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SOME QUESTIONS CONCERNING HARMONIC MEASURE*

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The purpose of this note is to discuss some conjectures concerning harmonic measure. We will start by considering harmonic measure on a simply connected plane domain Ω, but eventually we will also consider some multiply connected and higher dimensional domains. Most of these questions are trivial if $\partial\Omega$ has tangents a.e. and many are easy if Ω is only a quasicircle. Thus they are really questions about very non-smooth domains.

Let Ω be a simply connected plane domain and let ω denote harmonic measure on Ω. For our purposes the choice of base point is unimportant. Fix $x \in \partial\Omega$ and define a continuous branch of $\arg(z-x)$ on Ω. We say x is a *twist point* of Ω if both

$$\liminf_{z\to x, z\in\Omega} \arg(z-x) = -\infty$$

and

$$\limsup_{z\to x, z\in\Omega} \arg(z-x) = +\infty.$$

On the other hand, we say Ω has an *inner tangent* at x if there is a unique $\theta_0 \in [0, 2\pi)$ such that for every $0 < \epsilon < \pi/2$ there is a $\delta > 0$ such that :

$$\{x + re^{i\theta} : 0 < r < \delta, |\theta - \theta_0| < \pi/2 - \epsilon\} \subset \Omega$$

Up to a set of Λ_1 (one dimensional Hausdorff) measure zero, this is the same as simply requiring x to be the vertex of a cone in Ω. McMillan's twist point theorem [22] states that (with respect to harmonic measure) almost every boundary point of Ω is of one of these two types. We will let $\mathrm{Tan}(\Omega)$ and $\mathrm{Tw}(\Omega)$ denote the set of inner tangents and twist points respectively.

On the set of inner tangents, harmonic measure is mutually absolutely continuous with Λ_1 [22]. ¿From the point of view of this note, this is the "trivial" case. What is more interesting to us is what harmonic measure looks like on the twist points. Only a few things are known. First, there is a subset $A \subset \mathrm{Tw}(\Omega)$ such that $\omega(A) = \omega(\mathrm{Tw}(\Omega))$ but such that $\Lambda_1(A) = 0$ (see [20], [24]). Precise estimates concerning the Hausdorff dimension of A have been given by Makarov (see our remarks below). Also, $\Gamma \cap \mathrm{Tw}(\Omega)$ has zero harmonic measure whenever Γ is rectifiable ([10]). This says that even though harmonic measure on the twist points is concentrated

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on a set of zero length, this set is so dispersed in the plane that it cannot lie on any rectifiable curve.

In [24] it is shown that

$$\limsup_{r\to 0} \frac{\omega(D(x,r))}{r} = \infty$$

at almost every (ω) twist point. (At almost every tangent point this ratio has a finite, nonzero limit.) Our first conjecture is

CONJECTURE 1. *At almost every* (ω) *twist point*

$$\liminf_{r\to 0} \frac{\omega(D(x,r))}{r} = 0.$$

Thus at twist points the ratio must oscillate between 0 and ∞ as $r \to 0$. This is sometimes called the "lower density" conjecture. This is true if Ω is a quasidisk. We should also point out that this conjecture implies the result of [10] about twist points and rectifiable curves. For suppose $E \subset \mathrm{Tw}(\Omega)$ has positive harmonic measure and $E \subset \Gamma$. Choose ϵ very small and note that a.e. $x \in E$ is contained in arbitrarily small disks D such that $\omega(D)/\mathrm{rad}(D) < \epsilon$. We can choose a covering of a.e. point of E by a collection of such disks which has bounded overlap, say each point in at most M disks. Then

$$\Lambda_1(\Gamma) \geq M^{-1} \sum \mathrm{rad}(D_j) \geq M^{-1}\epsilon^{-1} \sum \omega(D_j) \geq M^{-1}\epsilon^{-1}\omega(E).$$

Taking $\epsilon \to 0$ shows $\Lambda_1(\Gamma) = \infty$.

An equivalent formulation of Conjecture 1 is:

CONJECTURE 2. *Suppose $E \subset \partial\Omega$ has the following property. For any disjoint collection of disks $\{D_j\}$ with centers in E and radii bounded by diam(E), $\sum \mathrm{rad}(D_j) \leq M(E) < \infty$. Then $\Lambda_1(E) = 0$ implies $\omega(E) = 0$.*

This condition is a generalization of being a subset of a rectifiable curve (if $E \subset \Gamma$, the sum of radii in the conjecture is bounded above by the length of Γ), but also includes other sets, such as the "1/4" square Cantor set. To see that it implies Conjecture 1, suppose $E \subset \mathrm{Tw}(\Omega)$ is a set where the liminf is greater than some $\epsilon > 0$. Then E satisfies the hypothesis of Conjecture 2 since

$$\sum \mathrm{rad}(D_j) \leq \epsilon^{-1} \sum \omega(D_j) \leq \epsilon^{-1} \sum \omega(E).$$

Since $E \subset \mathrm{Tw}(\Omega)$ it can be chosen to have zero Λ_1 measure so Conjecture 2 implies it has zero harmonic measure. To prove (1) $\Rightarrow$ (2) use (1) to cover E by a disjoint collection of disks $\{D_j\}$ with $\omega(D_j) \leq \epsilon r_j$. Then $\omega(E) \leq \epsilon M$ is as small as we wish.

Our next few questions require some notation. We will now consider a fixed closed Jordan curve Γ in the plane and its two complementary components Ω_1 and Ω_2. A point $x \in \Gamma$ is a tangent of Γ if it is an inner tangent of both sides and is a twist point of Γ if it is a twist point of either side. Harmonic measures for these domains are denoted ω_1 and ω_2. For $r < t < 1$ let $\theta_i(t)$ denote the angle measure of the largest arc in $\partial D(x,t) \cap \Omega_i$ for $i = 1, 2$. Then the Ahlfors distortion theorem implies

$$\omega_i(\Gamma \cap D(x,r)) \leq C \exp\{-\pi \int_r^1 \frac{dt}{t\theta_i(t)}\},$$

Since $\theta_1(t) + \theta_2(t) \leq 2\pi$ this easily implies

$$\omega_1(\Gamma \cap D(x,r))\omega_2(\Gamma \cap D(x,r)) \leq Cr^2.$$

Now define

$$\epsilon(x,t) = \max_{i=1,2}\{|\pi - \theta_i(t)|\}.$$

A simple calculation shows

$$\frac{1}{\theta_1(t)} + \frac{1}{\theta_2(t)} \geq \frac{2}{\pi} + \frac{2}{\pi}(\frac{\epsilon(x,t)}{\pi})^2,$$

so

$$\frac{\omega_1(D(x,r))\omega_2(D(x,r))}{r^2} \leq C_1 \exp(-C_2 \int_r^1 \epsilon^2(x,t)\frac{dt}{t}).$$

On the tangent points of Γ the left hand side is bounded away from zero so the integral on the right must converge. Our next conjecture (due to L. Carleson) is that the converse holds.

CONJECTURE 3. *Except for a set of zero Λ_1 measure, x is a tangent point of Γ iff*

$$\int_0^1 \epsilon^2(x,t)\frac{dt}{t} < \infty.$$

This is some times called the "epsilon squared" conjecture. Note that it does not mention harmonic measure at all, although it has equivalent reformulations that do. For example: if $E \subset \Gamma$ then $\omega_1 \perp \omega_2$ on E iff

$$\int_0^1 \epsilon^2(x,t)\frac{dt}{t} = \infty,$$

at a.e. (Λ_1) point of E. A weaker version of this conjecture has been proved in [11]. It should also be compared with the Stein-Zygmund theorem of L^2 differentiability of functions [26, Theorem VIII.6].

A related question for simply connected domains has been asked by McMillan.

CONJECTURE 4. *For a.e. $x \in \partial\Omega$ (ω), $\liminf_{r\to 0} \theta(x,r) \leq \pi$.*

Conjecture 4 is easy if Ω is a quasidisk. By definition this holds at inner tangent points, so this is really only a question about the twist points.

The estimate on the product of harmonic measures given above for the plane is also true in higher dimensions, i.e., if Ω_1 and Ω_2 are disjoint domains in $\mathbf{R}^n$ then (with appropriate normalizations)

$$\omega_1(D(x,r)\cap\partial\Omega_1)\omega_2(D(x,r)\cap\partial\Omega_2)\le Cr^{2(n-1)}.$$

For $n>2$ this follows from an estimate of Friedland and Hayman [15] concerning eigenvalues of the Laplacian. To describe this result suppose u is positive and subharmonic on $\mathbf{R}^n$ and vanishes on $\partial\Omega$, and for $r>0$ let $S(x,r)=\partial D(x,r)$ and define

$$m_r(u)=(\int_{S(x,r)}u^2d\sigma)^{1/2}$$

where σ is surface measure on the sphere normalized to have mass 1. They show

$$m_r(u)\le Cm_1(u)\exp(-\int_r^{1/2}\alpha(t)\frac{dt}{t})$$

where $\alpha(t)$ is the characteristic constant of the $n-1$ dimensional set $\Omega(t)$ which is the radial projection of $\Omega\cap S(x,t)$ onto the unit sphere. This can be defined by $\alpha(\alpha+n-2)=\lambda$ where

$$\lambda(\Omega(t))=\inf\frac{\int|\nabla_S f|^2d\sigma}{\int|f|^2d\sigma},$$

where the "inf" is over all Lipschitz, nonnegative functions vanishing off $\Omega(t)$ and $\nabla_S f$ denotes the spherical gradient of f. The constant $-\lambda$ is also the first eigenvalue for the Dirichlet problem with vanishing boundary conditions, at least if $\Omega(t)$ is smooth enough. If f is the eigenfunction corresponding to λ then $u(x)=|x|^\alpha f(x/|x|)$ is harmonic in the cone defined by $\Omega(t)$ iff $\alpha(\alpha+n-2)=\lambda$. See [15] for details.

Now suppose $\Omega_1,\dots,\Omega_m$ are disjoint domains in $\mathbf{R}^n$ and $x\in\cap_j\partial\Omega_j$. Then

$$\prod_{j=i}^m\omega_j(D(x,r))\le Cr^{m(n-2)}\exp(-\int_r^1\sum_{j=1}^m\alpha_j(t)\frac{dt}{t})$$

where α_i are the characteristic constants for the Dirichlet problem on the domains $\Omega_j(t)$. It follows from results in [15] that if Ω_1,Ω_2 are disjoint domains on the sphere then $\alpha_1+\alpha_2\ge 2$, which proves the product formula mentioned above. For $n=2$ and $m>2$ it easy to see that the sum of the α_j's is minimized when each Ω_j is a arc of length $2\pi/m$, but in higher dimensions the extremal configuration is not obvious. For example, when $n=m=3$ I suspect the worst case cases should consist of three spherical domains constructed by connecting two antipodal points on the sphere by three geodesics meeting at 120 degrees. In this case $\alpha=3/2$ and $\lambda=15/4$ so we expect:

Conjecture 5. *If Ω_1, Ω_2 and Ω_3 are disjoint domains in $\mathbf{R}^3$, $x\in\partial\Omega_1\cap\partial\Omega_2\cap\partial\Omega_3$ and $D=D(x,r)$ then*

$$\omega_1(D)\omega_2(D)\omega_3(D)\le Cr^{15/2}.$$

Conjecture 6. *If Ω_1, Ω_2 and Ω_3 are disjoint domains in $\mathbf{S}^2$, the unit sphere of $\mathbf{R}^3$, then*

$$\lambda_1+\lambda_2+\lambda_3\ge\frac{45}{4},$$

where λ_i is the first eigenvalue of the Laplacian for Ω_i with Dirichlet boundary values. (The minimum should be attained by joining antipodal points by three geodesics meeting at 120 degrees.)

Given a domain Ω we define

$$\dim(\omega)=\inf\{\alpha:\exists E\subset\partial\Omega \text{ such that } \omega(E)=1 \text{ and } \dim(E)=\alpha\}.$$

Here $\dim(E)$ refers to the Hausdorff dimension of E. Makarov has proven that $\dim(\omega)=1$ for every simply connected domain Ω in the plane. Peter Jones and Tom Wolff [16] have shown that $\dim(\omega)\le 1$ for any planar domain, verifying a conjecture of Øksendal. Øksendal had also conjectured that $\dim(\omega)\le n-1$ for any domain $\Omega\subset\mathbf{R}^n$, but Wolff [27] has shown this is false. In the other direction, Jean Bourgain [12] has shown that there exists an $\epsilon>0$ such that $\dim(\omega)\le n-\epsilon$ for every $\Omega\subset\mathbf{R}^n$ (and his ϵ depends on n). The obvious problem is to determine what the best value of ϵ is.

Conjecture 7. *For any domain $\Omega\subset\mathbf{R}^n$, $\dim(\omega)\le d_n\equiv n-1+(n-2)/(n-1)$.*

There is no strong reason for choosing this value. It has been suggested only because of the fact that if f is harmonic on $\mathbf{R}^n$ then $|\nabla f|^q$ is subharmonic for $q\ge(n-2)/(n-1)$. Suggestions for a better value are welcome.

If Ω_1 and Ω_2 are disjoint (not necessarily simply connected) domains in $\mathbf{R}^2$, then it is known exactly when their harmonic measures will be mutually singular (see below), but this is not understood in higher dimensions. In particular, if the harmonic measures for $\Omega_1,\Omega_2\subset\mathbf{R}^2$ are not mutually singular then one can show $\partial\Omega_1\cap\partial\Omega_2$ must intersect a Lipschitz graph in positive length and that the 2 harmonic measures be mutually absolutely continuous to Λ_1 on this intersection.

Conjecture 8. *If $\Omega_1,\Omega_2\subset\mathbf{R}^n$ are disjoint domains with harmonic measures ω_1,ω_2 that are mutually absolutely continuous on a set $E\subset\partial\Omega_1\cap\partial\Omega_2$, of positive measure, then there exists $F\subset E$ of positive measure such that ω_1 and ω_2 are mutually absolutely continuous with Λ_{n-1} on F*

This may be false, but it would be interesting even to prove Conjecture 7 under the additional hypothesis of mutually continuity, i.e., if $\omega_1\ll\omega_2\ll\omega_1$ on E then there exists a set $F\subset E$ of positive measure with $\dim(F)\le d_n$.

There is another refinement of the Jones-Wolff result suggested by the results in [8]. The theorem on mutually singularity of harmonic measures mentioned above is stated in terms of a Wiener type condition involving the logarithmic capacity of $\partial\Omega$ as follows. For $x \in \mathbf{R}^2$, $\delta > 0$, $\epsilon > 0$ and $\theta \in [0, 2\pi)$ we define the cone and wedge

$$C(x,\delta,\epsilon,\theta) = \{x + re^{i\psi} : 0 < r < \delta, |\psi - \theta| < \epsilon\}$$

$$W(x,\delta,\epsilon,\theta) = C(x,\delta,\epsilon,\theta) \cap \{z : \delta/2 \le |z - x|\}.$$

We also let $\mathrm{cap}(E)$ denote the logarithmic capacity of E. For a fixed x, ϵ and θ let

$$\gamma(k) = \mathrm{cap}(2^{k-2}(W(x, 2^{-k}, \epsilon, \theta)\backslash\Omega)),$$

i.e., $\gamma(k)$ is the capacity of $\Omega^c \cap W(x, 2^{-k}, \epsilon, \theta)$ after we have dilated it to have diameter about 1/2. We say a point $x \in \partial\Omega$ satisfies a weak cone condition (WCC) if there exists ϵ and θ such that

$$\sum_{k=1}^{\infty} \gamma(k) < \infty.$$

We refer to this as a "weak" condition because it generalizes the cone condition stated in [6], [9] which requires that

$$C(x,\delta,\epsilon,\theta) \subset \Omega.$$

It is clear that this condition implies the WCC since all but finitely many of the terms in the sum will be zero.

THEOREM. *Suppose Ω_1 and Ω_2 are disjoint subdomains in $\mathbf{R}^2$ and let ω_1 and ω_2 be their harmonic measures. Then $\omega_1 \perp \omega_2$ iff the set of points in $\partial\Omega_1 \cap \partial\Omega_2$ satisfying a weak cone condition with respect to both Ω_1, Ω_2 has zero 1 dimensional measure, Λ_1. Moreover, if ω_1 and ω_2 are mutually absolutely continuous on a set E then there is Besicovitch regular $F \subset E$ with $\omega_i(F) = \omega_i(E)$ and ω_i mutually absolutely continuous with Λ_1 on F for $i = 1, 2$.*

Wolff has proven that

$$F = \{x \in \partial\Omega : \limsup_{r\to 0} \frac{\omega(D(x,r) \cap \partial\Omega)}{r} > 0\}$$

has sigma finite length and full harmonic measure for any planar domain (unpublished). It is shown in [8] that if $E \subset \partial\Omega$ and every point of E is the vertex of a cone with convergent capacity series (as above) then

$$\lim_{r\to 0} \frac{\omega(D(x,r) \cap \partial\Omega)}{r} < \infty$$

Λ_1 a.e. on E. It seems possible that the converse is also true, i.e.,

CONJECTURE 9. *If $E \subset \partial\Omega$, $0 < \Lambda_1(E) < \infty$ and no point of E satisfies the weak cone condition then for ω a.e. $x \in E$*

$$\limsup_{r\to 0} \frac{\omega(D(x,r) \cap \partial\Omega)}{r} = \infty.$$

In particular, there exists $F \subset E$ with $\omega(F) = \omega(E)$, but $\Lambda_1(F) = 0$.

Wolff's theorem tells us that we can split the boundary of any planar domain into two pieces E, $\partial\Omega\backslash E$ such that harmonic measure is mutually continuous with Λ_1 on E and singular to Λ_1 on $\partial\Omega\backslash E$. If Conjecture 9 is true it gives a geometric characterization of this set. The conjecture is consistent with what is known in the simply connected case. It also has the following consequence which is of interest in its own right:

CONJECTURE 10. *If $\Omega \subset \mathbf{R}^2$ is a domain and $E \subset \partial\Omega$ is Besicovitch irregular then there exists $F \subset E$ with $\omega(F) = \omega(E)$ and $\Lambda_1(F) = 0$.*

Peter Jones has pointed out that this is true in the case when $E = \partial\Omega$ satisfies a capacitary "thickness" condition: there exists $\epsilon > 0$ such that for every $x \in \partial\Omega$ and $0 < r < r_0$, $\mathrm{cap}(r^{-1}(D(x, r/4) \cap \partial\Omega)) \ge \epsilon$.

Our next question is not about harmonic measure, but rather about Brownian motion. We will say a Brownian path γ in $\Omega = \mathbf{R}^2\backslash E$ separates E if there are $0 \le s < t < \tau$ (τ is the time the particle hits E) such that $\gamma(s) = \gamma(t)$ and so that the closed curve $\Gamma = \gamma([s,t])$ separates E (i.e., has points of E in more than one of its complementary components). For example, if $E \subset \mathbf{R}$ and $|E| > 0$ then it is clear that a Brownian path (say starting at $(0,1)$) has a positive probability of hitting E the first time it hits $\mathbf{R}$ (so it does not separate E). If E is the middle third Cantor set then almost every Brownian path separates E. One might think that the probability of a path separating E is 1 whenever $|E| = 0$. However, this in not the case. There is an example of a compact set $E \subset \mathbf{R}$ with $|E| = 0$ but such that a Brownian particle in $\Omega = \mathbf{R}^2\backslash E$ has a positive probability of hitting E without separating E [7].

The idea of a Brownian path separating the boundary is related to the problem of characterizing the compact sets E in $\mathbf{R}^2$ such that $E \cap \partial\Omega$ has zero harmonic measure in Ω for any simply connected Ω. We shall call such a set a SC-null set. If $\Omega = \mathbf{R}^2\backslash E$ and Brownian paths separate E a.s. then E has this property, since Brownian paths will have to hit $\partial\Omega\backslash E$ a.s. before hitting E. The example mentioned above was originally constructed to show the converse is false. This is because a theorem of Øksendal says that a subset of $\mathbf{R}$ is SC-null iff it has zero length. Thus the set constructed here must have zero harmonic measure in every simply connected domain even though it is not a.s. separated by Brownian paths. As mentioned earlier, a subset of a rectifiable curve is SC-null iff it has zero length [10]. Makarov [20] has proven that if $\Lambda_h(E) = 0$ where Λ_h is the Hausdorff measure associated to the function

$$h(t) = t\exp(C\sqrt{\log(1/t)\log\log\log(1/t)})$$

then E is SC-null, and that this is sharp except for the choice of $C > 0$. Neither of these results is easy, and a characterization of SC-null sets is probably quite difficult.

K. Burdzy has pointed out to me that if the set E has small enough Hausdorff dimension, then Brownian motion in the complement of E necessarily separates E. Consider a Brownian path starting at distance ϵ from the origin and let it run until the first time it hits the unit circle. Let $P(\epsilon)$ denote the probability that the origin and and infinity are in the same connected component of the path's complement, and suppose that it satisfies $P(\epsilon) \leq C\epsilon^{\alpha}$ for some fixed $\alpha > 0$. If E has finite α-dimensional Hausdorff measure, then an application of the Borel-Cantelli lemma shows that E is necessarily separated by Brownian paths hitting it. In [13] Burdzy and Lawler have shown that $P(\epsilon) \leq \epsilon^{\pi^{-2}}$. In light of Makarov's theorem we expect

CONJECTURE 11. *If $E \subset \mathbf{R}^2$ is compact and $\dim(E) < 1$ then Brownian paths separate it almost surely.*

This has also been conjectured by Terry Lyons. The conjecture is true if $E \subset \mathbf{R}$ or if we assume $\dim(E) < 1/2$ [7]. Another set of interesting questions concern the analogs of various theorems on harmonic measure in the discrete case, i.e., theorems about the hitting probabilities of random walks on lattices. Not very much is known about harmonic measure for random walks. One exception is a result of Harry Keston [19] which is a random walk analogy of a classical estimate of Beurling. Keston uses his result to estimate the rate of growth of Diffusion Limited Aggregation (DLA) [18].

DLA has a continuous and a discrete version. The continuous version can be described as follows. Fix a unit disk at the origin. Start another disk near infinity and move it along a Brownian path until the first time it hits the first disk and then stop it. Successively add new disks in the same way and try to describe what the resulting collection looks like. In the discrete version (as in Kesten's work), we move along a lattice by a random walk until we reach a vertex which is adjacent to a previously occupied vertex. We stop here and call this new vertex occupied. Among the questions we can ask, one of the simplest is what the average rate of growth of the diameter, D, is as a function of the number of disks, N. Trivially, $C\sqrt{N} \leq D \leq N$ and Kesten proved that $D \leq CN^{2/3}$ almost surely [18].

CONJECTURE 12. *There exists $\epsilon > 0$ such that $D \geq CN^{1/2+\epsilon}$ almost surely.*

On the the basis of computer pictures this seems obvious. Proving it, however, seems very difficult. One should also attempt to establish the exact order of growth. One interesting aspect of the computer simulations is that when DLA is grown on a lattice and a large enough number of disks are considered, the resulting object is not the same in all directions (as had been expected). Instead the growth is fastest along the axis directions, so the DLA resembles a cross. Proving this actuals happens would be another interesting problem. See [25] for background on this and related physical processes.

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