MINIMAL WEIGHT STEINER TRIANGULATIONS NEED NOT EXIST

CHRISTOPHER J. BISHOP

Abstract. We give an example of a finite planar point set with no minimal weight Steiner triangulation. The example has five points, three of which are colinear, so the case of points in general position remains open.

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1. Introduction

A triangulation of a finite set V in the plane is a maximal set of line segments with endpoints in the set and which are pairwise disjoint except for the endpoints. This corresponds to writing the convex hull of the points as a union of non-degenerate closed triangles with disjoint interiors. A Steiner triangulation of a set V is a triangulation of some finite set W containing V.

There are only finitely many triangulations of V, so at least one attains the minumum possible sum of edge lengths (or weights); we let this minimum be denoted $\mathrm{MWT}(V)$ (no Steiner points). Let $\mathrm{MWT}(V, n)$ denote the infimum of $\mathrm{MWT}(W)$ over all finite sets W containing V and at most n Steiner points. Clearly this is a non-increasing sequence in n and $\inf_n \mathrm{MWT}(V, n) = \lim_n \mathrm{MWT}(V, n)$ exists and is positive.

Finding a minimal triangulation for V has recently been shown to be NP-hard [2]. There are algorithms for computing Steiner triangulations that are within a bounded factor of the infimum [1], but it is not clear whether the infimum is a minimum, i.e., whether a minimal weight Steiner triangulation (MWST) exists. In fact, we shall show

Theorem 1.1. *The infimum need not be attained. There is a set V with five points so that* $\inf_n \mathrm{MWT}(V, n) = \mathrm{MWT}(V, 1) < \mathrm{MWT}(V)$, *and* $\mathrm{MWT}(W) > \mathrm{MWT}(V, 1)$ *for every finite set W containing V.*

The minimum fails to exist because our five points form the vertices of a convex polygon with four extreme points, i.e., three of the points are colinear, and the minimizing sequence of Steiner triangulations has a triangle that degenerates (it limits on a line segment). It would be very interesting to know whether there is a finite set V so that each of the finite minimums $\mathrm{MWT}(V, n)$ is attained, but $\inf_n \mathrm{MWT}(V, n)$ is not, e.g., $\{\mathrm{MWT}(V, n)\}_{n=1}^{\infty}$ is not eventually constant.

2. The Example

Consider the five points shown on the left side Figure 1 where the vertices are labeled $V = \{a, b, c, d, e\}$. In terms of coordinates we can take

$$a = (0, 0), b = (-1, s + t), c = (-1, s), d = (-1, 0), e = (-r, 0).$$

We shall assume r is large and s is small and $t \leq s/8$.

Let L be the length of the convex hull boundary of these five points. The right side of Figure 1 shows a Steiner triangulation with one extra point x on the edge de. As x tends to d along this edge, the total length of the triangulation tends to length $L+1+t+s$ as $x \to d$. We will show $\mathrm{MWT}(W) > L+1+s+t$ for any finite set W containing V, and hence the infimum is never attained.

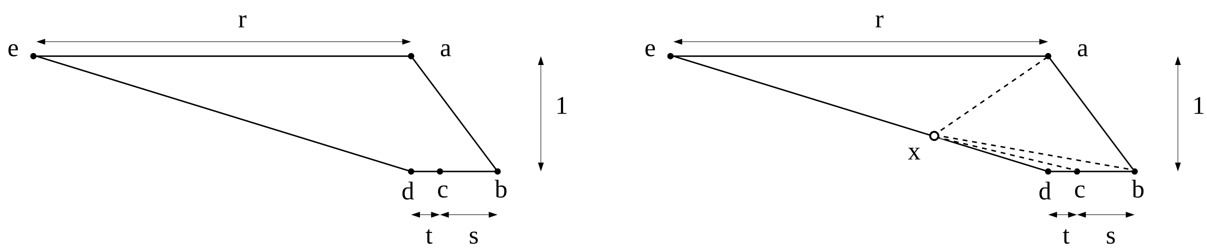

FIGURE 1. The set V and a minimizing family of Steiner triangulations.

Suppose W is a finite set containing V and suppose $\mathrm{MWT}(W) \leq L+1+s+t$. Let $\mathcal{T}$ be a minimal weight triangulation of W attaining this value. Since W has at least six points, $\mathcal{T}$ has at least four triangles. Since W contains V, its convex hull boundary has length $\geq L$ and hence the interior edges of $\mathcal{T}$ have total length $\leq 1+s+t$.

We will repeatedly use a simple fact: if a planar set is projected orthogonally onto a line, the length of the projection is bounded above by the length of the original set. In particular, if every line in an infinite strip hits a set, then the length of that set is at least the width of the strip.

Consider Figure 2. It shows a vertical strip of width $2 > 1+s+t$ separating vertices a and e and having vertex a on its boundary. If every vertical line inside the strip hits an interior edge of $\mathcal{T}$ then the interior edges would have total length $> 1+s+t$, a contradiction. Thus one of these lines L hits exactly two convex hull boundary edges e_1, e_2.

This can only happen if e_1 and e_2 are adjacent and the vertex where they meet is separated by L from the other vertices of W. However, L already separates vertex e from the other vertices of V, so e must be the only vertex of W that is to the left of L, and e must be the vertex where the edges e_1 and e_2 meet. Let w, x denote the other two vertices of this triangle. Note that w and x must lie on or outside the

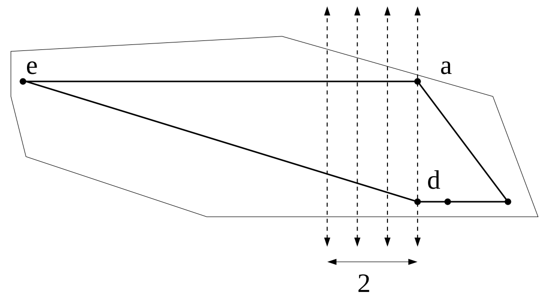

FIGURE 2.

convex hull boundary of V and the line through w and x separates the vertex e from all the other points of W (some may lie on this line).

The edge wx has to be an interior edge of $\mathcal{T}$; the other two edges are boundary edges and if the third edge was also a boundary edge, this single triangle would be the entire triangulation. Note that the length of wx is $\geq 1 - 2/r$. Choose r so large that wx has length $\geq 1 - t$ and so that x is no more than s above the line through b and d. Thus the remaining interior edges of $\mathcal{T}$ have total length $\leq t + s + t < 2s$. Since ew and ex are both boundary edges, removing the triangle Δewx from $\mathcal{T}$ leaves a triangulation of a convex region (segment xw is now a boundary edge).

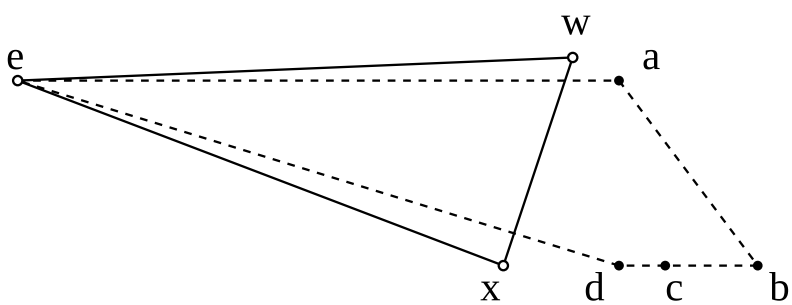

FIGURE 3. The triangulation must contain a triangle Δexw

We now play the same trick with the horizontal strip of width $2s$ that separates a from b, c, d and has lower edge that is distance s above a, b, c; this implies x is below the lower edge of the strip and hence the edge xw crosses the strip. See Figure 4. Each horizontal ray that leaves xw towards the right must cross the boundary of $\mathcal{T}$. If every such line also crosses an interior edge of $\mathcal{T}$ then the total length of interior edges besides xw would be $> 2s$, which is impossible. Hence one of these rays connects xw to a boundary edge e_3.

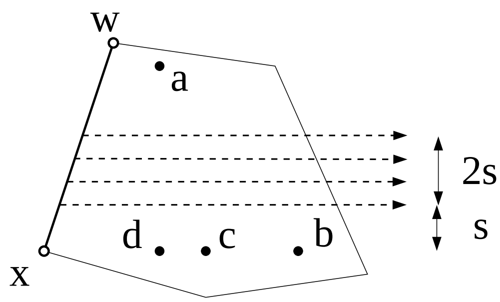

FIGURE 4. The horizontal strip. The figure is not too scale since we are assuming $s \ll 1 = |d - a|$

Thus xw and e_3 form two sides of triangle in $\mathcal{T}$. By our earlier argument, they must meet a point of $W' = W \setminus \{e\}$ that is separated from the other points of W' by the line containing the connecting ray. Thus they must meet at vertex a, i.e., $w = a$. See Figure 5. Note that $w = a$ implies x is left of the vertical line through d and is below the segment de.

Let y be the third vertex of the triangle formed by e_3 and xa. Note that y is on or outside the convex hull boundary of V and that the line through x and y separates a from c, d. Also note that xy must be an interior edge of $\mathcal{T}$ (if it were a boundary edge, then $\mathcal{T}$ would have only two triangles).

If s is small enough, then the length of xy must be $\geq 3s/4$ (both points are below the line parallel to bd and s units above it; the part of the convex hull of V below this line can't be crossed in less than length $3s/4$ if s is small). Hence the total remaining interior length in $\mathcal{T}$ must be $\leq t + t + s/4 < s/2$ (recall $t < s/8$).

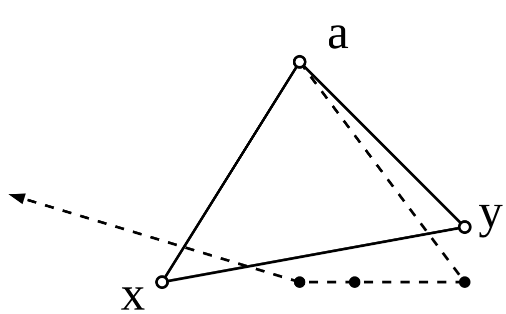

FIGURE 5. The second forced triangle.

We now play the strip trick a third time, with a vertical strip of width $s/2$ separating b from c and containing b on its boundary. Every downward vertical rays starting on xy must hit the boundary of $\mathcal{T}$, but if they all hit an interior edge as well, then we get a contradiction. Therefore some ray in the strip connects xy to a boundary edge and this edge must meet xy at $y = b$. See Figures 6 and 7. The third point of this triangle is denoted z and x, z must be a interior edge of $\mathcal{T}$ (otherwise the triangulation would have only three triangles) and xz with length $> t$ (we could only have equality if $x = d$, which is not allowed because of vertex c; here we are using the collinearity of c with b, d). Thus the interior edges of $\mathcal{T}$ have total length

$$\geq |x-a| + |x-b| + |x-z| \geq |x-a| + |x-b| + |d-c|, \tag{2.1}$$

since z must be the the right of c and x must be to the left of d. Also note $|d-c| = t$.

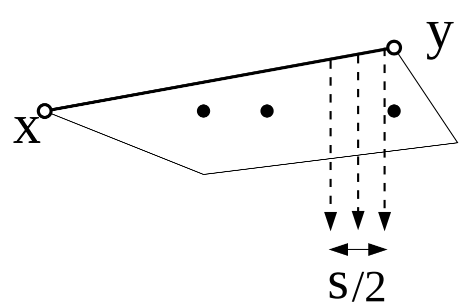

FIGURE 6.

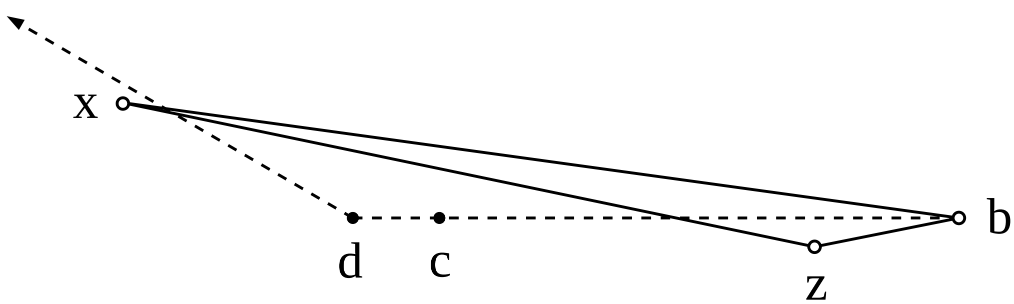

FIGURE 7.

We claim that $|x-a|+|x-b|$ is minimized at $x = d$, at least for the constraints we have on x (it is left of d and below de). See Figure 8. If x is below the line through b and d, move it to the right until it is below d and then move it upwards to d. Each of these moves strictly decreases both $|x-a|$ and $|x-b|$.

If x is above the line through c and d, then move x to the right until it hits the segment de. This decreases both $|x-a|$ and $|x-b|$.

Now x is on the segment de. Let p be the point on de such that pa is perpendicular to de (this is the closest point on de to a). If x is to the left of this point, then move x along the segment de to p; this decreases both $|x-a|$ and $|x-b|$. If x equals p or is on de to the right of p, move x along de from p to d. This always decreases $|x-b|$ but increases $|x-a|$; however if $r > 1$ (so the slope of de is < 1), it is easy to check that $|x-a|$ increases by $< \text{dist}(x, da)$ and and $|x-b|$ decreases by $> \text{dist}(x, da)$. Thus the sum decreases. Thus the starting value must be larger than than the final value of $|d-a| + |d-b| = 1 + s$.

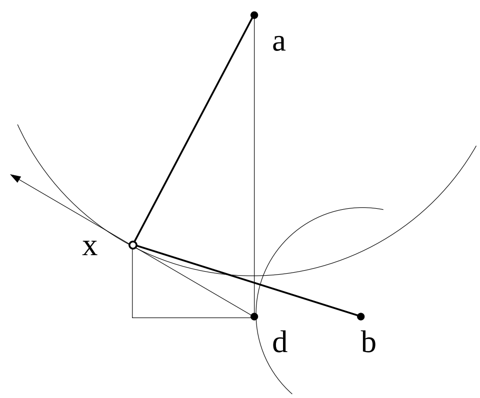

FIGURE 8. The final movement might increase $|x-a|$, but decreases $|x-a| + |x-b|$.

Thus $\text{MWT}(W) > 1 + s + t$, as desired. This proves that the infimum is never attained by a finite triangulation.

References

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C.J. Bishop, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651

Email address: `bishop@math.sunysb.edu`