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# HARMONIC MEASURE: ALGORITHMS AND APPLICATIONS

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This is a brief survey of results related to planar harmonic measure, roughly from Makarov's results of the 1980's to recent applications involving 4-manifolds, dessins d'enfants and transcendental dynamics. It is non-chronological and rather selective, but I hope that it still illustrates various areas in analysis, topology and algebra that are influenced by harmonic measure, the computational questions that arise, the many open problems that remain, and how these questions bridge the gaps between pure/applied and discrete/continuous mathematics.

## 1 Conformal complexity and computational consequences

• **Three definitions:** First, the most intuitive definition of harmonic measure is as the boundary hitting distribution of Brownian motion. More precisely, suppose $\Omega \subset \mathbb{R}^n$ is a domain (open and connected) and $z \in \Omega$. We start a random particle at $z$ and let it run until the first time it hits $\partial\Omega$. We will assume this happens almost surely; this is true for all bounded domains in $\mathbb{R}^n$ and many, but not all, unbounded domains. Then the first hit defines a probability measure on $\partial\Omega$. The measure of $E \subset \partial\Omega$ is usually denoted $\omega(z, E, \Omega)$ or $\omega_z(E)$. For $E$ fixed, $\omega(z, E, \Omega)$ is a harmonic function of $z$ on $\Omega$, hence the name "harmonic measure".

Next, if $\Omega$ is regular for the Dirichlet problem, then, by definition, every $f \in C(\partial\Omega)$ has an extension $u_f \in C(\overline{\Omega})$ that is harmonic in $\Omega$, and the map $z \to u_f(z)$, $z \in \Omega$ is a bounded linear functional on $C(\partial\Omega)$. By the Riesz representation theorem, $u_f(z) = \int_{\partial\Omega} f d\mu_z$, for some measure $\mu_z$, and $\mu_z = \omega_z$. For domains with sufficient smoothness, Green's theorem implies harmonic measure is given by the normal derivative of Green's function times surface measure on the boundary. Thus the key to many results are estimates related to the gradient of Green's function.

Finally, in the plane (but not in higher dimensions) Brownian motion is conformally invariant, so $\omega_z$ for a simply connected domain $\Omega$ is the image of normalized Lebesgue measure on the unit circle $\mathbb{T} = \{w : |w| = 1\}$ under a conformal map $f : \mathbb{D} = \{w : |w| < 1\} \to \Omega$ with $f(0) = z$. Because of the many tools from complex analysis, we generally have the best theorems and computational methods in this case.

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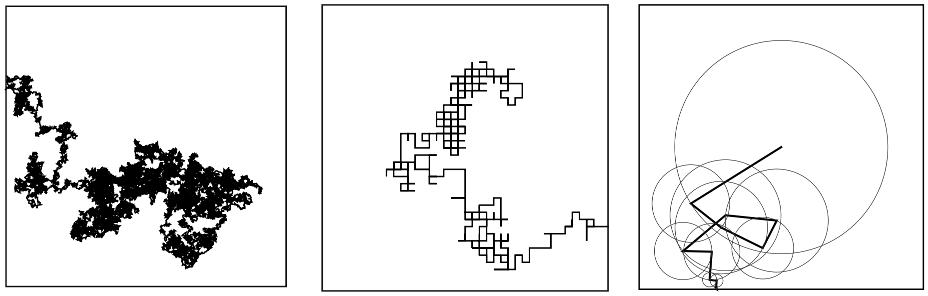

Figure 1: Continuous Brownian motion and two discrete approximations. In the center is a random walk on a grid; this is slow to use. On the right is the "walk-on-spheres" or "Kakutani's walk"; this is much faster to simulate.

• **The walk on spheres:** Suppose we want to compute the harmonic measure of one edge of a planar polygon. The most obvious approach is to approximate a Brownian motion by a random walk on a $\frac{1}{n}\times\frac{1}{n}$ grid. See Figure 1. However, it takes about $n^2$ steps for this walk to move distance 1, so for $n$ large, it takes a long time for each particle to get near the boundary. A faster alternative is to note that Brownian motion is rotationally invariant, so it first hits a sphere centered on its starting point $z$ in normalized Lebesgue measure. Fix $0 < \lambda < 1$ and randomly choose a point on

$$S_\lambda(z) = \{w : |w - z| = \lambda \cdot \mathrm{dist}(z, \partial\Omega)\}.$$

Now repeat. This random "walk-on-spheres" almost surely converges to a boundary point exponentially quickly, so only $O(\log n)$ steps are needed to get within $1/n$ of the boundary; see Binder and Braverman [2012]. I learned this process from a lecture of Shizuo Kakutani in 1986 and refer to it as Kakutani's walk.

However, even Kakutani's walk is only practical on small examples. Long corridors can make some edges very hard to reach, so we need a huge number of samples to estimate their harmonic measure. This is called the "crowding phenomena" (because the conformal pre-images of these edges are tiny; see below). For example, in a $1 \times r$ rectangle a Brownian path started at the center has only probability $\approx \exp(-\pi r/2)$ of hitting one of the short ends; for $r = 10$, the probability[1] is $\omega \approx 3.837587979 \times 10^{-7}$. See Figure 2. Thus random walks are not a time efficient method of computing harmonic measure (but they are memory efficient; see the work of Binder, Braverman, and Yampolsky [2007]).

• **The Schwarz-Christoffel formula:** Conformal mapping gives the best way of computing harmonic measure in a planar domain. See Figure 3. Many practical methods exist; surveys of various techniques include DeLillo [1994], Papamichael and Saff [1993], Trefethen [1986], Wegmann [2005]. Some fast and flexible current software includes `SCToolbox` by Toby Driscoll, `Zipper` by Don Marshall, and `CirclePack` by

[1] In fact, $\omega = \frac{2}{\pi}\arcsin((3-2\sqrt{2})^2(2+\sqrt{5})^2(\sqrt{10}-3)^2(5^{1/4}-\sqrt{2})^4)$; see page 262 of Bornemann, Laurie, Wagon, and Waldvogel [2004].

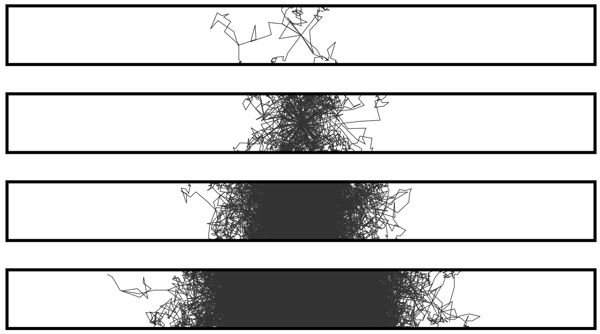

Figure 2: 10, 100, 1000 and 10000 samples of the Kakutani walk inside a $1 \times 10$ polygon. This illustrates the exponential difficulty of traversing narrow corridors.

Ken Stephenson. To quote an anonymous referee of Bishop [2010a]: “Algorithmic conformal mapping is a small topic – one cannot pretend that thousands of people pay attention to it. What it does have going for it is durability. These problems have been around since 1869 and they have proved of lasting interest and importance.”

When $\partial\Omega$ is a simple polygon, the conformal map $f : \mathbb{D} \to \Omega$ is given by the Schwarz-Christoffel formula (e.g., Christoffle [1867], Schwarz [1869], Schwarz [1890]):

$$f(z) = A + C \int_0^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

where $\{\alpha_1\pi, \dots, \alpha_n\pi\}$, are the interior angles of the polygon and $\mathbf{z} = \{z_1, \dots, z_n\} \subset \mathbb{T} = \{z : |z| = 1\}$ are the preimages of the vertices (we call these the SC-parameters or the pre-vertices). For references, variations, and history of this formula, see Driscoll and Trefethen [2002].

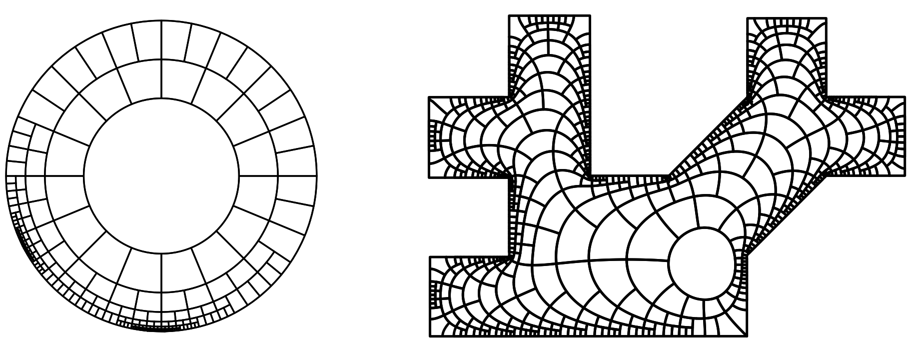

Figure 3: A conformal map to a polygon. The disk is meshed by boxes to a scale where vertex preimages are well separated. Counting boxes, we can estimate that the horizontal edge at top left has harmonic measure $\approx 2^{-16}$, another illustration of crowding.

The Schwarz-Christoffel formula does not really give us the conformal map; one must still solve for the $n$ unknown SC-parameters, and this is a difficult problem. There are various heuristic methods that work as follows: make a parameter guess, compute the corresponding map, compare the image with the desired domain and modify the guess accordingly. Davis [1979] uses a simple side-length comparison: if a side is too long (or short), one simply decreases (or increases) the gap between the corresponding parameters proportionally. The more sophisticated CRDT algorithm of Driscoll and Vavasis [1998] uses cross ratios of adjacent Delaunay triangles to make the updated guess. However, neither Davis' method nor CRDT comes with a proof of convergence, nor a bound on how many steps are needed to achieve a desired accuracy.

• **The fast mapping theorem:** However, such bounds are possible (see Bishop [2010a]):

**Theorem 1.** *Given $\epsilon > 0$ and an $n$-gon $P$, there is $\mathbf{w} = \{w_1, \dots, w_n\} \subset \mathbb{T}$ so that*

1. *$\mathbf{w}$ can be computed in at most $Cn$ steps, where $C = O(1 + \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$,*
2. *$d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$ where $\mathbf{z}$ are the true SC-parameters.*

Here a step means an infinite precision arithmetic operation or function evaluation. The error in Theorem 1 is measured using a distance between $n$-tuples defined by

$$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists\ K\text{-quasiconformal } h : \overline{\mathbb{D}} \to \overline{\mathbb{D}} \text{ such that } h(\mathbf{z}) = \mathbf{w}\}.$$

A homeomorphism $h : \mathbb{D} \to \mathbb{D}$ is $K$-quasiconformal ($K$-QC) if it is absolutely continuous on almost all lines (so partial derivatives make sense a.e.) and $|\mu_h| \le k < 1$, where $\mu_h = h_{\overline{z}}/h_z$ is the complex dilatation of $h$ (e.g., see Ahlfors [2006]). Geometrically, this says that infinitesimal circles are mapped to infinitesimal ellipses with eccentricity bounded by $K = (k+1)/(k-1) \ge 1$. In general, QC maps are non-smooth and can even map a line segment to fractal arc; see Bishop, Hakobyan, and Williams [2016] and its references.

The possible boundary values of a QC map $h : \mathbb{D} \to \mathbb{D}$ are exactly the quasisymmetric (QS) circle homeomorphisms. We say $h : \mathbb{T} \to \mathbb{T}$ is $M$-QS if $|h(I)| \le M|h(J)|$ whenever $I$ and $J$ are disjoint, adjacent intervals of the same length on $\mathbb{T}$.

A map $f : \mathbb{D} \to \mathbb{D}$ is called a quasi-isometry (QI) for the hyperbolic metric $\rho$ if there is an $A < \infty$ so that $A^{-1} \le \rho(f(z), f(w))/\rho(z, w) \le A$ whenever $\rho(z, w) \ge 1$; thus $f$ is bi-Lipschitz at large scales, but we make no assumptions at small scales, not even continuity. Nevertheless, such an $f$ does extend to a homeomorphism of the boundary circle, and the class of these extensions is again the QS-homeomorphisms. Thus QC and QI self-maps of $\mathbb{D}$ have the same set of boundary values.

Using the QC-metric on $n$-tuples has several advantages: it implies approximation in the Hausdorff metric and ensures points occur in the correct order on $\mathbb{T}$. When $K$ is close to 1, the QS formulation holds with $M \approx 1$ and implies that the relative gaps between points are correct in a scale invariant way. We also have $d_{QC}(\mathbf{w}, \mathbf{z}) = 0$ iff the $n$-tuples are Möbius images of each other; this occurs iff the corresponding polygons are similar, which makes $d_{QC}$ a natural metric for comparing shapes (to be precise, $d_{QC}$ is only a metric if we consider $n$-tuples modulo Möbius transformations). Finally, this metric is easy to bound by computing any vertex-preserving QC map between the corresponding polygons, e.g., the obvious piecewise linear map coming from two compatible triangulations. See Figure 4. Using this, we can bound the QC-distance to the true SC-parameters without knowing what those parameters are. Computing the exact QC-distance between $n$-tuples is much harder, e.g. see Goswami, Gu, Pingali, and Telang [2015].

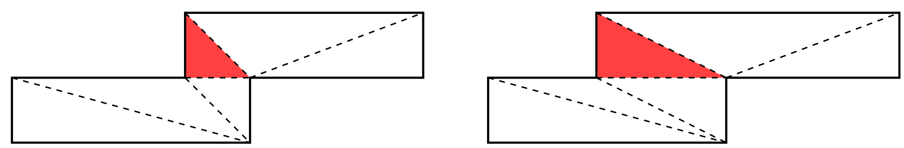

Figure 4: Equivalent triangulations of two polygons define a piecewise linear QC map and give an upper bound for the QC distance. Here $K = 2$ and the most distorted triangle is shaded.

• **Applications to computational geometry:** We will first discuss some applications of the fast mapping theorem (FMT), and then discuss its proof. As explained below, the proof of the FMT depends on ideas from computational geometry (CG), and it returns the favor by solving certain problems in CG. Optimal meshing is the problem of efficiently decomposing a domain $\Omega$ into nice pieces. Assume $\partial\Omega$ is an $n$-gon. "Efficient" means we want the number of mesh elements to be bounded by a polynomial in $n$ (independent of $\Omega$). "Nice" means the pieces are triangles or quadrilaterals that have angles strictly bounded between $0°$ and $180°$, whenever possible. Some results that use the FMT (or ideas from its proof) include:

► **Thick/thin decomposition:** Every polygon can be written as a union of disjoint thick and thin pieces that are analogous to the thick/thin pieces of a hyperbolic manifold (regions where the injectivity radius is larger/smaller than some $\epsilon$). See Figure 5. For an $n$-gon, each thin piece is either a neighborhood of a vertex (parabolic thin parts), or corresponds to a pair of sides that have small extremal distance within $\Omega$ (hyperbolic thin parts); the thin parts are in 1-to-1 correspondence with the thin parts of the $n$-punctured Riemann sphere formed by gluing two copies of the polygon along its (open) edges. Despite there being $\simeq n^2$ pairs of edges, there are $O(n)$ thin parts, and they can be found in time $O(n)$ using the FMT with $\epsilon \simeq 1$; see Bishop [2010a].

► **Optimal quad-meshing:** Any $n$-gon has an $O(n)$ quadrilateral mesh where every angle is less than $120°$ and all the new angles are at least $60°$; see Bishop [2010b], Bishop [2016b] Here "new" means that existing angles $< 60°$ remain, but are not subdivided. Both the complexity and angle bounds are sharp. The thick/thin decomposition plays a major role here: the thin parts are meshed with an ad hoc Euclidean construction and the thick parts are meshed by transferring a hyperbolic mesh from $\mathbb{D}$ by a nearly conformal map. Is there a similar approach in 3 dimensions, perhaps using decompositions into pieces that are meshed using some of the eight natural 3-dimensional geometries?

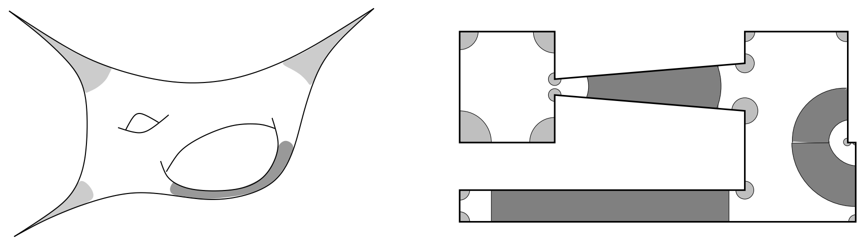

Figure 5: Thin parts of a surface and a polygon are shaded (light = parabolic, dark = hyperbolic), and the thick pieces are white.

► **The NOT theorem:** Every planar triangulation with $n$ elements can be refined to a **n**on**o**btuse **t**riangulation (all angles $\leq \theta = 90°$, called a NOT for brevity) with $O(n^{2.5})$ triangles; see Bishop [2016a]. No polynomial bound is possible if $\theta < 90°$ and the previous best result was with $\theta = 132°$, due to Tan [1996]. See also Mitchell [1993]. A gap remains between the $O(n^{2.5})$ algorithm and the $n^2$ worst known example. The proof of the NOT theorem involves perturbing a natural $C^1$ flow associated to the triangulation, in order to cause collisions between certain flow lines. Is there any connection to closing lemmas in dynamics, e.g., Pugh [1967]? Perhaps the gap could be reduced using dynamical ideas, or ideas from the NOT theorem applied to flows on surfaces.

The NOT theorem has an amusing consequence: suppose several adjoining countries have polygonal boundaries (with $n$ edges in total) and the governments all want to place cell towers so that a cell phone always connects to a tower (the closest one) in the same country as the phone. Is this possible using a polynomial number of towers? More mathematically, we are asking for a finite point set $S$ whose Voronoi cells conform to the given boundaries (the Voronoi cells of $S$ are the points closest to each element of $S$). If the countries are nonobtuse triangles this is easy to do, so the NOT theorem implies this is possible in general using $O(n^{2.5})$ points, the first polynomial bound for this problem stated in Salzberg, Delcher, Heath, and Kasif [1995].

• **Proof of the FMT:** Like the other methods mentioned earlier, the fast mapping algorithm iteratively improves an initial guess for the conformal map. However, whereas Davis' method and CRDT use conformal maps onto an approximate domain, and try to improve the domain, the fast mapping algorithm uses approximately conformal maps onto the correct target domain and improves the degree of conformality. More precisely, each iteration computes the dilatation $\mu_f$ of a QC map $f : \mathbb{D} \to \Omega$, and attempts to solve the Beltrami equation $g_{\overline{z}} = \mu_f g_z$ with a homeomorphism $g : \mathbb{D} \to \mathbb{D}$. If $g$ was an exact solution, then $F = f \circ g^{-1}$ would be the desired conformal map. The exact solution is given by a infinite series involving the Beurling transform (see e.g., Ahlfors [2006]) but the FMT uses only the leading term of this series and approximately solves the resulting linear equation (thus it is a higher dimensional version of Newton's method). Iterating gives a sequence of QC maps that converge quadratically to a conformal map, assuming the initial dilatation $\mu$ is small enough. A variation of this method was implemented by Green [2011].

To bound the total time, we have to estimate the time needed for each iteration, and the time needed to find a starting guess for which we can uniformly bound the number of iterations needed to reach accuracy $\epsilon$ (it is not obvious that such a point even exists). The first step involves representing the map as a collection of series expansions on the disk, and applying discretized integral operators using the fast multipole method and structured linear algebra. The second part is less standard: we use computational geometry to make a "rough-but-fast" QC approximation to the Riemann map and use 3-dimensional hyperbolic geometry to prove that this guess is close to the correct answer, with a dilatation bound independent of the domain. It is (fairly) easy to reduce from "bounded dilatation" to "small dilatation" by a continuation argument, so we will only discuss proving the uniform bound.

## 2 Disks, domes, dogbones, dimension and dendrites

• **The medial axis flow:** The medial axis (MA) of a planar domain $\Omega$ is the set of all interior points that have $\geq 2$ distinct closest points on $\partial\Omega$. For polygons, these are the centers of maximal disks in $\Omega$, but the latter set can be strictly larger in general; see Bishop and Hakobyan [2008]. If $\partial\Omega$ is a polygon, then the medial axis is a finite tree. See Figure 6.

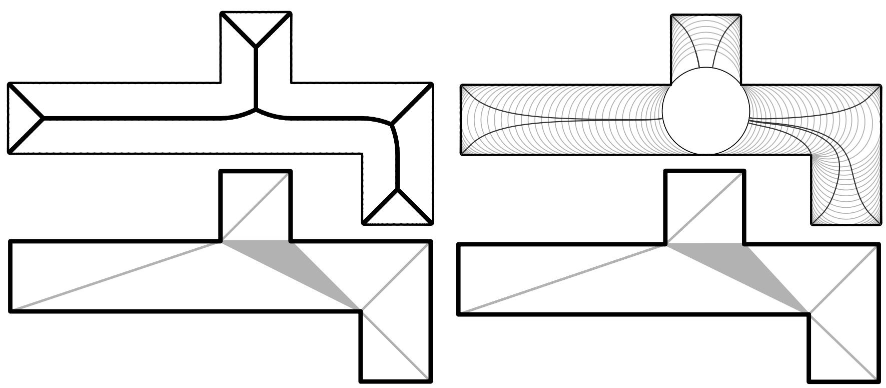

Figure 6: The top shows the medial axis of a domain (left) and the medial axis foliation and flow (right). The bottom show triangulations of the target polygon and initial guess using the MA-flow parameters. Here $K = 1.24$, (the most distorted triangle is shaded), but the polygons appear almost identical.

If we fix one medial axis disk $D_0$ as the "root" of this tree, then arcs of the remaining disks foliate $\Omega \setminus D_0$. Each boundary point can be connected to $D_0$ by a path orthogonal to this foliation; see Figure 6. The medial axis flow defines Möbius transformations between medial axis disks, hence preserves certain cross ratios, and given the medial axis, we can use this to compute the images of all $n$ boundary vertices in $O(n)$ time.

The medial axis itself can be computed in linear time by a result of Chin, Snoeyink, and Wang [1999], so the MA-flow gives a linear time (i.e., "fast") initial guess for the SC-parameters.

• **The convex hull theorem:** Why is our "fast guess" an accurate guess? The answer is best understood by moving from 2 to 3 dimensions. The "dome" of a planar domain $\Omega$ is the surface $S = S(\Omega) \subset \mathbb{H}^3 = \mathbb{R}^3_+ = \{(x, y, t) : t > 0\}$ that is the boundary of the union of all hemispheres whose base disk is contained in $\Omega$. In fact, it suffices to consider only medial axis base disks. See Figure 7.

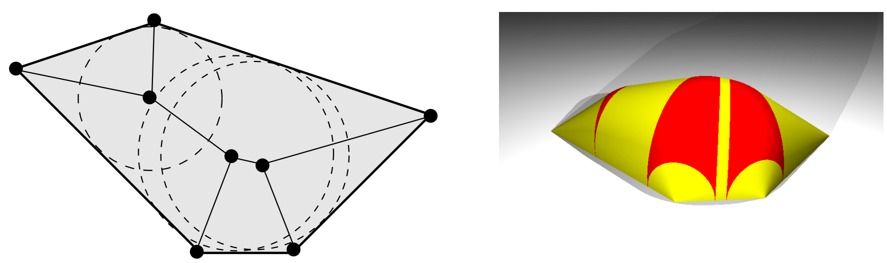

Figure 7: A polygonal domain and its dome. The red patches on the dome each correspond to the dome of a vertex disk of the medial axis; the yellow regions correspond to domes of edge disks.

Recall that $\mathbb{H}^3$ has a hyperbolic metric $d\rho = ds/t$. Each hemisphere below the dome $S$ is a hyperbolic half-space, and the region above $S$ is the intersection of their complements, hence is hyperbolically convex. Thus the dome of $\Omega$ is also the boundary of the hyperbolic convex hull in $\mathbb{H}^3$ of $\Omega^c = \mathbb{C} \setminus \Omega$. We define the "nearest point retraction" $R : \Omega \to S(\Omega)$ by expanding a horo-sphere in $\mathbb{R}^3_+$ tangent to $\mathbb{R}^2$ at $z \in \Omega$ until it first hits $S$ at a point $R(z)$. See Figure 8. Dennis Sullivan's convex hull theorem (CHT) states that $R$ is a quasi-isometry from the hyperbolic metric on $\Omega$ to the hyperbolic path metric on the dome. Sullivan [1981b] originally proved the CHT in the context of hyperbolic 3-manifolds (see below) and the version above is due to Epstein and Marden [1987]. See also Bishop [2001], Bishop [2002], Bridgeman and Canary [2010].

The dome $S$ with its hyperbolic path metric is isometric to the hyperbolic disk. The isometry $\iota : S \to \mathbb{D}$ can be visualized by thinking of $S$ as bent along a disjoint collection of geodesics, and "flattening" the bends until we get a hyperbolic plane (the hemisphere above $D_0$; this is clearly isomorphic to $\mathbb{D}$). Remarkably, the restriction of this map to $\partial S = \partial\Omega$ equals the MA-flow map $\partial\Omega \to \partial D_0$. Figure 9 gives the idea of the proof. Since $\iota \circ R : \Omega \to \mathbb{D}$ is a quasi-isometry (and because QI and QC maps of $\mathbb{D}$ have the same boundary values), the MA-flow map $\partial\Omega \to \partial D_0$ has a uniformly QC extension $\sigma : \Omega \to D_0$. Thus our "fast guess" is indeed a "good guess" with uniform bounds.

• **Convex hulls and 3-manifolds:** As mentioned above, Sullivan's CHT was first discovered in the context of hyperbolic 3-manifolds. By definition, such a manifold $M$ is

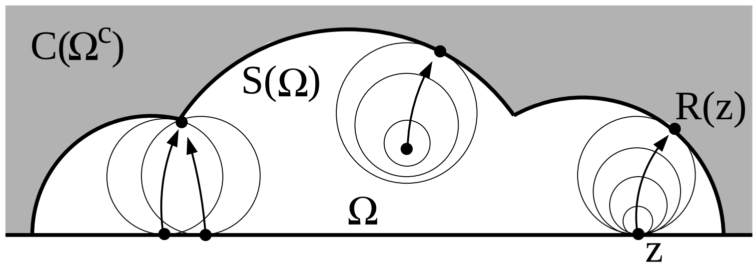


Figure 8: The dome $S$ of $\Omega$ is the boundary of the hyperbolic convex hull of $\Omega^c$ (shaded). The retraction map $R : \Omega \to S$ defined by expanding horoballs need not be 1-to-1, but is a quasi-isometry.

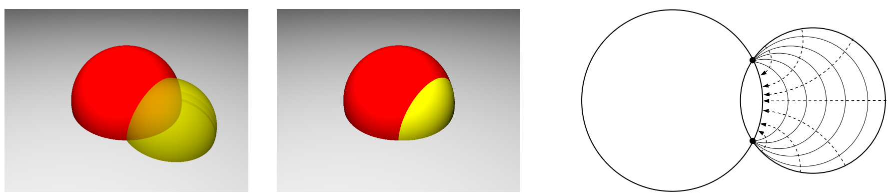

Figure 9: The dome of two overlapping disks consists of two hyperbolic half-planes joined along a geodesic (left). Flattening this bend means rotating one half-plane around the geodesic until it is flush with the other (center). On $\mathbb{R}^2$, this rotation corresponds to the medial axis flow in the base domain. The same observation applies to all finite unions of disks, and the general case follows by a limiting argument.

the quotient of $\mathbb{H}^3$ by a Kleinian group, i.e., a discrete group $G$ of orientation preserving hyperbolic isometries. This is completely analogous to a Riemann surface being the quotient of the hyperbolic disk by a Fuchsian group. The accumulation set of any $G$-orbit on $\partial\mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\}$ is called the limit set $\Lambda$ of $G$; this is often a fractal set. The complement of $\Lambda$ is called the ordinary set $\Omega$. In this paper we will always assume $\Omega \neq \varnothing$. We let $C(\Lambda) \subset \mathbb{H}^3$ denote the hyperbolic convex hull of $\Lambda$. It is $G$-invariant, so its quotient defines a region $C(M) \subset M$ called the convex core of $M$; this is also the convex hull of all the closed geodesics in $M$. We define the "boundary at infinity" of $M$ as $\partial_\infty M = \Omega/G$; this is a union of Riemann surfaces, one for each connected component of $\Omega$. The dome of each component of $\Omega$ is a boundary component of $C(\Lambda)$, and corresponds to a boundary component of $C(M)$. The original formulation of Sullivan's CHT (which he attributes to Thurston) is that $\partial_\infty M$ is uniformly QC-equivalent to $\partial C(M)$.

A case of particular interest is when $M$ is homeomorphic to $\Sigma\times\mathbb{R}$ for some compact surface $\Sigma$ and $C(M)$ is compact (this is called a co-compact quasi-Fuchsian manifold). See Figure 10. Then $\Lambda$ is a Jordan curve, so $\partial C(M)$ has two components, $\Omega_1$ and $\Omega_2$. Since $u = \omega(z, \Omega_2, \mathbb{H}^3)$ is invariant under $G$, it defines a harmonic function

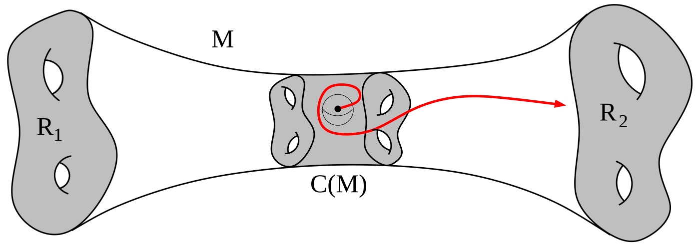


Figure 10: A co-compact quasi-Fuchsian manifold. The tunnel vision function is the harmonic measure of one component of $\partial_\infty M$.

$u(z) = \omega(z, R_2, M)$ on $M$. (Here $u$ is harmonic for the hyperbolic metric on $\mathbb{H}^3$, not the Euclidean metric; the two concepts agree in 2 dimensions, but not in 3.) This is the "tunnel vision" function: for $z \in M$, $u(z)$ is the normalized area measure (on the tangent 2-sphere) of the geodesic rays starting at $z$ that tend towards $R_2 \subset \partial_\infty M$. Thus $u$ is the "brightness" at $z$ if $R_2$ is illuminated but $R_1$ is dark. It is easy to check that $u \geq 1/2$ on the component of $\partial C(M)$ facing $R_2$ and is $\leq 1/2$ on the other component. Thus the level set $\{z : u(z) = \frac{1}{2}\}$ is contained inside $C(M)$.

• **Dogbones and 4-manifolds:** The topology of the tunnel vision level sets has an interesting connection to 4-dimensional geometry. If $\Lambda$ is a circle, then the level sets $\{u(z) = \lambda\}$, $0 < \lambda < 1$, are topological disks, but if $\Lambda$ approximates $\partial\Omega$, where

$$\Omega = \{z : |z-1| < 1/2\} \cup \{z : |z+1| < 1/2\} \cup \{z = x+iy : |x| < 1, |y| < \epsilon\},$$

and $\epsilon$ is small, then they can be non-trivial and $u$ has a critical point. See Figure 11.

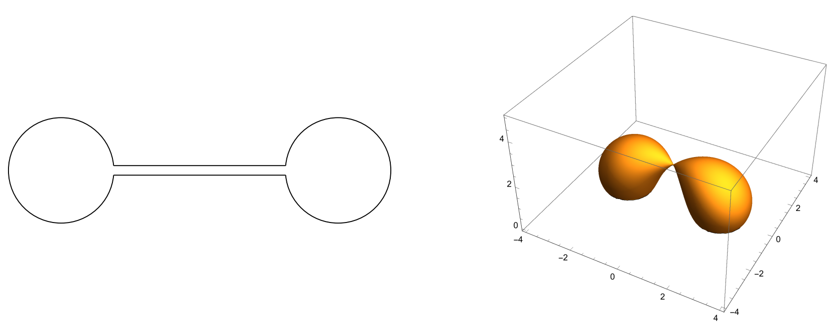

Figure 11: The dogbone domain (left) approximates two disjoint disks if the corridor is very thin. For two disks, the level surfaces $\{u(z) = \lambda\}$ evolve from two separate surfaces into a connected surface, so $u$ must have a critical point; the critical surface is shown at right.

This critical point has a surprising consequence. Claude LeBrun has shown how to turn the hyperbolic 3-manifold $M$ into a closed anti-self-dual 4-manifold $N$, so that

$N$ has an almost-Kähler structure if and only if $u$ has no critical points. For definitions and details, see Bishop and LeBrun [2017]. The simplest case is to take $M \times \mathbb{T}$ and collapse $\partial_\infty M$ to two points; this gives a conformally flat $N$, but a hierarchy of topologically distinct non-flat examples also exists. In Bishop and LeBrun [ibid.] we construct a co-compact Fuchsian group that can be deformed to a quasi-Fuchsian group with limit set approximating the dogbone curve. Thus the almost-Kähler metrics sweep out an open, non-empty, but proper subset of the moduli space of anti-self-dual metrics on the corresponding 4-manifold $N$, giving the first example of this phenomena. Thus harmonic measure solves a problem about 4–manifolds, and 4–manifolds raise new questions about harmonic measure: for which planar domains $\Omega$ does $\omega(z, \Omega, \mathbb{H}^3)$ have a critical point? The group in Bishop and LeBrun [ibid.] has a huge number of generators; how many are really needed to get an example with a critical point? Are critical points common near the boundary of Teichmüller space for any large $G$?

• **Heat kernels and Hausdorff dimension:** As above, suppose $M \simeq \Sigma \times \mathbb{R}$ is hyperbolic and $C(M)$ is compact. By compactness, a Brownian motion inside $C(M)$ hits $\partial C(M)$ almost surely; as noted earlier, it then has probability $\geq 1/2$ of hitting the corresponding component of $\partial_\infty M$. This implies Brownian motion on $M$ leaves $C(M)$ almost surely, which implies Brownian motion on $\mathbb{H}^3$ leaves $C(\Lambda)$ almost surely, which is equivalent to area$(\Lambda) = 0$. This observation can be made much more precise.

The heat kernel, $k_M(x, y, t)$, on a manifold $M$ gives the probability that a Brownian motion starting at $x$ at time 0 will be at $y$ at time $t$. Thus the probability of being in $C(M)$ at time $t$ is $p(x,t) = \int_{C(M)} k_M(x, y, t) dy$. The heat kernel can be written in terms of the eigenvalues and eigenfunctions of the Laplacian on M, $k_M(x, y, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y)$, so it seems reasonable that $p(x,t) = O(\exp(-\lambda_0 t))$. See Davies [1988], Grigor'yan [1995], which make this precise. The lift of $k_M$ to $\mathbb{H}^3$ is a sum over $G$-orbits of

$$k_{\mathbb{H}^3}(w, z, t) = (4\pi t)^{-3/2} \frac{\rho(z, w)}{\sinh(\rho(z, w))} \exp(-t - \frac{\rho(z, w)^2}{4t}).$$

Let $G_n = \{g \in G : n < \rho(0, g(0)) \leq n + 1\}$ and $N_n = \#G_n$. The critical exponent $\delta = \limsup_k (\log N_k)/k$, is always a lower bound for $\dim(\Lambda)$, and equality holds in many cases, e.g., when $G$ is finitely generated. See Bishop and P. W. Jones [1997a], Sullivan [1984].

Putting these estimates together (and dropping the non-exponential terms) gives

$$e^{-\lambda_0 t} \simeq k_M(x, x, t) \simeq \sum_n \sum_{g \in G_n} k_{\mathbb{H}}^3(0, g(0), t) \simeq e^{-t} \sum_n e^{-(1-\delta)n - n^2/4t}.$$

The final sum is dominated by the term $n = -2t(1-\delta)$, and comparing the exponents gives $\lambda_0 = \delta(2-\delta)$, a well known formula relating the geometry of $\Lambda$ to Brownian motion on $M$. Are other relations possible? If $C(M)$ is non-compact, but has finite volume, Sullivan [ibid.] showed the limit set has finite, positive packing measure (instead of Hausdorff measure, as happens when $C(M)$ is compact). Is this reflected by some property of Brownian motion or harmonic measure on $M$?

When $\mathrm{vol}(C(M)) = \infty$, Peter Jones and I proved that either (1) $\lambda_0 = 0$ and $\dim(\Lambda) = \delta = 2$ or (2) $\lambda_0 > 0$ and $\mathrm{area}(\Lambda) > 0$. See Bishop and P. W. Jones [1997a]. Again, this reduces to harmonic measure estimates: bounding $\omega(z, \partial C(M), M)$ at points deep inside $C(M)$. Both cases above can occur in general, but the second case ($\mathrm{area}(\Lambda) > 0$) is impossible for finitely generated groups with $\Omega \neq \emptyset$; this is the Ahlfors measure conjecture and was proven independently by Agol [2004] and by Calegari and Gabai [2006].

• **Dimension of dendrites:** We can strengthen the Ahlfors conjecture in some cases. Consider a singly degenerate manifold $M \simeq \Sigma \times \mathbb{R}$ where $C(M)$ contains one end of $M$, and also assume that $M$ has positive injectivity radius (i.e., non-trivial loops have length bounded away from zero). See Figure 12. Then the limit set $\Lambda$ is a dendrite (connected and does not separate the plane) of dimension 2 and area zero. Such limit sets are notoriously difficult to understand and compute.

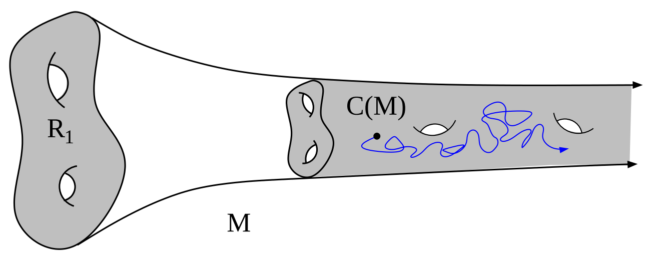


Figure 12: Co-compact quasi-Fuchsian manifolds can limit on a singly degenerate $M$: $C(M)$ contains a geometrically infinite end of $M$, and its complement is a geometrically finite end.

In this case, the tunnel vision function is constant, but there is an interesting alternative. By pushing the pole of Green's function $G$ to $\infty$ through the geometrically infinite end, normalizing at a fixed point, and using estimates of $|\nabla G|$ in terms of the injectivity radius, one can show there is a positive harmonic function $u$ on $M$ that is zero on $R_1 \subset \partial_\infty M$, and grows linearly in the geometrically infinite end, i.e., $u(z) \simeq 1 + \mathrm{dist}(z, \partial C(M))$ for $z \in C(M)$. See Bishop and P. W. Jones [1997b]. Note that $u$ lifts to a positive harmonic $U$ on $\mathbb{H}^3$, and $U$ must be the Poisson integral of a measure $\mu$ supported on $\Lambda$.

We expect Brownian motion, $B_t$, on the geometrically infinite end of $M$ to behave like a Brownian path in $[0, \infty)$. By the law of the iterated logarithm (LIL), we then expect $u(B_n)$ to be as large as $\sqrt{n \log\log n}$ infinitely often (i.o.). Since a Brownian path on $\mathbb{H}^3$ tends to the boundary at linear speed in the hyperbolic metric, this means that at $\mu$-a.e. $z \in \Lambda$, i.o. we have $U((1 - e^{-n}) \cdot z) \simeq \sqrt{n \log\log n}$. Estimates for the Poisson kernel then imply that $\mu$–a.e. point of $\Lambda$ is covered by disks such that

$$\mu(D(z,t)) \simeq \varphi(t) = t^2 \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}.$$

In fact, this optimistic calculation is actually correct; the paper Bishop and P. W. Jones [ibid.] shows that $\Lambda$ has finite, positive Hausdorff $\varphi$-measure, verifying a conjecture of Sullivan [1981a]. The optimal gauge $\varphi$ for the general case, where injectivity radius approaches zero, remains unknown. What about subsets of $\Lambda$ defined using geodesic rate of escape as in Gönye [2008], or Lundh [2003]?

## 3 Logarithms, length and Liouville

• **Makarov's theorems:** The LIL above for dendritic limit sets was much easier to discover because the connection between harmonic measure, random walks and Hausdorff dimension had already been uncovered by a celebrated result of Nick Makarov a decade earlier; see Makarov [1985]. Suppose $\Omega$ is planar and simply connected. He showed that if

$$\varphi_C(t) = t\exp\left(C\sqrt{\log\frac{1}{1-t}\log\log\log\frac{1}{1-t}}\right),$$

then there is a $C = C_1$ so that $\omega(E) = 0$ whenever $E$ has zero $\varphi_C$-measure. However, there is also a $C = C_2$, and a fractal domain $\Omega$, so that $\omega(E) = 1$ for some set $E \subset \partial\Omega$ of $\varphi_C$ measure zero. In fact, we can take $\Omega$ to be the interior of the von Koch snowflake, or any sufficiently "wiggly" fractal (some cases were known earlier, e.g., Carleson's paper Carleson [1985]). Makarov discovered that if $f : \mathbb{D} \to \Omega$ is conformal, then the harmonic function $g = \log|f'|$ behaves precisely like the dyadic martingale $\{u_n\}$ on $\mathbb{T}$ defined on each $n$th generation dyadic interval $I \subset \mathbb{T}$ by

$$u_n = \lim_{r\nearrow 1}\frac{1}{|I|}\int_I g(re^{i\theta})d\theta. \tag{1}$$

Distortion estimates for $f'$ imply this limit exists and $|g(z) - u_n(I)| = O(1)$, for any $z$ in the Whitney square corresponding to $I$. See Figure 13.

The $\{u_n\}$ have bounded differences, and the LIL for such martingales implies $|u_n(x)| = O(\sqrt{n\log\log n})$, for a.e. $x \in \mathbb{T}$. This, in turn, gives

$$|g(r\cdot x)| = O\left(\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}\right),$$

as $r \nearrow 1$ for a.e. $x \in \mathbb{T}$, and this implies Makarov's LIL. Makarov's discovery has since been refined and exploited in many interesting ways, e.g., it makes sense to talk about the asymptotic variance of $g = \log|f'|$ near the boundary and precise estimates for this have led to exciting developments in the theory of conformal and quasiconformal mappings, e.g., see the papers Astala, Ivrii, Perälä, and Prause [2015], Hedenmalm [2017], Ivrii [2016].

Makarov's LIL is just half of a remarkable theorem: $\dim(\omega) = 1$ for any simply connected planar domain, where $\dim(\omega) = \inf_E\{\dim(E) : \omega(E) = 1\}$. Since $\varphi_C(t) = o(t^\alpha)$ for any $\alpha < 1$, the LIL shows $\dim(E) < 1$ implies $\omega(E) = 0$. Hence $\dim(\omega) \geq$

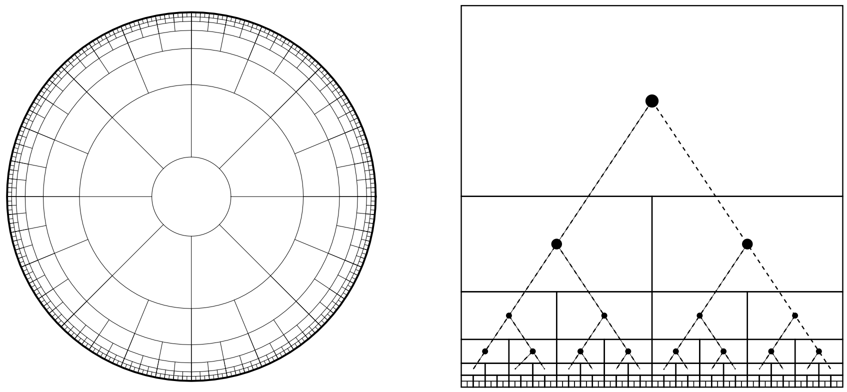

Figure 13: A Whitney decomposition of the disk and an enlargement near the boundary. Each box corresponds to a dyadic interval on the boundary. Although $g = \log|f'|$ is non-constant on each box, it is within $O(1)$ of the associated martingale value.

1. On the other hand, since $g = \log|f'|$ behaves like a martingale, along a.e. radius it is either bounded or oscillates between $-\infty$ and $\infty$. The boundary set where the former happens maps to $\sigma$-finite length (since this set is a countable union of sets where $|f'|$ is radially bounded) and the latter set maps to zero length (since $|f'| \to 0$ along some radial sequence). Thus $\dim(\omega) \leq 1$. See Pommerenke [1986]. For extensions to general planar domains, see P. W. Jones and Wolff [1988], Wolff [1993].

The obvious generalization to higher dimensions is that $\dim(\omega) = n$ for domains in $\mathbb{R}^{n+1}$. Bourgain [1987], proved $\dim(\omega) \leq n + 1 - \epsilon(n)$, Wolff [1995] constructed ingenious fractal "snowballs" in $\mathbb{R}^3$ where $\dim(\omega)$ can be strictly larger than or strictly smaller than 2, so the generalization above is false. In the plane, $\log|\nabla u|$ is subharmonic if $u$ is harmonic, and the failure of this key fact in $\mathbb{R}^3$ is the basis of Wolff's examples. However, in $\mathbb{R}^{n+1}$, $|\nabla u|^p$ is subharmonic if $p > 1 - 1/n$, and this suggests $\dim(\omega) \leq n + 1 - 1/n$ for all $\Omega \subset \mathbb{R}^{n+1}$, but this remains completely open.

• **Harmonic measure and rectifiability:** The F. and M. Riesz theorem (F. Riesz and M. Riesz [1920]) states that for a simply connected planar domain with a finite length boundary, harmonic measure and 1-measure are mutually absolutely continuous. Extending this has been a major goal in the study of harmonic measure for the last century.

For example, McMillan [1969] proved that for a general simply connected domain in $\mathbb{R}^2$, $\omega$ gives full measure to the union of two special subsets of the boundary: the cone points and the twist points. Cone points are simply vertices of cones inside $\Omega$, and on these points $\omega$ and Hausdorff 1-measure are mutually absolutely continuous. McMillan's theorem generalizes the F. and M. Riesz theorem since almost every point of a rectifiable curve is a tangent point, and hence is a cone point for each side.

A point $w \in \partial\Omega$ is a twist point if $\arg(z - w)$ on $\Omega$ is unbounded above and below in any neighborhood of $w$. More geometrically, any curve in $\Omega$ terminating at $w$ must twist arbitrarily far in both directions around $w$. On the twist points we have

$$\limsup_{r\to 0} \frac{\omega(D(x,r))}{r} = \infty, \qquad \liminf_{r\to 0} \frac{\omega(D(x,r))}{r} = 0. \tag{2}$$

The left side is due to Makarov [1985]; it implies that on the twist points, $\omega$ is supported on a set of zero length. The right side is due to Choi [2004].

Choi's theorem has an interesting consequence. Suppose $E$ consists of twist points, fix $\epsilon > 0$, and cover $\omega$-a.e. point of $E$ using disjoint disks such that $\omega(D(x_j, r_j)) < \epsilon r_j$ (use the Vitali covering lemma). Then any curve $\gamma$ containing $E$ has length at least

$$\ell(\gamma) \geq \sum_j r_j \geq \frac{1}{\epsilon} \sum_j \omega(D_j \cap E) \geq \frac{\omega(E)}{\epsilon},$$

i.e., $\ell(\gamma) = \infty$ if $\omega(E) > 0$. This implies the "local" F. and M. Riesz Theorem: if $E$ is a zero length subset of a rectifiable curve, then $\omega(E) = 0$ for any simply connected domain. A quantitative version of this, proven in Bishop and P. W. Jones [1990], Bishop and P. W. Jones [1994], was one of the first applications of Jones' $\beta$-numbers and his traveling salesman theorem characterizing planar rectifiable sets in terms of $\beta$-numbers P. W. Jones [1990]. There has been steady progress since this result on the relationship between harmonic measure and rectifiability, and even a sketch of this area would fill a survey longer than this one. A recent landmark, giving a converse to the local Riesz theorem in all dimensions, is due to Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa, and Volberg [2016]: if $\omega|_E \ll \mathcal{H}_n|_E$ then $\omega|_E$ is rectifiable (it's support can be covered by countably many Lipschitz graphs). Since $\omega$ is the normal derivative of Green's function $G$, $\omega \ll \mathcal{H}_n$ roughly means that $|\nabla G|$ is bounded near a subset of $E$, and this implies that the Riesz transforms (which relate the components of $\nabla G$) are bounded operators with respect to $\omega$ on a suitable subset. Several recent deep results on singular integral operators and geometric measure theory then imply rectifiability; e.g., see Léger [1999], Nazarov, Tolsa, and Volberg [2014], Nazarov, Treil, and Volberg [2003].

The left side of Equation (2) has an amusing corollary. If $x \in \partial\Omega_1 \cap \partial\Omega_2$, where $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ are disjoint with harmonic measures $\omega_1, \omega_2$ (fix some base point in each), then by Bishop [1991]

$$\omega_1(D(x,r)) \cdot \omega_2(D(x,r)) = O(r^{2n}). \tag{3}$$

Now assume $n = 1$ and $\gamma = \partial\Omega_1 = \partial\Omega_2$ is a closed Jordan curve. By the left side of Equation (2), $\omega$-a.e. twist point of $\Omega_1$ can be covered by disks where $\omega_1(D) \gg r$, so by Equation (3), these disks must also satisfy $\omega_2(D) \ll r \ll \omega_1(D)$. This implies $\omega_1 \perp \omega_2$ on the twist points of $\gamma$. On the tangent points of $\gamma$, $\omega_1$ and $\omega_2$ are mutually continuous to each other and to 1-measure, so $\omega_1 \perp \omega_2$ on $\gamma$ if and only if the set of tangent points of $\gamma$ has zero length; see Bishop, Carleson, Garnett, and P. W. Jones [1989]. This happens for the von Koch snowflake, as well as many other fractal curves. See Figure 14.

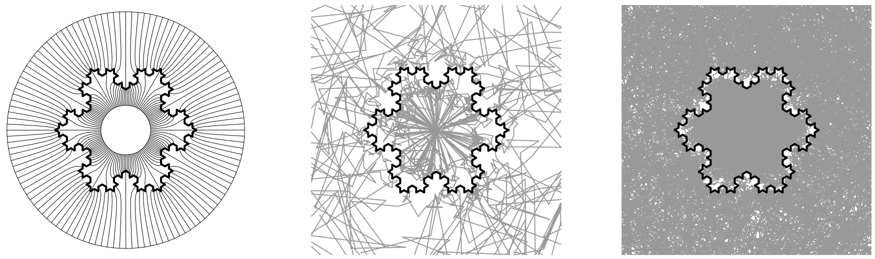

Figure 14: Conformal images of 120 equally spaced radial lines, illustrating the singularity of the inner and outer harmonic measures. On the right are 100 and 1000 Kakutani walks on each side; white shows points that are hard to hit from either side.

One way to think about Equation (2) is to consider a castle whose outer wall is a snowflake. If the fractal fortress is attacked by randomly moving warriors, then only a zero length subset of the wall needs to be defended, whereas if the fortress wall was finite length then it must all be defended. Thus a fractal fortress would be easier to defend (at least against a drunken army). However, because of the local Riesz theorem, it would take an officer infinite time to inspect all the defended positions.

• **Conformal welding:** We would like to compare $\omega_1, \omega_2$ for the two sides of a curve $\gamma$, but $\omega_1/\omega_2$ does not make sense in general. Instead, we consider the orientation preserving (o.p.) circle homeomorphism $h = g^{-1} \circ f$, where $f$ and $g$ are conformal maps from the two sides of the unit circle to the two sides of $\gamma$. Such an $h$ is called a "conformal welding" (CW). Not every circle homeomorphism is a conformal welding (see Figure 15), and a useful characterization is likely to be very difficult to find.

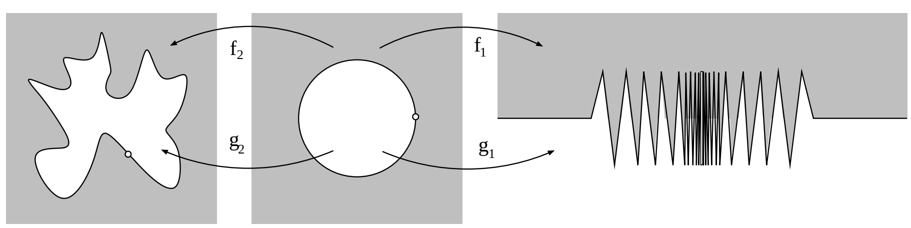


Figure 15: If $f_1, g_1$ map the two sides of $\mathbb{T}$ to the two sides of a $\sin(1/x)$ curve $\gamma$, then $h = g_1^{-1} \circ f_1$ is a homeomorphism, but is not a CW. Otherwise, $h = g_2^{-1} \circ f_2$ with maps corresponding to a Jordan curve, and then (by Morera's theorem) $f_2 \circ f_1^{-1}$ and $g_2 \circ g_1^{-1}$ would define a conformal map from the complement of a segment to the complement of a point, contradicting Liouville's theorem.

If $h(z) = z$, then the maps $f, g$ are equal on $\mathbb{T}$, so by Morera's theorem they define a 1-1 entire function, and this must be linear by Liouville's theorem. Thus only circles can have equal harmonic measures on both sides. If $h$ is bi-Lipschitz with constant near

1, David [1982] showed the corresponding curve is rectifiable, but for large constants the curve can have infinite length (see citeMR852832), or even dimension close to 2 (see Bishop [1988]). Nothing is known about where this transition occurs.

Every "nice" o.p. circle homeomorphism is a conformal welding, where "'nice" means quasisymmetric; this includes every diffeomorphism but also many singular maps. These send full Lebesgue measure on $\mathbb{T}$ to zero measure; this happens exactly when $\omega_1 \perp \omega_2$, as for the snowflake. Surprisingly, all sufficiently "wild" homeomorphisms are also conformal weldings, where "wild" means log-singular: there is a set $E$ of zero logarithmic capacity on the circle so that $\mathbb{T} \setminus h(E)$ also has zero logarithmic capacity. Zero logarithmic capacity sets are very small, e.g., Hausdorff dimension zero, so log-singular homeomorphisms are very, very singular. See Lundberg [2005]. Moreover, each log-singular map $h$ corresponds to a dense set of all closed curves in the Hausdorff metric, so the association $h \leftrightarrow \gamma$ is far from 1-to-1. See Bishop [2007].

To illustrate the gap between these two cases, consider the space of circle homeomorphisms with the metric $d(f,g) = |\{x : f(x) \neq g(x)\}|$. This space has diameter $2\pi$ and the set of QS-homeomorphisms and log-singular homeomorphisms are distance $2\pi$ apart. However, conformal weldings are known to be dense in this space; see Bishop [ibid.]. Are they a connected set in this metric? Residual? Borel? For some generalizations and applications of conformal welding, see the papers of Feiszli [2008], Hamilton [1991], and Rohde [n.d.].

## 4 True trees and transcendental tracts

• **Dessins d'enfants:** As noted above, a curve $\gamma$ with $\omega_1 = \omega_2$ must be a circle. Thus in terms of harmonic measure, a circle is the most "natural" way to draw a closed Jordan curve. What happens for other topologies? Can we draw any finite planar tree $T$ so harmonic measure is equal on "both sides"? More precisely, with respect to the point at infinity, can we draw $T$ so that

(1) every edge has equal harmonic measure,

(2) any subset of any edge has equal harmonic measure from both sides?

Perhaps surprisingly, the answer is yes, every finite planar tree $T$ has such drawing, called the "true form of the tree" (or a "true tree" for short). To prove this, consider Figure 16. Let $\tau$ be a quasiconformal map of the exterior $\Omega$ of $T$ to $\mathbb{D}^* = \{z : |z| > 1\}$, with each side of $T$ mapping to an arc of length $\pi/n$, and arclength on each edge mapping to a multiple of arclength in the image. Let $J(z) = \frac{1}{2}(z + \frac{1}{z})$ be the Joukowsky map; this is conformal from $\mathbb{D}^*$ to $U = \mathbb{C} \setminus [-1, 1]$. Then $q(x) = J(\tau(z)^n)$ is quasiregular off $T$ and continuous across $T$, so is quasiregular on $\mathbb{C}$.

By the measurable Riemann mapping theorem there is a QC "correction" map $\varphi : \mathbb{C} \to \mathbb{C}$ so that $p = q \circ \varphi$ is holomorphic. Since $p$ is also $n$–to–1, it must be a polynomial of degree $n$. Its only critical values are $\pm 1$, so it is a generalized Chebyshev, or Shabat, polynomial and $T' = \varphi(T) = p^{-1}([-1, 1])$ is a true tree.

It is easy to see that the polynomial $p$ can be normalized to have its coefficients in some algebraic number field. This connection is part of Grothendieck's' theory of

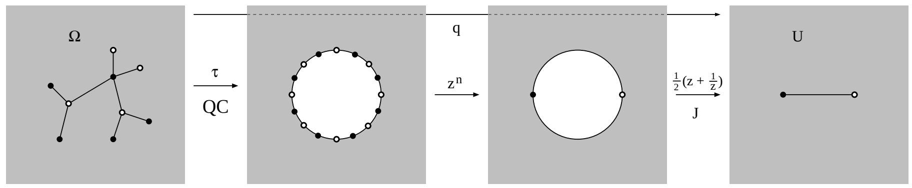


Figure 16: For a true tree, the conformal map $\tau : \mathbb{C} \setminus T \to \mathbb{D}^*$ sends sides of $T$ to arcs of equal length arcs. In general, we choose a QC map $\tau$ that sends normalized arclength on sides of $T$ to arclength on $\mathbb{T}$; then $q(z) = J(\tau(z)^n)$ is continuous across $T$ and quasiregular on $\mathbb{C}$.

*dessins d'enfants* and is closely connected to the spherical case of Belyi's theorem: a Riemann surface is algebraic iff it supports a meromorphic function ramified over three values. There are many fascinating related problems, e.g., Grothendieck proved that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of planar trees, but the orbits are unknown (some things are known, e.g., equivalent trees have the same set of vertex degrees). For more background see G. A. Jones and Wolfart [2016], Schneps [1994], Shabat and Zvonkin [1994].

It is a difficult problem to compute the correspondence between trees and polynomials, but this has been done by hand for trees with 10 or fewer edges, Kochetkov [2007], Kochetkov [2014]. It is possible to go much farther using harmonic measure. Don Marshall and Steffen Rohde have adapted Marshall's conformal mapping program `ZIPPER`; to compute the true form of a given planar tree (even with thousands of edges). See Marshall and Rohde [2007]. For small trees (less than 50 edges or so) they can obtain the vertices (and hence the polynomial) to thousands of digits of accuracy. Given enough digits of an algebraic integer $\alpha \in \mathbb{R}$ one can search for an integer relation among $1, \alpha, \alpha^2, \dots,$ that determines the field, e.g., using Helaman Ferguson's PSLQ algorithm; see Ferguson, Bailey, and Arno [1999].

Alex Eremenko asked if Shabat polynomials have special geometry. In Bishop [2014], I showed the answer is no in the sense that given any compact, connected set $K$ there are polynomials with critical values $\pm 1$ whose critical sets approach $K$ in the Hausdorff metric. In particular, the true tree $T = p^{-1}([-1, 1])$ can be $\epsilon$-close to any connected shape, i.e., "true trees are dense". See Figure 17.

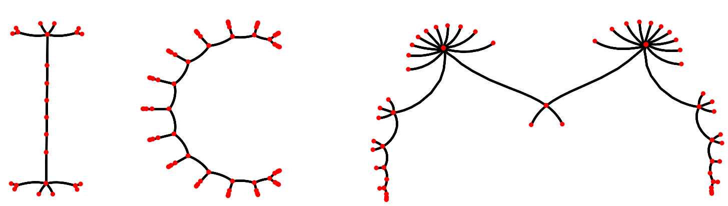

Figure 17: True trees approximating some random letters of the alphabet.

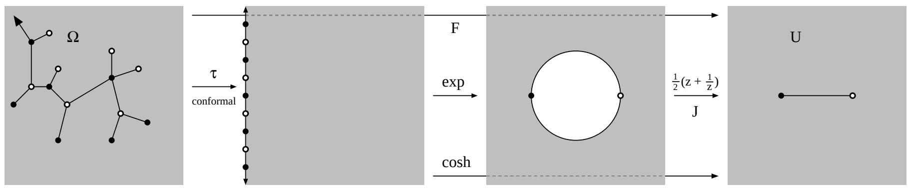


Figure 18: The transcendental version of Figure 16. $F$ is holomorphic off $T$ but not necessarily across $T$. QC-folding defines a quasiregular $g$ so that $g = F$ outside a "small" neighborhood of $T$.

Is there a higher dimensional analog of true trees? In what other settings does "equal harmonic measure from both sides" makes sense and lead to interesting problems? If we drop (1) from the definition of a true tree, then we get trees that connect their vertices using minimum logarithmic capacity. See Stahl [2012].

• **Dessins d'adolescents:** Given the connection between true trees and polynomials, it is natural to ask about a correspondence between infinite planar trees and entire functions, e.g., is every unbounded planar tree $T$ equivalent to $f^{-1}([-1,1])$ for some entire function $f$ with critical values $\pm 1$? The answer is no: one can show the infinite 3-regular tree is not of this form. However, a version of the "true trees are dense" construction does hold. Consider how to adapt the construction in Figure 16 to unbounded trees, as in Figure 18. Now, $\Omega = \mathbb{C} \setminus T$ is a union of unbounded, simply connected domains, called tracts, and each of these tracts can be mapped to $\mathbb{H}_r = \{x + iy : x > 0\}$, by a conformal map $\tau$. The power function $z^n$ is replaced by $\exp : \mathbb{H}_r \to \mathbb{D}^*$, but is still followed by the Joukowsky map, giving a holomorphic function $F(z) = J(\exp(\tau(z)))$ on each tract, but $F$ need not be continuous across $T$. Fixing this requires some assumptions (some regularity of $T$ that replaces finiteness). Via $\tau$, the vertices of $T$ define a partition of $i\mathbb{R} = \partial\mathbb{H}_r$ and we assume that this partition satisfies

(1) adjacent intervals have comparable length,

(2) interval lengths are all $\geq \pi$.

Under these hypotheses, the QC-folding theorem from Bishop [2015] gives a quasiregular $g$ that agrees with $F$ outside $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r \cdot \mathrm{diam}(e)\}$, where the union is over all edges in $T$. The tree $T' = g^{-1}([-1,1])$ satisfies $T \subset T' \subset T(r)$. The measurable Riemann mapping theorem gives a quasiconformal $\varphi$ so that $f = g \circ \varphi^{-1}$ is an entire function with critical values $\pm 1$ and no other singular values (the singular set $S(f)$ is the closure of the critical values and finite asymptotic values, i.e., limits of $f$ along curves to $\infty$).

Since $g$ is holomorphic off $T(r)$, $\mu_\varphi$ is supported in $T(r)$ and is uniformly bounded in terms of the assumptions on $T$. In many applications $T(r)$ has finite, even small, area, and $\varphi$ is close to the identity. Thus the QC-folding theorem converts an infinite planar tree $T$ satisfying some mild restrictions into an entire function $f$ with $S(f) = \{\pm 1\}$, and such that $T' = f^{-1}([-1,1])$ is "close to" $T$ in a precise sense.

Let $\mathcal{T}$ denote the transcendental entire functions (non-polynomials). The Speiser class is $\mathcal{S} = \{f \in \mathcal{T} : S(f) \text{ is finite}\}$, and the Eremenko-Lyubich class is $\mathcal{B} = \{f \in$

$\mathcal{T} : S(f)$ is bounded}. The QC-folding theorem (or simple modifications) gives a flexible way to construct examples in $\mathcal{S}$ and $\mathcal{B}$ with specified singular sets, including:

► A $f \in \mathcal{B}$ with a wandering domain. Wandering domains do not exist for rational functions by Sullivan's non-wandering theorem Sullivan [1985], nor in $\mathcal{S}$ by work of Eremenko and Lyubich [1992] and Goldberg and Keen [1986]. See Figure 19. See also the paper of Lazebnik [2017].

► A $f \in \mathcal{S}$ so that area$(\{z : |f(z)| > \epsilon\}) < \infty$ for all $\epsilon$. This is a strong counterexample to the area conjecture of Eremenko and Lyubich [1992].

► A $f \in \mathcal{S}$ whose escaping set has no non-trivial path components; this improves the counterexample to the strong Eremenko conjecture in $\mathcal{B}$ due to Rottenfusser, Rückert, Rempe, and Schleicher [2011].

► A $f \in \mathcal{S}$ so that $\limsup_{r\to\infty} \log m(r, f)/\log M(r, f) = -\infty$ where $m, M$ denote the min, max of $|f|$ on $\{|z| = r\}$. In 1916 Wiman had conjectured $\limsup \geq -1$, as occurs for $\exp(z)$. Beurling [1949] gave a partial positive result, but Hayman [1952] found a counterexample in general, and QC-folding now improves this to $\mathcal{S}$.

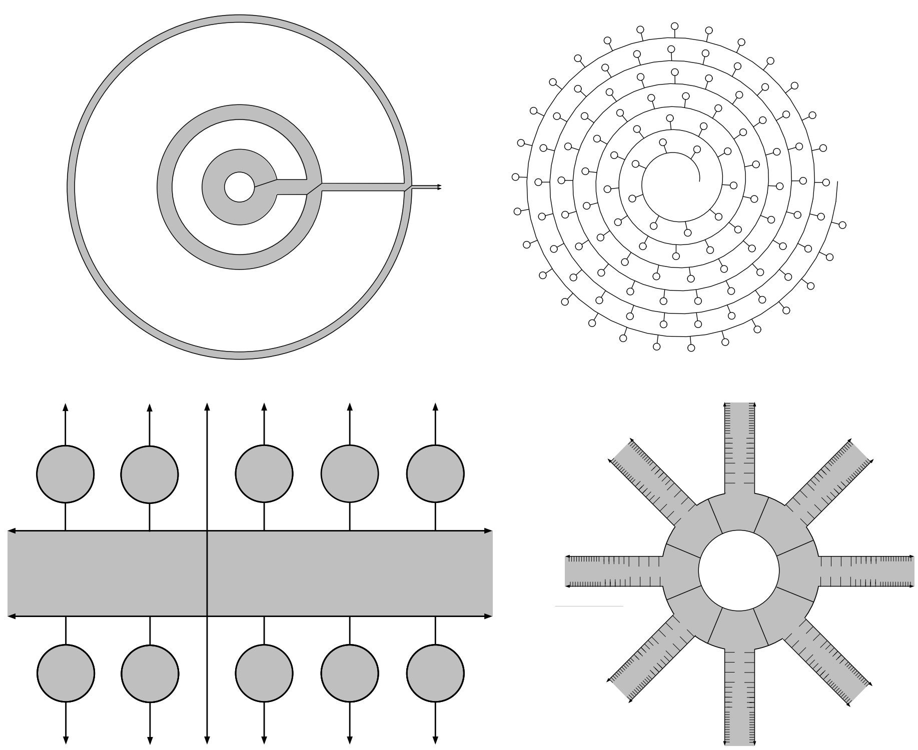

Figure 19: The folding theorem reduces constructing certain entire functions to drawing a picture. Here are the pictures associated to counterexamples for the area conjecture (upper left), Wiman's conjecture (upper right), an Eremenko-Lyubich wandering domain (lower left) and a Speiser class Julia set of dimension near 1.

► $f \in \mathcal{S}$ with Julia sets so that $\dim(\mathcal{J}) < 1+\epsilon$ Albrecht and Bishop [2017]. Examples in $\mathcal{B}$ are due to Stallard [1997], Stallard [2000], who also showed $\dim(\mathcal{J}) > 1$ for $f \in \mathcal{B}$. Baker [1975] proved $\dim(\mathcal{J}) \geq 1$ for all $f \in \mathcal{T}$, and examples with $\dim(\mathcal{J}) = 1$ (even with finite spherical linear measure) exist Bishop [2018], but it is unknown whether they can lie on a rectifiable curve on the sphere.

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