset [0,1]$ is compact and uncountable, $E \times [0,1]$ contains the graph of a continuous function f defined on E that is a CH-non-removable set. Extending f to be continuous on $\mathbb{R}$ and linear on the complementary intervals of E, gives a graph that is Jordan curve containing a CH-nonremovable graph, and hence is non-removable itself. Thus we might try to prove CH-removable curves are co-analytic complete by mapping trees to graphs of continuous functions (instead of product sets) and using Kaufman's theorem (instead of Carleson's or Gehring's). However, I have not seen how to make this work.

The difficulty is that Kaufman's construction starts by choosing a positive, non-atomic measure μ (all points have mass zero) on the uncountable set $E \subset I = [0,1]$. So it seems that we need a Borel map from trees to probability measures so that the non-wellfounded trees are the preimage of the non-atomic measures. However, it is easy to see that the non-atomic measures are co-analytic in $P([0,1])$, so such a map is impossible, since the preimage of a co-analytic set under a Borel map must be co-analytic. The space $P(I)$ of probability measures on $I = [0,1]$ can be made into a Polish space using the dual Lipschitz metric

$$d(\mu, \nu) = \sup_f |\int f d\mu - \int f d\nu|,$$

where the supreumum is over all 1-Lipschitz functions. This metrizes the weak* topolgy on measures; see Appendix A.3 of [11].

Question 2. *Are CH-removable continuous graphs co-analytic complete in 2^S?*

We also recall some questions from the introduction:

Question 3. *Do CH-removalble sets form a σ-algegra? Is the union of two CH-removable sets removable?*

9. How hard is conformal welding?

Another problem that seems ripe for the descriptive set theory treatment is conformal welding. If Γ is a closed Jordan curve in the plane the Riemann mapping theorem gives conformal maps f, g from the inside and outside of the unit circle to the inside and outside of Γ. By Carathéodory's theorem these maps extend to be homeomorphisms of $\mathbb{T}$ to Γ (this was actually first proven by his student Marie Torhorst in her 1918 doctoral dissertation using Carathéodory's theory of prime ends, so perhaps it is more appropriate to call it the Carathéodory-Torhorst theorem). Thus $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$ is a homeomorphism, and circle homeomorphisms that arise in this way are called conformal weldings.

Not every homeomorphism is a welding. Consider the graph of $\sin(1/x)$ for $x \neq 0$, together with the limiting segment $[-i, i]$. See Figure 7. This is closed set X dividing the plane into two simply connected domains and one can show that the conformal maps form either side of $\mathbb{T}$ to either side of X still define a circle homeomorphism h. However, h cannot correspond to any Jordan curve Γ; if it did, one could conformally map the two sides of X to the two sides of Γ so that the maps agree along the graph of $\sin(1/x)$. Since this smooth curve is removable for conformal homeomorphisms the map extends to be conformal from the complement $[-i, i]$ to the complement of a point. Since the complement of the segment is conformally equivalent to the unit disk, we would get conformal map between the disk and the plane, which would violate Liouville's theorem. Thus this homeomorphism is not a conformal welding.

It is a long standing, and apparently very difficult, problem to characterize conformal weldings among circle homeomorphisms. We explained in Section 7 that circle homeomorphisms are a G_δ set in $C(\mathbb{T}, \mathbb{T})$, and hence a Polish space.

Question 4. *Are conformal weldings Borel in the space of circle homeomorphisms?*

It's not hard to show

Lemma 9.1. *The set of conformal weldings is analytic in* $\mathrm{Homeo}(\mathbb{T}, \mathbb{T})$

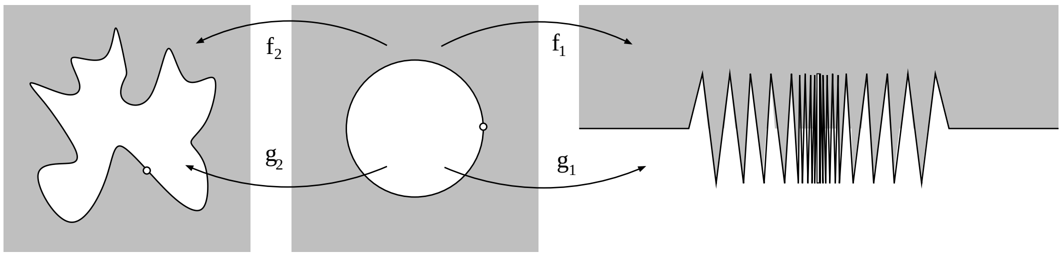

FIGURE 7. If f_1, g_1 map the two sides of $\mathbb{T}$ to the two sides of a $\sin(1/x)$ curve γ, then $h = g_1^{-1} \circ f_1$ is a homeomorphism, but is not a conformal welding. Otherwise, $h = g_2^{-1} \circ f_2$ with maps corresponding to a Jordan curve, and then (by Morera's theorem) $f_2 \circ f_1^{-1}$ and $g_2 \circ g_1^{-1}$ would define a conformal map from the complement of a segment to the complement of a point, contradicting Liouville's theorem.

Proof. Briefly, each closed curve gives a conformal welding and the mapping is continuous, so conformal weldings are the continuous image of a Polish space, hence analytic. However, there are actually a family of conformal weldings associated to each curve, so we have to be slightly more careful.

For each 1-to-1 map $\gamma : \mathbb{T} \to \mathbb{C}$ (the parameterization of a closed Jordan curve), let $\mathcal{F}_\gamma \subset C(\mathbb{C}, \mathbb{C})$ be the homeomorphisms of the plane that are holomorphic on $\mathbb{D}$ and map $\mathbb{T}$ to $\Gamma = \gamma(\mathbb{T})$, and let $\mathcal{G}_\gamma \subset C(\mathbb{C}, \mathbb{C})$ be the homeomorphisms that are analytic outside Γ and map Γ to $\mathbb{T}$. Then, since uniform limits of holomorphic functions are holomorphic, $\{(\gamma, \mathcal{F}_\gamma, \mathcal{G}_\gamma)\}$ is a closed set inside the product $\mathrm{Homeo}(\mathbb{T}, \mathbb{C}) \times \mathrm{Homeo}(\mathbb{C}, \mathbb{C}) \times \mathrm{Homeo}(\mathbb{C}, \mathbb{C})$. Map this closed set into $\mathbb{T} \times \mathbb{T} \times \mathbb{T}$ by $(\gamma, f, g) \mapsto (z, f(\gamma(z)), g(\gamma(x)))$. The projection of the image onto the latter to coordinates is the graph of a conformal welding homeomorphism, and every welding occurs for some choice of (γ, f, g) so the set of conformal weldings is the continuous image of a closed set in a product of Polish spaces, hence is analytic. $\square$

The best known sufficient condition for being a conformal welding (due to Beurling and Ahlfors [7]) is quasisymmetry: h is M-quasisymmetric if

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M,$$

whenever I, J are adjacent arcs on $\mathbb{T}$ of the same length, and $|I|$ denotes the length of an arc. For a fixed M, this is clearly a closed condition, so taking $M \to \infty$ along the integers shows quasisymmetric homeomorphisms are a F_σ set inside Homeo$(\mathbb{T},\mathbb{T})$. Quasisymemtric welding correspond precisely to closed curves that are quasicircles. i.e., images of the unit cirlce under quasiconformal maps of the plane. There are numerous characterizations of this class of curves, e.g., any two points $z, w \in \gamma$ are connected by a subarc with diameter boundedby by $O(|z-w|)$. See [2]. It is easy to see M-quaissymmetric maps are nowhere dense, so the set of quasisymmetric homeomorphisms is first category in the space of all circle homeomorphisms.

A more obscure sufficient condition is that h be log-singular, i.e., that there exist a set $E \subset \mathbb{T}$ of logarithmic capacity zero so that $\mathbb{T} \setminus f(E)$ also has logarithmic capacity zero. In [10] it is proven that h is log-singular if and only if the curve is flexible; this implies that the set of curves corresponding to h is dense in the space of all closed curves with the Hausdorff metric. See [10] for the precise definition. Quasisymmetric and log-singular circle homeomorphisms are easily seen to be disjoint sets (QS homeomorphisms preserve sets of zero logarithmic capacity).

If γ is a closed curve with complementary components Ω_1, Ω_2, we say $x \in \gamma$ is rectifiably accessible from Ω_k, $k = 1, 2$ if it is the endpoint of a rectifiable curve in Ω_k. By a result of Gehring and Hayman ([23] or Exercise III.16 of [21]) this occurs iff the hyperbolic geodesic ending at x has finite Euclidean length. A result of Charles Pugh and Conan Wu [41] says there is a residual set of closed curves γ so that no point on γ is rectifably accesible from both sides at once. In their terminology, γ in not pierced by any rectifiable arc. By a result of Beurling the set of points that are not rectifiably accessible from Ω_k, $k = 1, 2$ is the image of zero logarithmic capacity set on $\mathbb{T}$ under any conformal map $\mathbb{D} \to \Omega_k$; see [6], Exercise III.23 of [21], or [4]. If follows that every curve in this G_δ set has a conformal welding that is log-singular. Thus

Theorem 9.2. *The collection of CH-non-removable closed curves is residual in the space of all closed Jordan curves.*

Question 5. *Is the set of log-singular homeomorphisms residual in the space of all circle homeomorphisms?*

Question 6. *What is the Borel complexity of the log-singular homeomorphisms?*

It is not hard to show that they are at least analytic: h is log-singular if for every $n \in \mathbb{N}$ there is a compact set such that both E and $h(E^c)$ have logarithmic capacity less than $1/n$ (Lemma 11 of [10]). Thus the log-singular maps are a countable intersection of projections of the Borel sets $\{(h, E) : \text{cap}(E), \text{cap}(h(E^c)) < 1/n\}$ in $\text{Homeo}(\mathbb{T}, \mathbb{T}) \times 2^{\mathbb{T}}$. Can analytic be improved to Borel? Note that this would be true if the map from curves to weldings was 1-to-1 (bijective continuous images of Polish spaces are Borel). However, it is well known that this map is not 1-to-1.

Given a curve Γ the conformal maps of either side of Γ to either side of $\mathbb{T}$ can be post-composed with Möbius transformations fixing the unit disk. Thus the conformal weldings should really be considered as equivalence classes, modulo pre- and post-composition with such transformations. Similarly, any curve Γ has the same weldings as any Möbius image of Γ, so curves should be considered modulo Möbius transformations. If the mapping from equivalence classes of curves to equivalence classes of conformal weldings was 1-to-1, then conformal weldings would be Borel. Unfortunately, this is not the case.

For brevity, we say Γ' is a CH-image of Γ if $\Gamma' = f(\Gamma)$ where f is a homeomorphism of the sphere that is conformal off Γ and is a strict-CH-image if f is not a Möbius transformation. Saying Γ is C-non-removable means that strict-CH-images exist. Any CH-image of a curve has the same welding as that curve. Thus the correspondence is 1-to-1 if and only if the only CH-images of a curve are also Möbius images. This is far from true. For example, a curve Γ whose conformal welding is log-singular is a "flexible curve" in the sense that its not CH-images are dense in the space of all closed Jordan curves (and hence there must be non-Möbius images). Examples are constructed in [9], [10]. Thus not only is the map from curves to weldings not 1-to-1, there is a dense set of weldings, each of which has a dense pre-image. This also indicates that the set theoretic complexity of conformal weldings is intimately related to that of CH-removable curves.

Question 7. *Is the map from (equivalence classes of) curves to (equivalence classes of) conformal weldings* 1-*to*-1 *exactly on the* CH*-removable curves?*

It is very tempting to say the answer is obviously yes. So tempting that many authors (including myself) have erroneously stated it as a fact; see Malik Younsi's paper [48] for a list of some of these transgressions. However, Maxime Fortier Bourque pointed out that the image of Γ under a non-Möbius homeomorphism of the sphere might coincidentally agree with its image under some Möbius map. Moreover, Malik Younsi [49] has constructed a curve with a strict-CH-image that agrees with itself. In Younsi's example, there are other strict-CH-images that are not Möbius images, so the answer to Question 7 might still be yes. His example suggests we restate the question as:

Question 8. *Does a non-CH-removable curve always have a CH-image that is not a Möbius image?*

I expect this is true. The following is a stronger version.

Question 9. *Does every CH-non-removable curve have a CH-image of positive area?*

Some other related problems are:

Question 10. *Is the map from equivalence classes of curves to equivalence classes of weldings always either* 1*-to-*1 *or uncountable-to-*1*?*

Question 11. *Are CH-images of a curve a connected set in the Hausdorff metric?*

Question 12. *The CH-images of a flexible curve are dense in the space of closed Jordan curves, and hence are not a closed set. Is it Borel? (It must be analytic.)*

In [10] it is proven that every circle homeomorphism h and every $\epsilon > 0$, there is a conformal welding that agrees with h except on a set of Lebesgue measure less than ϵ. Thus conformal weldings are dense in all circle homeomorphisms in a very strong way. It this related to their Borel complexity? [10] also shows that given any circle homeomorphism there is an explicit way to construct conformal maps $\{f_n\}$, $\{g_n\}$ of $\{|z| < 1\}$ and $\{|z| > 1\}$ onto disjoint Jordan domains so that $f_n(x) - g_n \circ h(x)$ tends to zero everywhere except on a countable set. What can we say about the homeomorphisms where this difference tends to zero everywhere? It contains all conformal weldings, but what else does it contain? What is the Borel complexity of this set of homeomorphisms?

Yet another sufficient condition for being a welding map is given in Guy David's paper [17]. Roughly, it says that h is a welding if it has diffeomorphic extension H to the disk whose dilatation $\mu = H_{\overline{z}}/H_z$ satisifes $|\mu| > 1 - \epsilon$ only on a set of area

$O(\exp(-O(1/\epsilon)))$. These are also called trans-quasiconformal homemorphisms. Are these a Borel subset of all circle homeomorphisms?

10. What are natural ranks for removable sets?

This section requires greater familiarity with the transfinite ordinals than did earlier sections. Very briefly, each ordinal is a well ordered set (each element has a successor, although some elements have no predecessor). The ordinals themselves are well ordered and there is a first well ordering of an uncountable set, which is denoted ω_1. Every ordinal that becomes before ω_1 is, by definition, the well ordering of some countable set. The continuum hypothesis is the claim that $\omega_1 = c$, where c is the cardinality of $\mathbb{R}$, and is well known to be independent of ZFC.

If X is Polish and $A \subset X$ is co-analytic, then there always a co-analytic rank on A. This is a function ρ on X that assigns each each point of X to some ordinal $\leq \omega_1$ and such that

(1) $A = \{x \in X : \rho(x) < \omega_1\}$,
(2) $\{(x, y) \in A \times A : \rho(x) < \rho(y)\}$ is co-analytic in $X \times X$,
(3) $\{(x, y) \in A \times A : \rho(x) \leq \rho(y)\}$ is co-analytic in $X \times X$.

Given such a function ρ, one can show that for every countable ordinal α, every set $A_\alpha = \{x \in A : \rho(x) \leq \alpha\}$ is a Borel set and every analytic subset of A is contained in some A_α. Moreover, A is Borel if and only if every co-analytic rank is bounded above by some countable ordinal on A.

The standard example (dating back to Cantor and motivating his invention of transfinite ordinals) involves the derived sets of a compact set in $\mathbb{R}$. Given a compact K, the derived set K' is K with it isolated points removed; this is a compact subset of K, with at most countably many point removed. If K was finite then $K' = \emptyset$, and otherwise we can repeat the process to get the second derived set K''. Continuing, we get a nested sequence of sets that either becomes empty after $n < \infty$ steps (in which case we set $\rho(K) = n$) or we get an infinite, strictly decreasing sequence of nested compact sets whose intersection is a non-empty compact set K^ω. If the derived set of K^ω is empty, then set $\rho(K) = \omega$, and otherwise continue as before. We proceed with this using transfinite induction. If K is countable, then since we remove at least one point at each stage, we must reach the empty set at some countable ordinal, and take this ordinal to be the rank of K. Since we remove only countably many points at each stage, starting with an uncountable sets never gives the empty set at any countable ordinal. For such sets the rank is defined to be ω_1. This defines a rank for the co-analytic set of countable, compact subsets of $[0, 1]$.

In [34] Kechris and Woodin describe a natural rank on the set of everywhere differentiable functions in $C([0, 1])$. See also [35], [36], [42], for comparisons between their rank and other ranks on the same set. A thesis of [34] is that "natural" co-analytic sets should have natural ranks.

Question 13. *What is a natural rank on the space of conformally removable sets?*

For the special case of product sets $E \times [0, 1]$ with E countable, we can just take the usual rank on countable compact sets described above.

Question 14. *Can the derived set rank on* $E \times [0, 1]$ *be extended to a co-analytic rank on all removable sets in* $[0, 1]^2$*?*

The standard rank on the product set removes isolated connected components at each stage, but what happens when we add a horizontal line segment connecting all the components? This augmented set does not seem much more complicated than the original; should it have the same rank?

References

[1] Lars Ahlfors and Arne Beurling. Conformal invariants and function-theoretic null-sets. *Acta Math.*, 83:101–129, 1950.

[2] Lars V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard.

[3] Miklos Ajtai and Alexander S. Kechris. The set of continuous functions with everywhere convergent Fourier series. *Trans. Amer. Math. Soc.*, 302(1):207–221, 1987.

[4] Zoltan Balogh and Mario Bonk. Lengths of radii under conformal maps of the unit disc. *Proc. Amer. Math. Soc.*, 127(3):801–804, 1999.

[5] Howard Becker. Descriptive set-theoretic phenomena in analysis and topology. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 1–25. Springer, New York, 1992.

[6] Arne Beurling. Ensembles exceptionnels. *Acta Math.*, 72:1–13, 1940.

[7] Arne Beurling and Lars Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.

[8] Christopher J. Bishop. Constructing continuous functions holomorphic off a curve. *J. Funct. Anal.*, 82(1):113–137, 1989.

[9] Christopher J. Bishop. Some homeomorphisms of the sphere conformal off a curve. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 19(2):323–338, 1994.

[10] Christopher J. Bishop. Conformal welding and Koebe's theorem. *Ann. of Math. (2)*, 166(3):613–656, 2007.

[11] Christopher J. Bishop and Yuval Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.

[12] Benoît Bossard. A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces. *Fund. Math.*, 172(2):117–152, 2002.

[13] Andrew M. Bruckner, Judith B. Bruckner, and Brian S. Thomson. *Real Analysis, Second Edition*. ClassicalRealAnalysis.com, 2008.

[14] Lennart Carleson. On null-sets for continuous analytic functions. *Ark. Mat.*, 1:311–318, 1951.

[15] Roger Cooke. Uniqueness of trigonometric series and descriptive set theory, 1870–1985. *Arch. Hist. Exact Sci.*, 45(4):281–334, 1993.

[16] Joseph W. Dauben. The trigonometric background to Georg Cantor's theory of sets. *Arch. History Exact Sci.*, 7(3):181–216, 1971.

[17] Guy David. Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 13(1):25–70, 1988.

[18] Guy David. Analytic capacity, Calderón-Zygmund operators, and rectifiability. *Publ. Mat.*, 43(1):3–25, 1999.

[19] Sergei S. Demidov and Boris V. Lëvshin, editors. *The case of academician Nikolai Nikolaevich Luzin*, volume 43 of *History of Mathematics*. American Mathematical Society, Providence, RI, 2016. Translated from the 1999 Russian original by Roger Cooke, Research and commentary by N. S. Ermolaeva (Minutes), A. I. Volodarskii and T. A. Tokareva (Appendices).

[20] John Garnett. *Analytic capacity and measure*. Lecture Notes in Mathematics, Vol. 297. Springer-Verlag, Berlin-New York, 1972.

[21] John B. Garnett and Donald E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2005.

[22] F. W. Gehring. The definitions and exceptional sets for quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 281:28, 1960.

[23] F. W. Gehring and W. K. Hayman. An inequality in the theory of conformal mapping. *J. Math. Pures Appl. (9)*, 41:353–361, 1962.

[24] Szymon Głąb. On the complexity of continuous functions differentiable on cocountable sets. *Real Anal. Exchange*, 34(2):521–529, 2009.

[25] Gilles Godefroy, Mohammed Yahdi, and Robert Kaufman. The topological complexity of a natural class of norms of Banach spaces. In *Proceedings of the International Conference "Analyse & Logique", (Mons, 1997)*, volume 111, pages 3–13, 2001.

[26] Kurt Gödel. The consistency of the axiom of choice and of the generalized continuum-hypothesis. *Proceedings of the National Academy of Sciences*, 24(12):556–557, 1938.

[27] Juha Heinonen and Pekka Koskela. Definitions of quasiconformality. *Invent. Math.*, 120(1):61–79, 1995.

[28] W. Hurewicz. Zur theorie der analytischen mengen. *Fund. Math.*, 15:4–16, 1930.

[29] Peter W. Jones. On removable sets for Sobolev spaces in the plane. In *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 250–267. Princeton Univ. Press, Princeton, NJ, 1995.

[30] Peter W. Jones and Stanislav K. Smirnov. Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.*, 38(2):263–279, 2000.

[31] R. Kaufman. Fourier transforms and descriptive set theory. *Mathematika*, 31(2):336–339 (1985), 1984.

[32] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[33] Alexander S. Kechris and Alain Louveau. *Descriptive set theory and the structure of sets of uniqueness*, volume 128 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[34] Alexander S. Kechris and W. Hugh Woodin. Ranks of differentiable functions. *Mathematika*, 33(2):252–278 (1987), 1986.

[35] Haseo Ki. The Kechris-Woodin rank is finer than the Zalcwasser rank. *Trans. Amer. Math. Soc.*, 347(11):4471–4484, 1995.

[36] Haseo Ki. On the Denjoy rank, the Kechris-Woodin rank and the Zalcwasser rank. *Trans. Amer. Math. Soc.*, 349(7):2845–2870, 1997.

[37] Jerome H. Manheim. *The genesis of point set topology*. Pergamon Press, Oxford-Paris-Frankfurt; The Macmillan Co., New York, 1964.

[38] R. Daniel Mauldin. The set of continuous nowhere differentiable functions. *Pacific J. Math.*, 83(1):199–205, 1979.

[39] S. Mazurkiewicz. Über die menge der differenzierbaven functionen. *Fund. Math.*, 27:244–249, 1936.

[40] Hervé Pajot. *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral*, volume 1799 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[41] Charles Pugh and Conan Wu. Jordan curves and funnel sections. *J. Differential Equations*, 253(1):225–243, 2012.

[42] T. I. Ramsamujh. Three ordinal ranks for the set of differentiable functions. *J. Math. Anal. Appl.*, 158(2):539–555, 1991.

[43] Nikolaos Efstathiou Sofronidis. The set of continuous piecewise differentiable functions. *Real Anal. Exchange*, 31(1):13–21, 2005/06.

[44] S.M. Srivastava. How did Cantor discover set theory and topology? *Resonance*, pages 977–999, 2014.

[45] John R. Steel. What is ... a Woodin cardinal? *Notices Amer. Math. Soc.*, 54(9):1146–1147, 2007.

[46] Xavier Tolsa. Analytic capacity, rectifiability, and the Cauchy integral. In *International Congress of Mathematicians. Vol. II*, pages 1505–1527. Eur. Math. Soc., Zürich, 2006.

[47] Joan Verdera. L^2 boundedness of the Cauchy integral and Menger curvature. In *Harmonic analysis and boundary value problems (Fayetteville, AR, 2000)*, volume 277 of *Contemp. Math.*, pages 139–158. Amer. Math. Soc., Providence, RI, 2001.

[48] Malik Younsi. On removable sets for holomorphic functions. *EMS Surv. Math. Sci.*, 2(2):219–254, 2015.

[49] Malik Younsi. Removability and non-injectivity of conformal welding. *Ann. Acad. Sci. Fenn. Math.*, 43(1):463–473, 2018.

C.J. Bishop, Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651

Email address: bishop@math.stonybrook.edu