A CURVE WITH NO SIMPLE CROSSINGS BY SEGMENTS

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Abstract. We construct a closed Jordan curve $\gamma \subset \mathbb{R}^2$ so that $\gamma \cap S$ is uncountable whenever S is a line segment whose endpoints are contained in different connected components of $\mathbb{R}^2 \setminus \gamma$.

We say that a Jordan arc $\sigma \subset \mathbb{R}^2$ crosses a compact set $K \subset \mathbb{R}^2$ if the two endpoints of σ are in different connected components of $\mathbb{R}^2 \setminus K$. Clearly any arc crossing K must intersect K in at least one point of K. If the intersection consists of exactly one point, we say K has a simple crossing by σ. In this note we answer a question of Percy Deift by constructing a closed Jordan curve $\gamma \subset \mathbb{R}^2$ so that $\gamma \cap S$ is uncountable whenever S is a line segment crossing γ, i.e., γ has no simple crossings by a line segment. Very likely, such examples are known to a variety of people, but I am not aware of a reference in the literature.

We will construct a sequence of closed Jordan curves $\{\gamma_n\}$ and a decreasing sequence of positive real numbers $\{\epsilon_n\} \searrow 0$ so that so that if we set

$$\Gamma_n = \{z \in \mathbb{R}^2 : \text{dist}(z, \gamma_n) \leq \epsilon\},$$

then

(1) $\Gamma_{n+1} \subset \Gamma_n$ for $n = 0, 1, 2, \dots$ and $\gamma = \cap_{n=0}^{\infty} \Gamma_n$ is a closed Jordan curve.
(2) Any closed segment that crosses Γ_n contains at least two disjoint closed subsegments that each cross Γ_{n+1}.

If S is any closed segment that crosses γ, then there is a $\epsilon > 0$ so that both endpoints of S are at least distance ϵ from γ and hence these endpoints are in different complementary components of Γ_n for some n. Thus S crosses Γ_n. Claim (2) above then implies that each component of $S \cap \Gamma_{n+k}$ contains two components of $S \cap \Gamma_{n+k+1}$, and this implies that $S \cap \gamma$ contains a Cantor set and is uncountable.

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Thus it suffices to construct curves $\{\gamma_n\}$ and numbers $\{\epsilon_n\}$ so that (1) and (2) are satisfied. We can start with γ_0 being a circle and shall proceed in such a way that each γ_n is a finite union of circular arcs $\mathcal{A}_n = \{A^n_k\}$, that are disjoint except for their endpoints, and so that γ_n is continuously differentiable (so adjacent circular arcs have a common tangent where they meet). See Figure 1. Since γ_0 clearly has this form, it suffices to describe how to construct γ_n from γ_{n-1}.

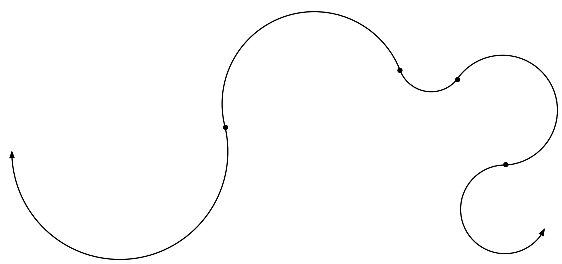

FIGURE 1. The curves γ_n will be finite unions of circles joined end-to-end to for a differentiable curve.

We let the radius of a circular arc A denote the radius of the circle containing A (if A is a line segment, we let the radius be ∞, although we will not use this case in this paper). Since γ_n is a finite union of circular arcs, there is a minimum radius r_n that occurs and this number must be positive. There is also a positive distance $d_n > 0$ so that any two non-adjacent arcs in $\mathcal{A}_n$ are distance at least d_n apart.

Given γ_{n-1} as above, choose $0 < \epsilon_n < \min(d_{n-1}, r_{n-1})$ and choose a finite collection of points $\{z^n_k\} \subset \gamma_n$, for $k = 1, \dots, N_n$. We will assume N_n is even. These points should include every endpoint of every arc in $\mathcal{A}_n$ and are chosen so that

$$|z^n_k - z^n_{k+1}| < \delta \cdot \epsilon_n,$$

where $0 < \delta < 1$ is a fixed constant that we will specify below. (Here and later, indices are considered modulo N_n, so the equation includes the case $|z^n_{N_n} - z^n_1| < \delta\epsilon_n$.)

If $\delta \leq 1$, the collection of open disks, $\mathcal{D}_n$, given by $D^n_k = D(z^n_k, \epsilon_n)$ covers γ_{n-1}. If δ is small, then each point of γ_n is contained in a large number of the disks $\{D^n_k\}$ (approximately $1/\delta$ disks). Let $C^n_k = \{z : |z - z^n_k| = \epsilon_n\}$ denote the boundary circle of D^n_k.

Since $\epsilon_n < d_n$, a disk of this form can only hit a disk that is centered on the same circular arc in $\mathcal{A}$ or is centered on one of the two adjacent arcs. Since $\epsilon_n < r_n$, the closure of D^n_k does not hit the closure of $D^n_{k+2} \setminus D^n_{k+1}$. See Figure 2.

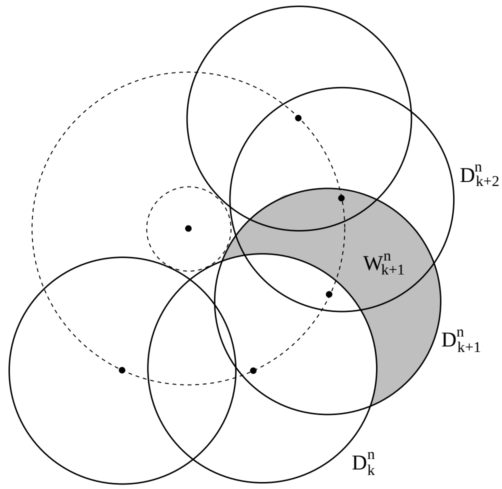

FIGURE 2. Disks centered along γ_n are nicely separated in the sense that $W^n_{k+1} = D^n_{k+1} \setminus D^n_k$ defines a crescent that only touches W^n_{k+2} and W^n_{k+2} and no other crescents in the chain.

Let $w^n_k = C^n_k \cap \gamma_n \cap D^n_{k+1}$ (our assumptions imply this is a single point). Let L^n_k be the line perpendicular to γ_{n-1} at w^n_k. Let H^n_k be the half-plane defined by L^n_k and not containing z^n_k. Because of our choice of δ, every ray with vertex w^n_k that lies in H^n_k crosses C^n_j for $j = k+1, \dots, k+3$. See Figure 3.

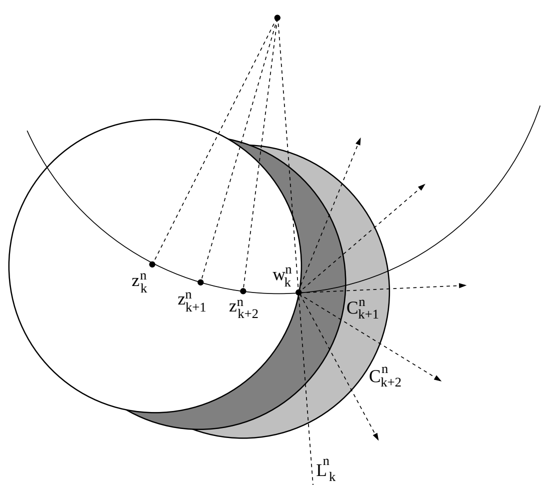

FIGURE 3. The line L^n_k through w^n_k, perpendicular to γ_n define a half-plane H^n_k. Every ray in H^n_k with base point w^n_k, hits the circular arcs $C^n_{k+1} \setminus D^n_k$ and $C^n_{k+2} \setminus D^n_{k+1}$.

Thus we can place a disk $\widetilde{D}_k^n$ inside the crescent $D_{k+1}^n \setminus D_k^n$ that is tangent to both side of the crescent and that is disjoint from H_k^n. For example, we could make $\widetilde{D}_k^n$ be tangent to both sides and also tangent to L_k^n. There are two possible locations for such a disk, one near either vertex of the crescent, i.e., on either side of γ_n. If n is even, we put the disk D_k^n in the bounded complementary component of γ_n, and if n is odd we place it in the unbounded component; thus the disks alternate sides we move along γ_n. See Figure 4.

The curve γ_{n+1} is formed by following the arc of C_k^n from w_k^n to the disk $\widetilde{D}_k^n$, follow the boundary of this disk until it hits C_{k+1}^n and then follow this circle to w_n^{k+1}; requiring the curve to be C^1 determines which direction we travel on each circle. We start the procedure at w_1^n and continue until we return to w_1^n. This gives the curve γ_{n+1}. See Figure 4.

By construction, any curve that crosses γ_n, crosses Γ_{n+1} at least twice. This is condition (2) above, and proves that the limiting curve has no simple crossing by a line segment.

We can modify the construction so there is a constant $C < \infty$ so that for any $\delta > 0$ we can take so that $\epsilon_{n+1} = \delta\epsilon_n$ and so that any line crossing Γ_n crosses Γ_{n+1} at least $(C\delta)^{-1}$ times with disjoint segments that each have length at least $\epsilon_n\delta$. By standard estimates this implies the intersection of any crossing line segment with γ has Hausdorff dimension at least $1 - (\log C)/(\log \frac{1}{\delta})$. By replacing a fixed δ by a sequence tending to zero, one can construct a curve γ whose intersection with any crossing segment is 1.

It seems likely that one can construct a curve γ (necessarily of positive area) that intersects any crossing segment in positive length, however, this is not immediate from the construction in this note. However, an analogous construction using variable sized disks might work. In the other direction we can ask about replacing line segments by more general curves, e.g., rectifiable curves. Is there a closed curve γ with no simple crossings by a rectifiable curve? Is there a curve γ that intersects every crossing rectifiable curve σ in positive length? Is there a relationship between the modulus of continuity of the parameterization of a closed curve γ and the modulus of continuity of an arc that crosses it simply? Images of radial segments under the Riemann maps onto the complementary components of of γ, give simple crossing curves, so

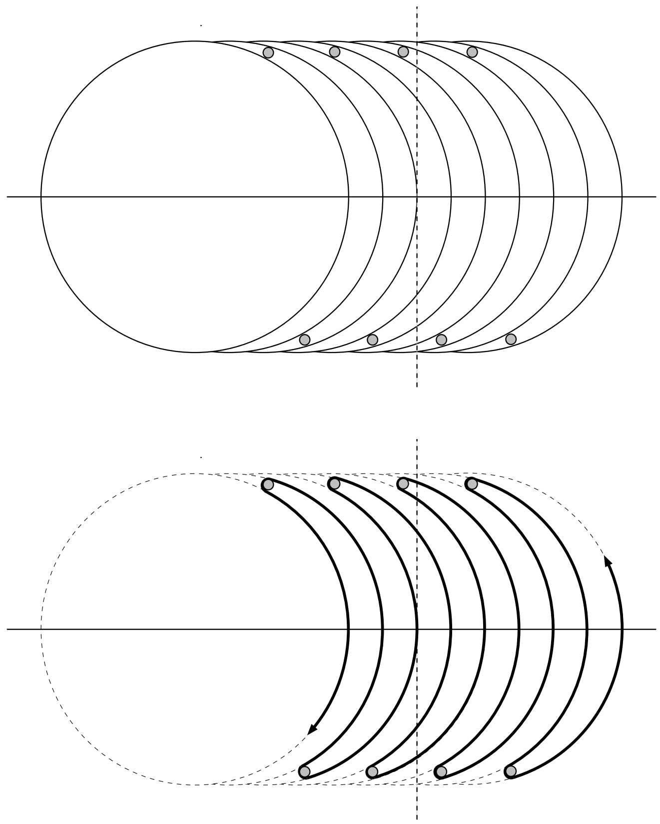

FIGURE 4. For clarity, we have drawn γ_{n-1} as a horizontal straight line, and the disks $\{D_k^n\}$ as equally spaced. Any segment crossing γ_{n-1} must cross γ_n at least twice (at points where the crossing angle is bounded away from zero, but we won't use this here).

the modulus of continuity of the Riemann map would give some bounds; see [2]. Are there "natural" examples of curves that can't be simply crossed by line segments, e.g., self-similar fractals or a SLE path? It is known that 2-dimensional Brownian motion cannot be simply crossed by a segment [1], but this is not a Jordan curve.

References

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