HARMONIC MEASURE, L^2 ESTIMATES AND THE SCHWARZIAN DERIVATIVE

By

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Abstract. We consider several results, each of which uses some type of "L^2" estimate to provide information about harmonic measure on planar domains. The first gives an a.e. characterization of tangent points of a curve in terms of a certain geometric square function. Our next result is an L^p estimate relating the derivative of a conformal mapping to its Schwarzian derivative. One consequence of this is an estimate on harmonic measure generalizing Lavrentiev's estimate for rectifiable domains. Finally, we consider L^2 estimates for Schwarzian derivatives and the question of when a Riemann mapping Φ has $\log \Phi'$ in BMO.

1. Introduction

In [49] Stein and Zygmund showed that a real valued function f on the real line, $\mathbf{R}$, has an ordinary derivative at almost every point of a set E iff both

$$f(x+t)+f(x-t)-2f(x)=O(|t|), \quad t\to 0,$$

$$\int_{|t|\le\delta} |f(x+t)+f(x-t)-2f(x)|^2\frac{dt}{|t|^3}<\infty,$$

for almost every $x\in E$ (also see [48, VIII.5.1]). This result is an excellent example of the power of so called "L^2 techniques" in studying classical smoothness properties of functions, and is the starting point of a whole series of results which relate "square Dini conditions" to the smoothness properties of various objects. Examples include Carleson's result on quasiconformal mappings [12], Dahlberg's theorem on the absolute continuity of elliptic measures [17] and similar results by Fefferman, Kenig and Pipher [21], [22]. Recently there have been attempts to study the smoothness properties of sets by using L^2 theory in a similar way. In large part this has been motivated by the desire to understand the L^2 boundedness of the Cauchy integral on a Lipschitz graph and on a general planar set of finite 1-dimensional Hausdorff measure, Λ_1. In particular, the sets in the plane which are subsets of rectifiable curves can be characterized precisely by a "geometric

* Supported in part by NSF Grant DMS-91-00671.

† Supported in part by NSF Grant DMS-86-025000.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 62 (1994)

square function" [31], in a way which is very reminiscent of use of classical square functions in real analysis.

In this paper we consider several distinct, but related questions. The first concerns the behavior of a curve near its tangent points. In particular, we obtain an a.e. characterization of tangent points analogous to the Stein–Zygmund theorem mentioned above. The second relates the derivative of a Riemann mapping Φ to certain "area integrals" involving its Schwarzian derivative. Next, we obtain some L^2 estimates involving these area integrals and relate them to geometrical properties of Φ. Finally, we state the pointwise versions of these results which give several conditions which are a.e. equivalent to the existence of nontangential limits for Φ'.

We start by recalling the definitions of inner tangent and tangent points. Suppose Ω is a simply connected domain. We call $x \in \partial\Omega$ an inner tangent point if there exists a $\theta_0 \in [0, 2\pi)$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\{x + re^{i\theta} : 0 < r < \delta, \, |\theta - \theta_0| < \pi - \epsilon\} \subset \Omega$$

and this is not true for any $\epsilon < 0$. In other words, x is the vertex of a cone in Ω and the angle can be taken arbitrarily close to π, but no larger. If $\Phi : \mathbf{D} \to \Omega$ is univalent then (except for a set of zero measure) Φ' has a finite, non-zero, non-tangential limit at $e^{i\theta}$ iff $x = \Phi(e^{i\theta})$ is an inner tangent of Ω [41, pages 305, 328]. If Γ is a closed Jordan curve then a point $x \in \Gamma$ is a tangent point of Γ if and only if it is an inner tangent for each of the two complementary domains.

We would like to have an almost everywhere characterization of these tangents points in terms of some average behavior of Γ that would be analogous to the characterization of differentiability described above. A conjecture of Carleson's along these lines is as follows. For any $x \in \Gamma$ and $t > 0$, let $\theta_i(t)$ denote the angle measure of the longest arc in $\{|z - x| = t\} \cap \Omega_i$, $i = 1, 2$. Let $\epsilon(x, t) = \max(|\pi - \theta_1(t)|, |\pi - \theta_2(t)|)$. Then except for a set of zero Λ_1 measure Γ should have a tangent at x iff

$$\int_0^1 \epsilon^2(x, t) \frac{dt}{t} < \infty.$$

The necessity of this integral condition follows from an estimate of Beurling on harmonic measure. Let Ω_1 and Ω_2 denote the two complementary components of Γ in $\overline{\mathbf{C}}$. Fix a point in each domain and let ω_1 and ω_2 denote the harmonic measures on Γ with respect to these points. Then it follows from the Ahlfors distortion theorem and the fact that $\theta_1 + \theta_2 \leq 2\pi$ that

$$\omega_1(D(x, r))\omega_2(D(x, r)) \leq A \exp\Big(-\pi \int_r^1 \Big\{\frac{1}{\theta_1(t)} + \frac{1}{\theta_2(t)}\Big\} \frac{dt}{t}\Big) \leq Ar^2,$$

where A depends only on the choice of base points for the harmonic measures. This is due to Beurling [6] and is also in [9]. The set of tangent points of a Jordan curve has sigma-finite Λ_1 (a tangent point must be the vertex of a cone in the complement; by taking unions of these cones one can construct a countable family of Lipschitz domains whose boundaries contain every tangent point). Furthermore, on the set of tangent points the harmonic measures ω_1 and ω_2 are both mutually absolutely continuous with Λ_1. Thus for Λ_1 almost every tangent point x

$$\lim_{r\to 0}\frac{\omega_i(\Gamma\cap D(x,r))}{r}>0, \quad i=1,2.$$

A simple calculation shows

$$\frac{1}{\theta_1(t)}+\frac{1}{\theta_2(t)}\geq\frac{2}{\pi}+\frac{2}{\pi}\Big(\frac{\epsilon(x,t)}{\pi}\Big)^2$$

so by Beurling's estimate

$$\frac{\omega_1(D(x,r))\omega_2(D(x,r))}{r^2}\leq C_1\exp\Big(-C_2\int_r^1\epsilon^2(x,t)\frac{dt}{t}\Big).$$

On almost every tangent point of Γ the left hand side is bounded away from zero so the integral on the right must remain bounded as $r\to 0$. The converse is open.

The first goal of this paper is to prove a version of "ϵ squared" conjecture using "β"'s coming from the second author's work on rectifiable curves. For $x\in\Gamma$ and $t>0$ define $\beta(x,t)$ as

$$\beta(x,t)=\inf_L\Big\{\sup\frac{\operatorname{dist}(z,L)}{t}:z\in\Gamma\cap D(x,4t)\Big\}$$

where the infimum is taken over all lines L passing through $D(x,t)$.

Theorem 1 *There exists $C>0$ such that if K is a compact set of diameter 1 and if for every $x\in K$,*

$$\int_0^1\beta^2(x,t)\frac{dt}{t}\leq M$$

then K lies on a rectifiable curve Γ of length at most Ce^{CM}.

The estimate e^{CM} is best possible, as shown by an example in Section 2. We will also show

Theorem 2 *Except for a set of zero Λ_1 measure, $x\in\Gamma$ is a tangent point of Γ iff*

$$\int_0^1\beta^2(x,t)\frac{dt}{t}<\infty.$$

Equivalently, ω_1 and ω_2 are mutually absolutely continuous exactly on the set where this integral is finite.

We introduce the second part of the paper by recalling the F. and M. Riesz theorem [44]: if Ω has a rectifiable boundary then harmonic measure, ω, and arclength, Λ_1, are mutually absolutely continuous on $\partial\Omega$. Using Jensen's inequality and the subharmonicity of $\log|\Phi'|$ (Φ the Riemann mapping onto Ω), Lavrentiev gave the estimate

$$\omega(E) \leq \frac{C \log \ell(\partial\Omega)}{|\log \ell(E)| + 1},$$

with the normalization that harmonic measure is taken with respect to a point z_0 satisfying $\text{dist}(z_0, \partial\Omega) \geq 1$. Since this estimate is crucial to what follows let us recall the proof.

Let $\Phi : \mathbf{D} \to \Omega$ be a Riemann mapping with $\Phi(0) = z_0$. Since $\text{dist}(z_0, \partial\Omega) \geq 1$ we have $|\Phi'(z)| \geq 1$ and since $\partial\Omega$ is rectifiable, $\Phi' \in L^1(\mathbf{T})$ with norm at most $M = 2\Lambda_1(\partial\Omega)$. Then $\log|\Phi'|$ is subharmonic so $0 \leq \int_{\mathbf{T}} \log|\Phi'|(e^{i\theta})d\theta$. Thus if $\log^{\pm} = \max(0, \pm\log)$ Jensen's inequality implies

$$\| \log^- |\Phi'| \|_{L^1(\mathbf{T})} \leq \| \log^+ |\Phi'| \|_{L^1(\mathbf{T})} \leq \log M.$$

Tchebyshev's inequality gives

$$|\{x \in \mathbf{T} : \log|\Phi'| \leq -\lambda\}| \leq \frac{M}{\lambda}.$$

Now suppose we have a set $E \subset \partial\Omega$ with $\omega(E) = \alpha > 0$ (i.e., $|F| = |\Phi^{-1}(E)| = \alpha$). If $\lambda = 2M/\alpha$ then $|\Phi'| \geq \exp(-2M/\alpha)$ on at least half of F. Thus

$$\Lambda_1(E) = \int_F |\Phi'(e^{i\theta})| d\theta \geq \frac{\alpha}{2} e^{-2M/\alpha} \geq e^{-(2M+2)/\alpha}.$$

Taking logarithms we get

$$\log \frac{1}{\Lambda_1(E)} \leq \frac{2 + 2\log \Lambda_1(\partial\Omega)}{\omega(E)}.$$

Rearranging (and using the fact that $\Lambda_1(\partial\Omega) \geq 2\pi$) gives the desired estimate.

Lavrentiev's estimate is sharp in the sense that given M, ϵ one can construct an Ω and $E \subset \partial\Omega$ with $\Lambda_1(\partial\Omega) = e^M$, $\omega(E) = \epsilon$ and $\Lambda_1(E) \sim e^{-M/\epsilon}$. Recall that an analytic function on the disk is said to be in the Bloch space if

$$\|f\|_* = \sup_{z \in \mathbf{D}} \frac{|f'(z)|}{1 - |z|^2} < \infty.$$

Furthermore, if $\|f\|_*$ is small enough, then $f = \log F'$ for some univalent F [41]. For a finite measure μ on $\mathbf{T}$ let u denote its harmonic extension to $\mathbf{D}$ and v the harmonic conjugate. Given M it is possible to construct a probability measure μ which is singular to Lebesgue measure but which is "smooth" in the sense that $M\|u + iv\|_*$ is small (see [20], [32], [47]). Thus $M(1-u) + iMv = \log \Phi'$ for some univalent Φ. By the construction of the measure we can find disjoint intervals $\{I_j\}$ such that $\sum |I_j| \sim \epsilon$ and $\mu(I_j) \sim |I_j|/\epsilon$. Thus if $\Omega = \Phi(\mathbf{D})$ and $E = \Phi(\bigcup I_j)$, then $\Lambda_1(\partial\Omega) = 2\pi e^M$, $\omega(E) \sim \epsilon$ and $\log \Lambda_1(E) \sim -M/\epsilon$. One can also give a geometrical construction of Ω by adapting the construction of the von Koch snowflake with a certain stopping time rule.

If $\Lambda_1(\partial\Omega) = \infty$ it is possible for there to be a set $E \subset \partial\Omega$ with $\omega(E) > 0$ but $\Lambda_1(E) = 0$ [35]. However, MacMillan showed that harmonic measure is mutually absolutely continuous to Λ_1 on the inner tangents of Ω and results of Makarov [36] and Pommerenke [43] imply that harmonic measure is singular to Λ_1 on the rest of the boundary, i.e., a Lavrentiev type example occurs whenever the inner tangents have less than full measure. In [10] we proved a generalization of the F. and M. Riesz theorem: if $E \subset \partial\Omega \cap \Gamma$ where Γ is a rectifiable curve then $\Lambda_1(E) = 0$ implies $\omega(E) = 0$. One of the main steps of this proof is to estimate Φ' in terms of the Schwarzian derivative of Φ, defined as

$$S(\Phi)(z) = \left[\frac{\Phi''(z)}{\Phi'(z)}\right]' - \frac{1}{2}\left[\frac{\Phi''(z)}{\Phi'(z)}\right]^2.$$

The Schwarzian can be considered as a type of second derivative, measuring the rate of change of the best approximating Möbius transformation (rather than linear function). Our next result is motivated by the results of [10] as well as the classical results for the area integral.

Theorem 3 *If Φ is univalent and*

$$A = A(\Phi) = |\Phi'(0)| + \iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|^2)^3 dxdy < \infty,$$

then $\Phi' \in H^{\frac{1}{2}-\eta}$ for every $\eta > 0$ and $\|\Phi'\|_{\frac{1}{2}-\eta} \le C(\eta)A$.

Considering the Möbius transformation τ from the disk to the half plane we note $S(\tau) \equiv 0$, $\tau' \in H^p$, $0 < p < 1/2$ but $\tau \notin H^{1/2}$. Thus the most we could expect is that the finiteness of this integral implies $\Phi' \in H^p$ for every $p < 1/2$.

The proof of this result will show that the finiteness of the Schwarzian integral implies that Ω can be approximated by rectifiable curves in the following sense.

Corollary 1 *Suppose $\Phi : \mathbf{D} \to \Omega$ is univalent and $A(\Phi) < \infty$. Then for every $\eta > 0$ there is a $C(\eta)$ such that for any $\epsilon > 0$ there exists a curve $\Gamma \subset \mathbf{D}$ with $\ell(\Gamma \cap \mathbf{T}) \ge 1 - \epsilon$ and $\ell(\Phi(\Gamma)) \le C(\eta)A\epsilon^{-1-\eta}$.*

The proof of Lavrentiev's estimate given above still works if $\Phi' \in L^p$ for some $p > 0$ (with a constant depending on p). Thus we obtain:

Corollary 2 *With Φ and A as above, for any $E \subset \partial\Omega$,*

$$\omega(E) \leq \frac{C \log A + 1}{|\log \ell(E)| + 1}.$$

By adapting the proof of Theorem 3 to Lipschitz domains and certain covering maps which arise from the argument in [10] we can also prove:

Corollary 3 *There exists a $C > 0$ such that if Ω is simply connected, Γ is a rectifiable curve and ω is measured with respect to a point z_0 with* $\operatorname{dist}(z_0, E) \geq 1$ *then $E \subset \partial\Omega \cap \Gamma$ implies*

$$\frac{\omega(E)}{|\log \omega(E)| + 1} \leq C \frac{\log^+ \ell(\Gamma) + 1}{|\log \ell(E)| + 1}.$$

In particular, if E is a subset of a rectifiable curve then $\Lambda_1(E) = 0$ implies $\omega(E) = 0$.

The $\log \omega(E)$ prevents this from being an actual generalization of Lavrentiev's estimate and is probably unnecessary. The final statement has an easier proof which does not require Theorem 3 and which will be sketched in Section 6. Replacing [10, Theorem 2] by Corollary 3 gives an easier proof of [10, Theorem 3]: Γ is an Ahlfors-David regular curve iff there exists $C > 0$ such that $\Lambda_1(\Phi^{-1}(\Gamma)) \leq C$ for every univalent Φ on $\mathbf{D}$. See [10] for details.

An integral similar to that in Theorem 3 occurs in work of Astala and Zinsmeister who show that $\log \Phi'$ is in BMO iff $|S(\Phi)(z)|^2(1 - |z|)^3 dxdy$ is a Carleson measure on $\mathbf{D}$. Recall that a Carleson square in $\mathbf{D}$ is a set of the form

$$Q = Q_I = \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r \leq 1\}.$$

We let z_Q denote the center of the top half of Q,

$$T(Q) = \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r \leq 1 - |I|/2\}.$$

A positive measure μ on the disk is called Carleson if there is a $C > 0$ such that for every Carleson square Q_I, $\mu(Q_I) \leq C|I|$. The following theorem extends the result of Astala and Zinsmeister and was inspired by their paper [4].

Theorem 4 *Suppose Ω is simply connected and $\Phi : \mathbf{D} \to \Omega$ is conformal. Then the following are equivalent:*

(1) $\varphi = \log \Phi'$ *is in* $\mathrm{BMO}(\mathbf{T})$.

(2) *There exists a* $\delta, C > 0$ *such that for every* $z_0 \in \Omega$ *there is a rectifiable subdomain* $D \subset \Omega$ *such that* $\ell(\partial D) \leq C\text{dist}(z_0, \partial\Omega)$ *and* $\omega(z_0, \partial D \cap \partial\Omega, D) \geq \delta$.

(3) *There exists a* $\delta, C > 0$ *such that for every* $z_0 \in \Omega$ *there is subdomain* $D \subset \Omega$ *which is chord-arc with constant* C *and such that* $D(z_0, \delta) \subset D$, $\ell(\partial D) \leq C\text{dist}(z_0, \partial\Omega)$ *and* $\ell(\partial D \cap \partial\Omega) \geq \delta d$.

(4) *There exists* $C > 0$ *such that for every Carleson square* Q,

$$\iint_Q |S(\Phi)(z)|^2 (1 - |z|)^3 dxdy \leq C\ell(Q).$$

(5) *There exists* $\delta, C > 0$ *such that for every* $w_0 \in \mathbf{D}$, *there exists a chord-arc domain* $D \subset \mathbf{D}$ *such that* $\omega(w_0, \partial D \cap T, D) \geq \delta$ *and*

$$\iint_D |\Phi'(z)||S(F)(z)|^2 (1 - |z|)^3 dxdy \leq C|\Phi'(w_0)|(1 - |w_0|).$$

Chord-arc means that there is a $C > 0$ such that the shorter arc connecting two points $z_1, z_2 \in \partial D$ has length at most $C|z_1 - z_2|$. These are also called Lavrentiev curves and are exactly the bi-Lipschitz images of circles. Since condition (3) is clearly bi-Lipschitz invariant, we get

Corollary 4 *The collection of domains satisfying* $(1) - (5)$ *is invariant under bi-Lipschitz homeomorphisms of the plane.*

Harmonic measure on a chord-arc curve is in the Muckenhaupt class A_∞ with respect to arclength [28], so condition (3) could be restated by saying D satisfies $\omega(z_0, \partial D \cap \partial\Omega, D) \geq \delta$.

A quasicircle is the image of the unit circle under a quasiconformal mapping. They can be characterized more geometrically by the condition that there is a $M > 0$ such that the smaller arc between any two points z_1, z_2 on the curve has diameter at most $M|z_1 - z_2|$. Given a quasicircle one can find a bi-Lipschitz involution fixing the curve and exchanging the two complementary components [3]. Thus

Corollary 5 *A quasidisk satisfies* $(1) - (5)$ *iff its complement does.*

These corollaries answer questions from [4]. The implication (1) $\Rightarrow$ (4) of Theorem 4 was noted by Zinsmeister, [50, Lemma 7] and the implication (4) $\Rightarrow$ (1) is due to Astala and Zinsmeister, [4, Lemma 1]. For a general domain let $\Delta(\Omega)$ denote those measures μ such that

$$||\mu||_C = \sup r^{-1} |\mu|(D(x, r)) < \infty$$

where the "sup" is over all $x \in \partial\Omega$ and $r > 0$. These are called the Carleson measures for Ω. The class of domains defined by Theorem 4 have previously been studied by Zinsmeister, who showed that $\log \Phi' \in$ BMO iff there is a $M > 0$ such that if $\mu \in \Delta(\Omega)$ then $\nu = |\Phi'|^{-1}\Phi^*\mu \in \Delta(\mathbf{D})$ with $\|\nu\|_C \leq M\|\mu\|_C$. This condition is equivalent to saying $\iint_\Omega |f| d\mu \leq M\|\mu\|_C \|f\|_{H^1(\Omega)}$ where $\|f\|_{H^1(\Omega)} = \|f \circ \Phi \cdot \Phi'\|_{H^1(\mathbf{D})}$. See [52] for the proof of these equivalences and some other properties of these domains.

The conditions of Theorem 4 do not imply

$$\iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy < \infty,$$

as is seen by taking an infinite strip $\Omega = \{z : -1 \leq \mathrm{Im}(z) \leq 1\}$.

A standard characterization of BMO($\mathbf{T}$) states that $\varphi \in$ BMO iff either (hence both)

$$|\varphi'(z)|^2(1-|z|)dxdy, \quad |\varphi''|^2(1-|z|)^3 dxdy$$

are Carleson measures. Since $S(\Phi) = \varphi'' - \frac{1}{2}(\varphi')^2$, the equivalence of (1) and (4) is not too surprising. The proof of (4) $\Rightarrow$ (1) will also show:

Corollary 6 *We have* $C^{-1}\|S(\Phi)\|_{A^{2,3}} \leq \|\varphi - \varphi(0)\|_{L^2(\mathbf{T})} \leq C + C\|S(\Phi)\|_{A^{2,3}}$ *for any univalent* Φ.

Here we have saved space by using the notation of weighted Bergman spaces [16], i.e.,

$$\|f\|_{A^{p,r}} = \left(\iint_{\mathbf{D}} |f|^p(1-|z|^2)^r dxdy\right)^{1/p}.$$

The only use of univalence in the previous result is that if Φ is univalent, then $\varphi = \log|\Phi'|$ is in Bloch, i.e., $\|\varphi'(z)\|_* \leq 6$ [19]. The argument remains true if we forget about Φ and think of it as a result about φ. The left hand inequality is not true for general analytic functions, but the proof of the right hand inequality can be made to work without the assumption that φ is in Bloch, so we get

Corollary 7 *We have* $\|\varphi - \varphi(0)\|_{L^2(\mathbf{T})} \leq C|\varphi'(0)| + C\|\varphi'' - (\varphi')^2/2\|_{A^{2,3}}$ *for any analytic* φ *on* $\mathbf{D}$.

Our final result is the "almost everywhere" pointwise result which follows naturally from the estimates of Theorems 3 and 4. First we define a version of the β's adapted to inner tangents. For $e^{i\theta} \in \mathbf{T}$ and $t > 0$ let $x = \Phi(e^{i\theta})$ and let C be the collection of all line hitting $D(x,t)$ such that at least one component of $D(x,4t)\backslash L$ both hits $\Phi([0,e^{i\theta}])$ and is contained in Ω and let

$$\eta(e^{i\theta}, t) = t^{-1} \min_C \left(\max_{z \in L \cap D(x,t)} \mathrm{dist}(z, \partial\Omega) \right).$$

If $C = \emptyset$ we set $\eta = 1$.

Theorem 5 *Suppose Ω is simply connected and $\Phi : \mathbf{D} \to \Omega$ is conformal. Then except for a set of zero measure the following conditions are equivalent:*
(1) *Φ' has a non-tangential limit at $e^{i\theta}$.*

$$\int_0^1 \eta^2(e^{i\theta}, t)\frac{dt}{t} < \infty. \tag{2}$$

$$\int_{W(\theta)} |S(\Phi)(z)|^2(1-|z|)^2 dxdy < \infty. \tag{3}$$

$$\int_{W(\theta)} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^2 dxdy < \infty. \tag{4}$$

Since (1) is a.e. equivalent to $\Phi(e^{i\theta})$ being an inner tangent of Ω, this is an analog of Theorem 2. It should be compared to more classical results, as in [48], which say Φ' has a limit a.e. where

$$\int_{W(\theta)} |\Phi''(z)|^2(1-|z|)^2 dxdy < \infty.$$

Also note that the integral in (2) is a version of the integral of Marcinkiewicz (e.g. [48]),

$$\mathcal{M}_F(y) = \int_{-1}^1 \operatorname{dist}(y-x, F)\frac{dx}{x^2}, \quad F \subset \mathbf{R}.$$

The main difference is that whereas η measures the approximation to a line in a L^∞ sense, the Marcinkiewicz integral measures in a L^1 sense.

Theorems 3, 4 and 5 each say that if either

$$\int_{\mathbf{D}} |S(\Phi)(z)|^2(1-|z|)^3 dxdy < \infty, \tag{1.1}$$

or

$$\int_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy < \infty, \tag{1.2}$$

then Ω is "almost" rectifiable in some sense. In particular, almost every (ω) point of $\partial\Omega$ is an inner tangent point. In fact, for bounded quasicircles (1.2) implies rectifiability (Lemma 4.9). Conversely, if $\partial\Omega$ is rectifiable then the second integral must be finite but the first need not (Lemma 4.7). However, by considering a map from the disk to a half plane, one sees that the finiteness of the integrals does not imply $\partial\Omega$ has finite length and one can even construct domains for which both integrals are finite but $\partial\Omega$ has positive area.

In Section 2 we prove Theorems 1 and 2. The proof of Theorem 1 is only a modification of an argument given in [31] so will only be sketched. In Section 3 we review some facts concerning the Schwarzian derivative. We will prove Theorem 3 in Section 4 and deduce its corollaries. In Section 5 we prove Theorem 4, and in Section 6 we prove Theorem 5.

An earlier version of this paper had a harmonic measure estimate analogous to Beurling's estimate with $\epsilon(x,t)$ replaced by $\beta(x,t)$. However, Juha Heinonen, Pekka Koskela and Steffen Rohde pointed out an error in its proof.

2. Proof of Theorems 1 and 2

We start this section be recalling some definitions and results from [10], [31]. Suppose $E \subset \mathbf{R}^2$, $x \in \mathbf{R}^2$ and $t > 0$. Let $\mathcal{C}$ be the collection of all lines hitting $D(x,t)$. As in the introduction we define

$$\beta_E(x,t) = t^{-1} \inf_{\mathcal{C}} \operatorname{dist}(E \cap D(x,4t), L).$$

This measures how close E is to lying on a straight line. Let L be a line for which β is minimized and let E^* denote the perpendicular projection of $E \cap D(x,4t)$ onto L. Let

$$\gamma_E(x,t) = t^{-1} \sup_{z \in L \cap D(x,4t)} \operatorname{dist}(z, E^*),$$

i.e., γ measures the largest "gap" in $L \backslash E^*$.

These definitions are sometimes stated in terms of dyadic squares, i.e., squares of the form

$$Q = [k2^n, (k+1)2^n] \times [j2^n, (j+1)2^n]$$

with $j, k, n \in \mathbf{Z}$. Let $\ell(Q)$ denote the side length of Q and for $\alpha > 0$, let αQ denote the concentric square with side length $\alpha\ell(Q)$. Given a dyadic square Q there is a unique dyadic square Q' containing Q such that $\ell(Q') = 2^n\ell(Q)$. We denote this square by Q^n. The functions defined above can also be given by replacing $D(x,t)$ by a dyadic square of size $\sim t$ containing x, i.e,

$$\beta(Q) = \ell(Q)^{-1} \inf_{\mathcal{C}} \operatorname{dist}(E, L \cap 3Q),$$

where $\mathcal{L}$ is the collection of lines hitting Q. By adjusting the constants in the definitions one gets

$$\int_0^1 \beta(x,t)^2 \frac{dt}{t^2} \sim \sum_{x \in Q} \beta(Q)^2,$$

$$\int_0^\infty \int_{\mathbf{R}^2} \beta(x,t)^2 dxdy \frac{dt}{t} \sim \sum_Q \beta(Q)^2 \ell(Q).$$

The dyadic square notation is used, for example, in [10] and [31].

Lemma 2.1 ([31, Theorem 1])

(1) *If* $\Gamma \subset \mathbf{C}$ *is connected then* $\sum_Q \beta_\Gamma^2(Q)\ell(Q) \le C\ell(\Gamma)$.

(2) *If* $E \subset \mathbf{D}$ *satisfies* $\beta^2(E) = \sum_Q \beta_E^2(Q)\ell(Q) < \infty$, *then there exists* Γ *connected with* $E \subset \Gamma$ *and* $\ell(\Gamma) \le 4\mathrm{diam}(E) + C_0\beta^2(E))$.

This result has been extended to sets in $\mathbf{R}^n$ by K. Okikiolu [39]. The hypothesis of Theorem 1 can be restated in this notation to read: for every $x \in E$

$$\sum_{x\in Q} \beta_E(Q)^2 \le M \tag{2.1}$$

(the sum is over all dyadic squares containing x). Note that Theorem 1 and Lemma 2.1 show that (2.1) implies $\beta(E) \le C + Ce^{CM}$. If we had a direct proof of this we could then deduce Theorem 1, but we know of no such argument. Note that such a direct argument cannot be merely arithmetic, but would have to use some geometry. (Let Q_0 be a unit square and define numbers $\beta(Q) = 1/n$ if $\ell(Q) = 2^{-n}$ and $Q \subset Q_0$ and zero otherwise. Then the sum in (2.1) is finite but the sum in Lemma 2.1 is not. This is possible since these numbers do not come from any set E.) We can also note that the e^{CM} is the correct constant by considering the snowflake formed as follows. Fix an $\epsilon > 0$ and for any line segment I consider the isosceles triangle with base I and height ϵ. Replace the line segment by the path formed by the 2 opposite sides and repeat the construction of each of them. After N iterations we have a curve of length $(1+\epsilon^2)^N \sim Ce^{N\epsilon^2}$ and such that the sum in (2.1) is bounded by $CN\epsilon^2$.

In proving Theorem 1, we assume the reader is familiar with [31]. Suppose E is a set of diameter less than 1 satisfying (2.1). In [31], one constructs a collection of finite sets $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n, \dots \subset E$ such that

$$|z_j - z_k| \le 2^{-n}, \quad z_j, z_k \in \mathcal{L}_n, \tag{2.2}$$

$$\inf_{z\in E} \mathrm{dist}(z, \mathcal{L}_n) \le 2^{-n}, \quad z \in E, \tag{2.3}$$

and a sequence of polygonal curves $\{\Gamma_n\}$ containing $\mathcal{L}_n$, whose lengths can be uniformly bounded in terms of $\beta(E)$. To prove Theorem 1 we modify the construction slightly and instead of estimating the length of Γ_n directly we build functions $\{f_n\}$ on Γ_n such that

$$0 \le f_n(x) \le C + C\sum_{x\in Q} \beta^2(Q), \quad x \in \Gamma_n, \tag{2.4}$$

$$\int_{\Gamma_n} \exp(-f_n)ds \leq C, \tag{2.5}$$

for some $C > 0$. Passing to the limit, we obtain a curve Γ which contains E and has length at most Ce^{CM}.

To define Γ_0 choose 2 points z_0, z_1 in E with $|z_0 - z_1| \geq \text{diam}(E)/2$ and let Γ_0 be the segment $[2z_0 - z_1, 2z_1 - z_0]$ containing these points and extending beyond them. Let $f_0 \equiv 0$. In general, suppose we have a finite set $\mathcal{L}_n \subset E$ satisfying (2.2) and (2.3) and Γ_n is a polygonal curve containing $\mathcal{L}_n$. Consider points $z_0, z_1 \in \mathcal{L}_n$ with $|z_0 - z_1| \sim 2^{-n}$. Let $L \subset \Gamma_n$ denote the line segment connecting these points. There are essentially 2 cases depending on whether E contains a point near the midpoint of L or not.

First suppose E contains a point z_2 with $\text{dist}(z_0, z_2) \sim \text{dist}(z_1, z_2) \sim 2^{-n}$. Let Q denote a dyadic cube of size 2^{-n+1} containing z_2. Then define Γ_{n+1} by replacing the line segment L by the 2 segments $[z_0, z_2]$ and $[z_2, z_1]$. These segments have total length less than $|z_0 - z_1|(1 + C\beta(Q)^2)$. Define $f_{n+1}(x) = f_n(x) + C\beta(Q)^2\chi_{3Q}(x)$. Thus

$$\int_{\Gamma_{n-1}} \exp(-f_{n+1}(x))ds \leq \int_{\Gamma_n} \exp(-f_n(x))ds \leq C.$$

In the second case, the only other points of E near L are very close to either z_0 or z_1. For example, suppose $z_2 \in E$ is close to z_0. Let Q be a dyadic square containing z_2 with $\ell(Q) \sim |z_0 - z_2|$. We define Γ_{n+1} by keeping L and adding the segment $[z_0, 2z_2 - z_0]$. Let I be the middle third of L and define

$$f_{n+1}(x) = f_n(x) + C\beta(Q)^2\chi_{3Q}(x) + C\chi_I(x).$$

Since $\ell(I) \gg \ell(Q)$ we get as above

$$\int_{\Gamma_{n+1}} \exp(-f_{n+1}(x))ds \leq \int_{\Gamma_n} \exp(-f_n(x))ds \leq C.$$

In the proof of [31, Theorem 1] there are actually 6 cases considered in passing from Γ_n to Γ_{n+1}. However, each of these cases can be dealt with by one of the two arguments above (more precisely, cases 1 and 2 of that paper are dealt with by the first argument and cases 3 to 6 by the second argument). With this modification we obtain a proof of Theorem 1.

The deduction of Theorem 2 from Theorem 1 consists of known arguments, although the details may not have been recorded together before. If E is the set of tangent points of Γ then it is known that E has sigma-finite length and Besicovitch regular. In fact, except for a set of measure zero we can write E as a union of set E_n so that each E_n (after rotation and dilation) is of the form $\{x + iy : f(x) = y = g(x)\}$ where f, g real valued Lipschitz functions on $[0, 1]$ with

$f \le g$ and $\Gamma \cap [0,1]^2 \subset \{x+iy : f(x) \le y \le g(x)\}$ (this all just says that locally Γ is trapped between two Lipschitz graphs which agree on the set E; this is a standard construction and is recorded in [7]). If we let Γ_1, Γ_2 denote the graphs of the functions f and g, they are clearly rectifiable and therefore the β-sum over all dyadic squares corresponding to $\Gamma_1 \cup \Gamma_2$ is finite. This easily implies the pointwise β-sum is finite at almost every point of $\Gamma_1 \cup \Gamma_2$ (integrating the pointwise sum over the curves gives a lower bound for the sum over all squares) and hence at a.e. point of E_n.

To deduce the other direction of Theorem 2, suppose E is a set where the β integral is bounded. Passing to a subset of positive measure we may assume E is compact and the integral is bounded by some M at each point of E. By Theorem 1 E lies on a rectifiable curve. We may use a point of density argument to assume it lies on a Lipschitz graph $G = \{(x, g(x))\}$ with very small constant. Let $x \in E$ be a point of density and choose $r > 0$ so small that $\Lambda_1(E \cap D(x,t)) \ge (1-\epsilon)2t$ for all $0 < t < r$ and $\int_0^r \beta^2(x,t)dt/t < \epsilon$. Rescale so that $x = 0$, $r = 1$ and let $z \in D(i, 1/4)$. Then $z \notin \Gamma$ for otherwise $\beta(x,t) \ge 1/2$ for $t \sim 1$. But this contradicts the integral estimate. A similar argument works for $z \in D(-i, 1/4)$ so we see x is the tip of two cones, one on either side of Γ. Standard arguments now imply that (except for a set of zero Λ_1 measure) x is a tangent point of Γ.

3. Some background on the Schwarzian derivative

In this section we state some basic results concerning the Schwarzian derivative and review certain results from [10]. The Schwarzian derivative of a locally univalent function F is defined by

$$\begin{aligned} S(F)(z) &= \left[\frac{F''(z)}{F'(z)}\right]' - \frac{1}{2}\left[\frac{F''(z)}{F'(z)}\right]^2 \\ &= \left[\frac{F'''(z)}{F'(z)}\right] - \frac{3}{2}\left[\frac{F''(z)}{F'(z)}\right]^2. \end{aligned}$$

If we write $F' = e^{\varphi}$ then it can be rewritten as

$$S(F)(z) = \varphi'' - \frac{1}{2}(\varphi')^2.$$

Recall that $S(F) \equiv 0$ iff F is a Möbius transformation and that S satisfies the composition law

$$S(F \circ G) = S(F)(G')^2 + S(G).$$

In particular, if G is Möbius then

$$S(F \circ G) = S(F)(G')^2,$$

$$S(G \circ F) = S(F).$$

In addition, given an $\epsilon > 0$, hyperbolic disk D and a compact neighborhood K of D, there is a $\delta > 0$ so that $|S(F)| \leq \delta$ on D implies F uniformly approximates a Möbius transformation on K to within ϵ. The relevance of the Schwarzian to the study of univalent mappings is described by (e.g., [19]):

Lemma 3.1 *Suppose F is analytic on $\mathbf{D}$. If F is univalent and $\varphi = \log F'$ then*

$$|S(F)(z)| \leq \frac{6}{(1-|z|^2)^2} \quad \text{and} \quad |\varphi'(z)| \leq \frac{6}{1-|z|^2}.$$

Conversely, if

$$|S(F)(z)| \leq \frac{2}{(1-|z|^2)^2} \quad \text{or} \quad |\varphi'(z)| \leq \frac{2}{(1-|z|^2)},$$

then F is univalent. If the 2 is replaced by a small constant, ϵ, then the image of the disk is a quasidisk with small constant.

If Ω is a domain, $\Phi : \mathbf{D} \to \Omega$ univalent and $e^{i\theta} \in T$ let $x = \Phi(e^{i\theta}) \in \partial\Omega$ and $t > 0$. Let $\mathcal{C}$ be the subcollection of lines hitting $D(x,t)$ such that a component of $D(x,4t)\backslash L$ is contained in Ω and hits the geodesic $\Phi([0,e^{i\theta}])$ and set

$$\eta_\Omega(x,t) = t^{-1} \inf_{\mathcal{C}} \sup_{z \in L \cap D(x,4t)} \operatorname{dist}(z, \partial\Omega).$$

The condition involving the ray is to prevent the situation where $\partial\Omega$ looks like the complement of an arc which looks flat from at least one side at every scale, but for which the flat side switches infinitely often as the scales change (this can occur).

There are some fairly obvious relations between the β, γ (defined in Section 2) and η which we state as a lemma.

Lemma 3.2 *Suppose Ω is simply connected, and $\beta = \beta_E$, $\gamma = \gamma_E$ and $\eta = \eta_\Omega$.*

(1) *If $E = \partial\Omega$ then $\eta(x,t) \leq \beta(x,t)$.*

(2) *If $E = \partial\Omega$ is a quasicircle then $\beta(x,t) + \gamma(x,t) \leq C\eta(x,t)$ where C depends only on the quasicircle constant of Ω.*

(3) *If $E = \partial\Omega \cap \{z : \operatorname{dist}(z,L) \leq \eta(x,t)\}$ then $\beta(x,t) + \gamma(x,t) \leq C\eta(x,t)$ (here L is a minimizing line in the definition of η).*

Next we note that this geometric quantity controls the Schwarzian.

Lemma 3.3 *Suppose $\Phi : \mathbf{D} \to \Omega$ is univalent on $D(w, \epsilon(1-|w|))$. Then*

$$|S(\Phi)(w)|(1-|w|^2)^2 \leq C\epsilon^{-2} \int_r^\infty \eta(x,t) \left(\frac{t}{r}\right)^{-\mu} \frac{dt}{t}$$

where $r = \mathrm{dist}(\Phi(w), \partial\Omega)$ *The number* μ *satisfies* $0 < \mu < 1$ *but can be taken as close to 1 as we wish. The constant* C *depends only on the choice of* μ.

This is proven in [10, Lemma 3.1] with η_Ω replaced by $\beta_E + \gamma_E$ and Φ the universal covering map of the complement of E. The lemma can be proven by repeating the proof (with minor changes) given in [10] or can be deduced from that result using part (3) of Lemma 3.2. The following results quantify the idea that Φ behaves like a Möbius transformation if $S(\Phi)$ is small.

Lemma 3.4 *Given* $\epsilon, n > 0$ *there exists* $C = C(\epsilon) > 0$, $\delta = \delta(\epsilon, n) > 0$ *such that if* Q *is a Carleson square and*

$$|S(\Phi)(z)| \leq \frac{\delta}{(1 - |z|^2)^2}$$

for all $z \in T(Q)$, *then there is a geodesic* γ *such that every point* $z \in Q$ *such that* $1 - |z| \geq 2^{-n}\ell(Q)$ *and* $|\varphi'(z)| \geq \epsilon/(1 - |z|^2)$ *lies within hyperbolic distance* C *of* γ. *Moreover,* Φ' *grows like* $(1 - |z|)^{-2}$ *along* γ. *In particular, for every* $\eta > 0$ *there is a* $\delta > 0$ *so that if* γ_0 *is an arc of* γ *from* $z_0 \in T(Q)$ *to* $z_1 \in Q \cap \{|z| = 1 - 2^{-n}\ell(Q)\}$ *then* $|\Phi'(z_1)| \geq 2^{2n(1-\eta)}|\Phi'(z_0)|$.

The proof is fairly easy and we will only sketch it. If δ is small enough (depending on n) then Φ is unformly approximated by a Möbius transformation τ on $Q \cap \{|z| \leq 1 - 2^{-n}\ell(Q)\}$ and so $|\Phi'(z)(1 - |z|^2)^2 - \tau'(z)(1 - |z|^2)^2| \leq \epsilon/2$ by the Cauchy estimate (a similar estimate clearly holds for the higher derivatives). The lemma is true for Möbius transformations (by a direct calculation) and so follows for Φ by the estimate above.

Lemma 3.5 ([10, Lemma 5.2]) *Given* $\epsilon > 0$ *there is a* $\delta > 0$ *with the following property. Suppose* $z_0 = r_0 e^{i\theta} \in \mathbf{D}$ *and let* $\gamma = [r_0 e^{i\theta}, r_1 e^{i\theta}]$ *be a radial line segment with endpoint* z_0. *Suppose* $|\varphi'(z_0)| \leq \epsilon/(1 - |z_0|^2)$ *and for all* $z \in \gamma$

$$|S(\Phi)(z)| \leq \frac{\delta}{(1 - |z|^2)^2}.$$

Then for all $z \in \gamma$

$$|\varphi'(z)| \leq \frac{\epsilon}{1 - |z|^2}.$$

Moreover, for $z \in \gamma$ *and* $\rho(z, z_0)$ *large enough (depending only on* δ *and* ϵ*) we have*

$$|\varphi'(z)| \leq \frac{C\sqrt{\delta}}{1 - |z|^2}.$$

The following is a well known version of Green's theorem

Lemma 3.6 ([14], [28], [33]) *Suppose* $0 \in \mathcal{D}$ *is chord-arc with constant* M, *that we have* $\operatorname{dist}(0, \partial\mathcal{D}) \sim \operatorname{diam}(\mathcal{D}) = 1$ *and that* F *is holomorphic on* $\mathbf{D}$ *and satisfies*

$$\iint_{\mathcal{D}} |F'(z)|^2 d(z)\,dxdy < \infty,$$

with $d(z) = \operatorname{dist}(z, \partial\mathcal{D})$. *Then* $F \in H^2(\mathcal{D})$ *and*

$$\int_{\partial\mathcal{D}} |F(z)|^2 ds(z) \sim |F(0)|^2 + \iint_{\mathcal{D}} |F'(z)|^2 d(z)\,dxdy$$

with constants depending only on M. *Similarly,*

$$\int_{\partial\mathcal{D}} |F(z)|^2 ds(z) \sim |F(0)|^2 + |F'(0)|^2 + \iint_{\mathcal{D}} |F''(z)|^2 d(z)^3\,dxdy.$$

Moreover, the constants grow at most like polynomials in M.

See [14] for a simple proof when $\mathcal{D}$ is a Lipschitz domain. We shall use this result with $F = (\Phi')^{1/2}$, Φ univalent. Then if $\varphi = \log(\Phi')$,

$$F' = \frac{1}{2}(\Phi')^{1/2}(\varphi'),$$

and

$$\begin{aligned} F'' &= \frac{1}{2}(\Phi')^{1/2}(\varphi'' + \frac{1}{2}(\varphi')^2) \\ &= \frac{1}{2}(\Phi')^{1/2}(S(\Phi) + (\varphi')^2). \end{aligned}$$

Lemma 3.7 *If* $\Phi : \mathbf{D} \to \Omega$ *is univalent and* $\partial\Omega$ *is rectifiable then*

$$\iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|)^3\,dxdy < \infty$$

To prove this note

$$\begin{aligned} \iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)|^2(1 - |z|)^3\,dxdy &\le C\iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z) + (\varphi'(z))^2|^2(1 - |z|)^3\,dxdy \\ &\quad + C\iint_{\mathbf{D}} |\Phi'(z)||\varphi'(z)|^4(1 - |z|)^3\,dxdy \\ &\le C\iint_{\mathbf{D}} |F''(z)|^2(1 - |z|)^3\,dxdy \\ &\quad + C\|\varphi\|_*^2 \iint_{\mathbf{D}} |F'(z)|^2(1 - |z|)\,dxdy \end{aligned}$$

Φ is univalent so $\|\varphi\|_*$ is uniformly bounded and so by the formulas above, each of the two terms on the right is bounded by $C\|\Phi'\|_{H^1}$. That proves the lemma.

To construct a map Φ onto a rectifiable domain Ω such that

$$\iint_{\mathbf{D}} |S(\Phi)(z)|^2(1-|z|)^3 dxdy = \infty,$$

we mimic the proof of sharpness of Lavrentiev's estimate. Consider a positive measure μ on $\mathbf{T}$ which is singular to Lebesgue measure but which is "smooth", i.e., such that its harmonic extension φ to the disk satisfies $\|\varphi'(z)\|_* \leq \epsilon$. The construction of such measures is given in [20], [32], [40]. The function Φ defined by $\mathrm{Re}(\log|\Phi'|) = -\varphi$ is univalent by Lemma 3.2 and $|\Phi'|$ is bounded so $\Phi(\mathbf{D})$ is rectifiable. However, $\log\Phi'$ is only in weak L^1 and certainly not in L^2, so by Corollary 6 the integral above cannot be finite.

Suppose Φ is univalent on the unit disk and that $\mathcal{D} \subset \mathbf{D}$ is a Lipschitz subdomain. Let $\{Q_j\}$ be the dyadic squares in the Whitney decomposition of $\Omega = \Phi(\mathbf{D})$ (see [48, Chapter VI]). Let $\epsilon, \delta > 0$ and define a collection of "bad squares" $\mathcal{B}_{\epsilon,\delta}$ by putting $Q_j \in \mathcal{B}_{\epsilon,\delta}$ if

$$|\varphi'(z)|(1-|z|^2) \geq \epsilon,$$

for some $z \in \Phi^{-1}(Q_j)$ and

$$|S(\Phi)(z)|(1-|z|^2)^2 \leq \delta,$$

for every $z \in \Phi^{-1}(Q_j)$. Also let $d(z) = \mathrm{dist}(z, \partial\mathcal{D})$ for $z \in \mathcal{D}$.

Lemma 3.8 ([10, Lemma 4.2]) *There are universal constants $\epsilon_0 > 0$ and $C > 0$ such that whenever $\epsilon < \epsilon_0$ and $\delta > 0$ then*

$$\int_{\partial\mathcal{D}} |\Phi'(z)|ds \leq C|\Phi'(0)| + C(1+\delta^{-2}) \iint_{\mathcal{D}} |\Phi'(z)||S(\Phi)(z)|^2 d(z)^3 dxdy$$
$$+ C \sum_{Q_j \in \mathcal{B}_{\epsilon,\delta}} \ell(Q_j).$$

(Our definition of "bad square" above is slightly different from the one in [10] where we only required $S(\Phi)$ to be small at one point of Q_j. This does not effect Lemma 3.8, since if the Schwarzian is large at one point of Q_j then its integral over Q_j is large (by normal families) and this is all the proof of Lemma 3.8 uses.) In [10] it is mentioned without proof that the final term is unnecessary if Ω is assumed to be a quasicircle. More precisely,

Lemma 3.9 *Suppose Ω is a quasidisk and $\Phi : \mathbf{D} \to \Omega$ is conformal. Then there is a $C > 0$ (depending only on the quasidisk constant) such that*

$$\ell(\partial\Omega) \sim \operatorname{diam}(\Omega) + \iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2 d(z)^3 dxdy.$$

The same holds for a Lipschitz subdomain of $\mathbf{D}$.

To prove this we claim that all the bad squares (if any exist) lie within a bounded hyperbolic distance of some fixed hyperbolic geodesic, γ, so the sum of their lengths is at most a constant times the length of $\Phi(\gamma)$. Since Ω is a quasidisk $\ell(\Phi(\gamma))$ is bounded by $C\operatorname{diam}(\Omega)$, as desired.

To prove the claim, first suppose 0 is mapped to the "center" of the quasidisk, i.e., $\operatorname{dist}(\Phi(0), \partial\Omega) \sim \operatorname{diam}(\Omega)$. We will show that all bad squares occur within a bounded hyperbolic distance of $\Phi(0)$. For suppose Q is a bad square with $\ell(Q) \ll \operatorname{diam}(\Omega)$. Then if δ is small enough in the definition of bad squares, Ω looks like a half plane near Q and Φ^{-1} looks like an inversion at a point $z_0 \in \partial\Omega$ with $\operatorname{dist}(z_0, Q) \sim \ell(Q)$. This means that if $I, J \subset \partial\Omega$ are two arcs with $\operatorname{diam}(I) \sim \operatorname{diam}(J) \sim \ell(Q)$ and with $z_0 \in I$ and $\operatorname{dist}(J, z_0) \sim \ell(Q)$ then the harmonic measure of I will be much larger then the harmonic measure of J and this contradicts the doubling property of harmonic measure on quasicircles.

Now suppose 0 is mapped to a point very near $\partial\Omega$. Let $w \in \mathbf{D}$ be a point such that $\operatorname{dist}(\Phi(w), \partial\Omega) \sim \operatorname{diam}(\Omega)$ and let $\tau : \mathbf{D} \to \mathbf{D}$ be a Möbius transformation mapping 0 to w. Then $\Phi \circ \tau$ satisfies the conditions of the previous case so all its bad squares occur in a bounded hyperbolic region. Using the composition law for the Schwarzian and the conformal invariance of $|\tau'(z)|(1 - |z|^2)$ we see that $S(F)(z)(1 - |z|^2)^2$ is invariant under Möbius transformations. Moreover,

$$\frac{(F \circ \tau)''(z)}{(F \circ \tau)'(z)}(1 - |z|^2) = \frac{F'' \circ \tau(z)}{F' \circ \tau(z)}\tau'(z)(1 - |z|^2) + \frac{\tau''(z)}{\tau(z)}(1 - |z|^2).$$

The first term on the right is invariant and the second term is large only near the ray from 0 to w. Therefore the only bad points for Φ can occur in a bounded hyperbolic neighborhood of w and within a bounded hyperbolic distance of the geodesic from 0 to w. This proves the claim and finishes the proof of Lemma 3.9.

4. Proof of Theorem 3

Theorem 3 *If Φ is univalent and*

$$A = |\Phi'(0)| + \iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|)^3 dxdy < \infty,$$

then $\Phi' \in H^{\frac{1}{2}-\eta}$ *for every* $\eta > 0$ *and* $\|\Phi'\|_{\frac{1}{2}-\eta} \le C(\eta)A$.

Divide the dyadic Carleson squares into three classes,

$$\begin{aligned} \mathcal{L} &= \{Q : |S(\Phi)(z)| \ge \delta/(1-|z|^2)^2 \text{ some } z \in T(Q)\}, \\ \mathcal{B} &= \{Q \notin \mathcal{L} : |\varphi'(z)| \ge \epsilon/(1-|z|^2) \text{ some } z \in T(Q)\}, \\ \mathcal{G} &= \{Q \notin \mathcal{L} \cup \mathcal{B}\}. \end{aligned}$$

The letters stand for "large", "bad" and "good". For any Carleson square Q we let z_Q denote the point in the center of $T(Q)$.

Given $M, \eta > 0$ we will construct a region $\mathcal{R} \subset \mathbf{D}$ and a constant $C > 0$ (depending on η but not on M) such that

$$\iint_{\mathcal{R}} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \le CM \iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy + C|\Phi'(0)|, \tag{4.1}$$

and

$$|\mathbf{T} \backslash \partial\mathcal{R}| \le CM^{-1+\eta}. \tag{4.2}$$

If $Q \in \mathcal{B} \cup \mathcal{G}$ define $\mathcal{D}_Q = Q \backslash \cup_{Q' \subset Q, Q' \in \mathcal{L}} Q'$. Each dyadic square is contained in a "mother" square of twice the size and we shall call a square in each class maximal if its mother is of a different type. We shall build $\mathcal{R}$ and verify (4.1), (4.2) by taking unions of three types of domains: $T(Q)$, for all $Q \in \mathcal{L}$, and $\mathcal{D}_Q$, for maximal squares in $\mathcal{G} \cup \mathcal{B}$.

Since φ is in Bloch, (4.1) is trivially true on $T(Q)$ for any $Q \in \mathcal{L}$ with constant C/δ.

To handle $\mathcal{D}_Q, Q \in \mathcal{G}$, apply Lemma 3.6 (as well as the identities following it) to $|\Phi'|^{1/2}$,

$$\begin{aligned} \iint_{\mathcal{D}_Q} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \le & C|\Phi'(z_Q)| + C \iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy \\ & + C \iint_{\mathcal{D}_Q} |\Phi'(z)||\varphi'(z)|^4(1-|z|)^3 dxdy. \end{aligned}$$

Since $Q \in \mathcal{G}$ then Lemma 3.5 implies every subsquare hitting $\mathcal{D}_Q$ is also in $\mathcal{G}$. Thus the last term above is at most $C\epsilon^2$ times the left hand side. Subtracting we get

$$\begin{aligned} \iint_{\mathcal{D}_Q} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \le & C \iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy \\ & + C|\Phi'(z_Q)|\ell(Q). \end{aligned}$$

However, since $Q \in \mathcal{G}$ is maximal the first term on the right dominates the second since Q is adjacent to a square in $\mathcal{L}$, and so $|S(\Phi)(z)|$ is large on $T(Q)$ (by normal families).

Thus we are reduced to considering bad squares. Suppose $Q \in \mathcal{B}$ and $\nu > 0$ is small (to be chosen below). First assume

$$\iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy \leq \nu|\Phi'(z_Q)|\ell(Q).$$

If ν is small enough (depending on ϵ and δ) this cannot occur if Q is the daughter of a square in $\mathcal{L}$ for then by normal families the Schwarzian is large on most of $T(Q)$. Thus this case can only occur if Q is the square containing the origin and so (since $\ell(Q) \sim 1$)

$$\iint_{\mathcal{D}_Q} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \leq C|\Phi'(0)|.$$

Next suppose that

$$\iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy = B \geq \nu|\Phi'(z_Q)|\ell(Q).$$

Using Lemma 3.4 there is a geodesic ray γ so that any subsquare $Q' \in \mathcal{B}$ intersecting $\mathcal{D}_Q$ must be within a bounded hyperbolic distance of γ, i.e., they lie with in some Stolz cone S. Drop down along γ until we either leave Q or leave $\cup_{\mathcal{B}} T(Q)$ or reach a square $Q' \in \mathcal{B}$ such that $z_{Q'}$ satisfies $|\Phi'(z_{Q'})|\ell(Q') \geq MB \geq M\nu|\Phi'(z_Q))|\ell(Q)$. Let $\mathcal{U}$ denote the collection of "ugly" squares Q' which arise in this way from maximal "bad" squares Q. Let $\hat{\mathcal{D}}_Q = \mathcal{D}_Q \backslash Q'$. We claim that

$$\iint_{\hat{\mathcal{D}}_Q} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \leq CM \iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy.$$

Define $\mathcal{R} = \mathbf{D} \backslash \cup_{\mathcal{U}} Q'$. The claim clearly implies (4.1).

To prove the claim we would like to apply Lemma 3.6 to $\hat{\mathcal{D}}_Q \backslash S$, but this may not be possible because the chord-arc constant of this domain might be too large (the edge of the cone might pass near the top of some $\mathcal{L}$ square in the definition of $\mathcal{D}_Q$). We get around this by dividing $\hat{\mathcal{D}}_Q \backslash S$ into chord-arc domains with fixed constant by adding vertical lines to the boundary. More precisely, recall that $\mathcal{D}_Q = Q \backslash \cup Q_j$ for some dyadic squares $\{Q_j\}$ and suppose L denotes one edge of the cone S which meets the circle at a point s_0. Suppose Q_j is to the left of s_0. We will joint Q_j to L by a vertical line segment from Q_j's upper right corner if $\text{dist}(Q_j, L) \le \ell(Q_j)/2$. The resulting domains are certainly chord-arc with a fixed constant and only a bounded number of Q_j of the same size are ever chosen (since L has positive slope). Thus the diameters of the new domains $\mathcal{D}_j$ can be chosen to have geometrically decreasing diameters (depending on S). Let z_j denote points in the top of each domain. Lemma 3.6 now applies to each $\mathcal{D}_j$ so

$$\begin{aligned}\iint_{\hat{\mathcal{D}}_Q} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy &\le \sum_j \iint_{\mathcal{D}_j} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \\ &\le C\sum_j |\Phi'(z_j)|(1-|z_j|) \\ &\quad + C\sum_j \iint_{\mathcal{D}_j} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy.\end{aligned}$$

The values $|\Phi'(z_j)|(1-|z_j|)$ are dominated by the values at nearby points along γ. These are growing geometrically, so the sum is dominated by the last term, namely, $C|\Phi'(z_{Q'})|\ell(Q')$. This proves the claim and hence (4.1). To prove (4.2) we need two lemmas.

Lemma 4.1 *For any $\eta > 0$ there exists δ and ϵ such that if $Q \in \mathcal{B}$ is maximal and Q' is the corresponding ugly square, then $\ell(Q') \le M^{-1+\eta}\ell(Q)$.*

This follows because if δ is small enough on the "bad" geodesic γ then Φ' must be growing almost like $(1-|z|)^{-2}$. In particular, by Lemma 3.4 and induction we have

$$|\Phi'(z)| \ge C|\Phi'(z_Q)|\left(\frac{1-|z|}{1-|z_Q|}\right)^{-2+\eta},$$

if δ is small enough. Given the definition of ugly squares, this proves the lemma.

Lemma 4.2 *For every $\eta > 0$ there is a $C = C(\eta)$ such that $\sum_{\mathcal{U}} \ell(Q') \le CM^{-1+2\eta}$ if $M \ge C$.*

To prove this we associate to each square $Q' \in \mathcal{U}$ the set $\partial\mathcal{D}_Q \cap \mathbf{T}$. Since these sets are disjoint, by Lemma 4.1 it suffices to show $\ell(\mathcal{D}_Q \cap \mathbf{T}) \ge \mathbf{CM^{-\eta}\ell(Q)}$. Normalize so $\ell(Q) = 1$ and $\Phi'(z_Q) = 1$. Choose a square Q'' with $Q' \subset Q'' \subset Q$ and

$$|\Phi'(z)|\ell(Q'') \sim M^{\eta}B = M^{\eta}\iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy,$$

for $z \in T(Q'')$. Let $\mathcal{D}'' = Q'' \cap \mathcal{D}_Q$. We claim that $\ell(\partial\mathcal{D}'' \cap \mathbf{T}) \ge \frac{1}{2}\ell(Q'')$ and hence $\ge CM^{-\eta}\ell(Q)$. Let $z'' = z_{Q''}$. Clearly $\text{diam}(\Phi(\mathcal{D}'')) \ge C|\Phi'(z'')|\ell(Q'')$. Also by Lemma 3.8,

$$\ell(\Phi(\partial\mathcal{D}'')) \le C|\Phi'(z'')|\ell(Q'') + CB \le C|\Phi'(z'')|\ell(Q''),$$

since $B \leq C\nu^{-1}|\Phi'(z_Q)|\ell(Q) \leq C|\Phi'(z'')|\ell(Q'')$. Let E denote the union on the top edges of the maximal large squares in Q (i.e., the squares removed in the definition of $\mathcal{D}_Q$). Since $S(\Phi)$ is large near the $\mathcal{L}$ squares,

$$\ell(\Phi(E)) = \int_E |\Phi'(z)|ds \leq C \iint_{\mathcal{D}_Q} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy = CB.$$

However since $\Phi(\partial\mathcal{D}'')$ has both diameter and length $\sim |\Phi'(z'')|\ell(Q'') \sim CM^\eta B$, Lavrentiev's estimate says that $\Phi(E)$ has small harmonic measure if M is large enough. Thus E has small harmonic measure in $\mathcal{D}''$ and since $\mathcal{D}''$ is a chord-arc domain this means E has small length compared to $\ell(Q'')$, say $\leq \ell(Q'')/2$. This proves Lemma 4.2.

We have now verified (4.1) and (4.2) and proceed to deduce Theorem 3. Suppose

$$1 = A = |\Phi'(0)| + \iint_{\mathbf{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy.$$

(It is sufficient to consider $A = 1$ by scaling.) Note that if $x \in \mathbf{T}\backslash 3\mathcal{U} = \mathbf{T}\backslash \cup_{\mathcal{U}} 3Q$ then the Stolz cone $W(x)$ with vertex at x is contained in $\mathcal{R}$. Recall that the area integral of a holomorphic function f on the disk,

$$A(f)(x) = \Big(\iint_{W(x)} |f'(z)|^2 dxdy\Big)^{1/2},$$

characterizes the Hardy spaces, i.e., $f \in H^p$ iff $A(f) \in L^p$. Motivated by this and applying the identities following Lemma 3.6, we define

$$A(x) \equiv A((\Phi')^{1/2})(x) = \Big(\iint_{W(x)} |\Phi'(z)||\varphi'(z)|^2 dxdy\Big)^{1/2}, \quad x \in \mathbf{T}\backslash \dot{3}\mathcal{U},$$

and $A(x) = \infty$ otherwise.

Suppose $\lambda > 0$ (to be chosen later). There is a $C = C(\eta)$ such that

$$\begin{aligned} |\{x \in \mathbf{T} : A(x) \geq \lambda\}| &\leq |3\mathcal{U}| + \frac{1}{\lambda^2}\int_{\mathbf{T}\backslash 3\mathcal{U}} A^2(x)dx \\ &\leq CM^{-1+\eta} + \frac{1}{\lambda^2}\iint_{\mathcal{R}} |\Phi'(z)||\varphi'(z)|^2(1-|z|)dxdy \qquad (4.3)\\ &\leq CM^{-1+\eta} + \frac{M}{\lambda^2}. \end{aligned}$$

Choose λ so these two terms are the same size, i.e., $\lambda^2 = M^{2-\eta}$. Then

$$|\{x \in \mathbf{T} : A(x) \geq \lambda\}| \leq CM^{-1+\eta} = C\lambda^{(-2+2\eta)/(2-\eta)} \leq C\lambda^{(-1+\eta/2)}.$$

Thus $A(x)^{1-\eta} \in L^1$ with norm at most $C + C\eta^{-1}$ and so $\|\Phi'\|_{\frac{1}{2}-\eta} \leq C(\eta)$. This completes the proof of Theorem 3.

To prove Corollary 1, fix $\eta, \epsilon > 0$. By (4.2) there is a domain $\mathcal{R}$ such that

$$\ell(\mathbf{T}\backslash\partial\mathcal{R}) \leq CM^{-1+\eta}.$$

Using (4.1), Lemma 3.6 and the usual identities for $(\Phi')^{1/2}$,

$$\ell(\Phi(\partial\mathcal{R})) \leq \iint_{\mathcal{R}} |\Phi'(z)||\varphi(z)|^2(1-|z|)dxdy \leq\leq CMA.$$

Choosing M so $\epsilon = CM^{-1+\eta}$ gives $\ell(\Phi(\partial\mathcal{R})) \leq CA\epsilon^{-1-2\eta}$, as desired.

To prove Corollary 2, simply repeat the proof of Lavrentiev's estimate in Section 1, using the fact that $\| \log^+ |\Phi'|\|_{L^1(\mathbf{T})} \leq \|\Phi'\|_{L^p(\mathbf{T})}$.

The proof of Corollary 3 requires some modifications to the proof of Theorem 3. We assume the reader is familiar with [10]. Suppose Ω is simply connected, $E \subset \partial\Omega$ and $\omega(z_0, E, \Omega) = \epsilon$ for some z_0 with $\operatorname{dist}(z_0, \partial\Omega) \geq 1$. Let $\Phi : \mathbf{D} \to \mathbf{C}\backslash E$ be a uniformizing map with $\Phi(0) = z_0$. By results of Pommerenke [42] there is a (hyperbolically) convex fundamental domain $\mathcal{F}$ for Φ with the property that $|\mathbf{T} \cap \partial\mathcal{F}| \geq C\epsilon$. To prove [10, Theorem 2] we passed to a subset $\mathcal{W} \subset \mathcal{F}$ which was a $C\epsilon^{-1}$-Lipschitz domain $\mathcal{W} \subset \mathbf{D}$ such that Φ is 1 to 1 on $\mathcal{W}$, $|\mathbf{T} \cap \partial W| \geq \epsilon/2$, the hyperbolic distance from $\mathcal{W}$ to $\partial\mathcal{F}$ is at least $C\epsilon$.

What we wish to observe here is that we can actually take $\mathcal{W}$ to be Lipschitz with a constant close to 1 by modifying E slightly and dropping the condition that Φ be univalent on $\mathcal{W}$, instead requiring that it satisfy estimates similar to those for univalent functions. We will add a set to E as follows. Consider a Whitney decomposition of $\mathbf{C}\backslash E$. We need only consider boxes with size $\leq \operatorname{diam}(E)$. For each square Q that hits $\partial\Omega$ choose a point $z \in Q \cap \partial\Omega$. By throwing away some of the points, if necessary, we may assume

$$|z_j - z_k| \geq C \min(\operatorname{dist}(z_j, E), \operatorname{dist}(z_k, E)), \quad j \neq k,$$

and for all $z \in \partial\Omega$ there is a z_j such that

$$|z - z_j| \leq C\operatorname{dist}(z, E) \quad \text{and} \quad \operatorname{dist}(z_j, E) \geq C\operatorname{dist}(z, E).$$

We now take a subset of these points by choosing z_j if $\operatorname{dist}(z_j, E) \leq 1$ and there is a point $x \in E$ such that

$$\{z : |x - z_j|/10 \leq |x - z| \leq 10|x - z_j|\} \cap E = \emptyset.$$

By iterating the construction we could take E to be uniformly thick with respect to logarithmic capacity, instead of countable. More precisely, fix z_j and scale so $\operatorname{dist}(z_j, E) = 1$. Choose another point $\tilde{z}_j \in \partial\Omega$ with $|z_j - \tilde{z}_j| = \frac{1}{10}$. For each of these two points choose points in $\partial\Omega$ at distance $\frac{1}{100}$. Continuing in this way we associate to z_j a Cantor set $F_j \subset \partial\Omega$ of small Hausdorff dimension (as small as we wish by replacing $\frac{1}{10}$ by smaller numbers). Moreover each F_j lies on a rectifiable curve Γ_j with length comparable to $\operatorname{diam}(F_j) \sim \operatorname{dist}(z_j, E)$ (it only requires length $2^n 10^{-n}$ to connect the nth generation points to the $(n-1)$st points).

Let F be the union of E and the sets $\{F_j\}$. Let Φ be the covering map of $\mathbf{C}\backslash F$ and $\mathcal{F}$ the fundamental domain described above. It will also be convenient to assume that the base point z_0 for harmonic measure satisfies

$$\operatorname{dist}(z_0, E) \sim \operatorname{dist}(z_0, F\backslash E),$$

but this is easy to get. Moreover, by the maximum principle we may assume $\operatorname{dist}(z_0, E) = 1$ (if z_0 is far from E we can find points closer to E which give E at least as much harmonic measure).

The set F lies on a rectifiable curve $\tilde{\Gamma}$ whose length is at most $C\Lambda_1(\Gamma) + 1$. This is because we can build $\tilde{\Gamma}$ by adding the curves Γ_j and connecting them to E by a line segment of length $\operatorname{dist}(z_j, E)$. To show the sum of these lengths is bounded, associate to each z_j an arc $A_j \subset \Gamma$ with $\Lambda(A_j) \sim \operatorname{dist}(A_j, E) \sim \operatorname{dist}(z_j, E)$ and $\operatorname{dist}(z_j, A_j) \leq C\operatorname{dist}(z_j, E)$. This can be done because Γ must cross the annulus in definition of the collection $\{z_j\}$. Moreover each A_j can be used at most a bounded number of times (because there are only a bounded number of Whitney squares of the right size and distance which can contain a z_j). If $\operatorname{diam}(\Gamma) \ll 1$ there may be some points z_j with no associated arc, but these have distances to E which sum with a uniform bound. Thus $\Lambda_1(\tilde{\Gamma}) \leq C\Lambda_1(\Gamma) + C$.

Since each F_j has small Hausdorff dimension, it has zero harmonic measure in Ω [6], [13], [36]. So F has the same harmonic measure in Ω as E does. Therefore $\partial\mathcal{F}$ hits the circle in length at least $\omega(E)$. Furthermore, if z is any point in the disk, by our construction there must exist points $x, y \in F$ such that

$$|\Phi(z) - x| \sim |\Phi(z) - y| \sim |x - y|.$$

Comparing the hyperbolic metric on $\mathbf{C}\backslash F$ to the metric on $\mathbf{C}\backslash\{x, y\}$ we see that Φ must be univalent on some uniform hyperbolic ball centered at z (this is equivalent to saying F is uniformly thick with respect to logarithmic capacity). In particular, we have

$$|\varphi'(w)| \le \frac{C}{1-|w|^2},$$
$$|S(\Phi)(w)| \le \frac{C}{(1-|w|^2)^2},$$
$$|\Phi'(w)| \sim |\Phi'(z)|, \qquad w \in D(z, \tfrac{1}{2}(1-|z|)).$$

In [10] we constructed a $C\epsilon^{-1}$ Lipschitz domain $\mathcal{W}$ inside the fundamental domain $\mathcal{F}$ and so that $S = \partial\mathcal{W} \cap \mathbf{T}$ has length $\ge \epsilon/C$. We want to enlarge $\mathcal{W}$ by replacing the cones of angle ϵ by larger cones of angle $\pi - \epsilon$, i.e., let $\mathcal{V}$ be the union of Stolz cones of angle $\pi - \epsilon$ with vertices in S. Then $\mathcal{V}$ is a $1 + C\epsilon$ Lipschitz domain on which Φ satisfies the usual estimates for univalent functions (with larger, but uniform, constants depending on the choice of F).

We claim that Φ is at most $C\epsilon^{-C}$ to 1 on the domain $\mathcal{V}$. To see this, first note that $\mathcal{W}\backslash\mathcal{V}$ consists of a countable number of regions, each of which is the difference of two tents, one with slope ϵ and the other of slope ϵ^{-1}. Thus any point of $\mathcal{V}$ is within hyperbolic distance $C\log\epsilon^{-1}$ of $\mathcal{W}$. Suppose B is a ball of hyperbolic radius ~ 1 in $\mathcal{W}\backslash\mathcal{V}$ and draw the geodesic Γ from B to $\mathcal{W}$ (which has length $\le C\log\epsilon^{-1}$ by our previous remark). We take a disjoint collection of balls $\{B_j\} = \{B(z_j, \epsilon)\} \subset \mathcal{W}$ with hyperbolic area $\sim \epsilon^2$ and $\operatorname{dist}(z_j, \partial\mathcal{W}) = \epsilon$ and associate to γ the ball B_j closest to its endpoint on $\partial\mathcal{W}$.

Let $\{w_1, \dots, w_M\}$ be points in $\mathcal{V}$ with the same image under Φ. Because of our choice of F, Φ is 1 to 1 on a hyperbolic ball of diameter ~ 1 around each point. Draw the geodesic from w_k to $\mathcal{V}$ and associate to w_j the ball $B_j = B_{j(k)}$ as above. Since each w_j is a uniform hyperbolic distance from the other points, a simple volume estimate shows that each B_j is associated to at most ϵ^{-C} different points. Moreover, the collection of B_j's used have disjoint images which all lie within distance $C\log\epsilon^{-1}$ of a given point, so $\operatorname{area}(\cup B_j) \le \epsilon^{-C}$. Thus

$$M \le C\epsilon^{-2}\sum_{k=1}^{M} \operatorname{area}\left(B_{j(k)}\right) \le C\epsilon^{-C}\operatorname{area}\Big(\bigcup_{k=1}^{M} B_{j(k)}\Big) \le C\epsilon^{-2C}.$$

This proves the claim that Φ is at most $C\epsilon^{-C}$ to 1 on $\mathcal{V}$.

The argument in [10] now shows that

$$\iint_{\mathcal{V}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3\,dxdy \le C\epsilon^{-C}A \le C\epsilon^{-C}(\Lambda_1(\Gamma)+1).$$

By standard conformal mapping estimates for Lipschitz domains,

$$\omega(0, S, \mathcal{W}) \ge C|S|^{1+C\epsilon} \ge C\epsilon^{1+C\epsilon} \ge C\epsilon.$$

Repeat the proof of Lavrentiev's estimate (using harmonic measure on $\mathcal{V}$ instead of $d\theta$ on $\mathbf{T}$) to get

$$C\epsilon \sim \omega(0, S, \mathcal{V}) \leq \frac{C\log(C\epsilon^{-C}A)}{|\log \Lambda_1(E)|}.$$

Since $\epsilon = \omega(E, \Omega)$, this easily implies the desired inequality

$$\frac{\omega(E)}{\log \omega(E)^{-1} + 1} \leq C\frac{\log A + 1}{|\log \Lambda_1(E)| + 1}.$$

5. Proof of Theorem 4

Theorem 4 *Suppose $\Phi : \mathbf{D} \to \Omega$ is univalent. Then the following are equivalent:*

(1) *$\varphi = \log \Phi'$ is in BMO.*

(2) *There exists a $\delta, C > 0$ such that for every $z_0 \in \Omega$ there is a rectifiable subdomain $\mathcal{D} \subset \Omega$ such that $\ell(\partial\mathcal{D}) \leq C\text{dist}(z_0, \partial\Omega)$ and $\omega(z_0, \partial\mathcal{D} \cap \partial\Omega, \mathcal{D}) \geq \delta$.*

(3) *There exists a $\delta, C > 0$ such that for every $z_0 \in \Omega$ there is subdomain $\mathcal{D} \subset \Omega$ which is chord-arc with constant C and such that $\ell(\partial\mathcal{D}) \leq C\text{dist}(z_0, \partial\Omega)$ and $\ell(\partial D \cap \partial\Omega) \geq \delta d$.*

(4) *There exists $C > 0$ such that for every Carleson square Q,*

$$\iint_Q |S(\Phi)(z)|^2(1 - |z|)^3 dxdy \leq C\ell(Q).$$

(5) *There exists $\delta, C > 0$ such that for every $w_0 \in \mathbf{D}$, there exists a Lipschitz domain $\mathcal{D} \subset \mathbf{D}$ such that $\omega(w_0, \partial\mathcal{D} \cap T, \mathcal{D}) \geq \delta$ and*

$$\iint_{\mathcal{D}} |\Phi'(z)||S(F)(z)|^2(1 - |z|)^3 dxdy \leq C|\Phi'(w_0)|(1 - |w_0|).$$

We will prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1); (1) $\Leftrightarrow$ (4); (1), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2): Let $w_0 = \Phi^{-1}(z_0)$. Since φ is in BMO we can find a chord-arc domain $\mathcal{D}$ containing w_0 of diameter $\sim (1 - |w_0|)$ such that $\omega(w_0, \partial\mathcal{D} \cap \mathbf{T}, \mathcal{D}) \geq 1/2$ and

$$|\varphi(w) - \varphi(w_0)| \leq C\|\varphi\|_{\text{BMO}},$$

for all $w \in \mathcal{D}$. One just applies the usual stopping time construction of throwing out dyadic squares where $|\varphi(w) - \varphi(w_0)| > M$ is too large on the top half. The lengths of the omitted squares is controlled by the maximal function of the characteristic function of the set where $|\varphi(e^{i\theta}) - \varphi(w_0)| \leq CM$ which is small if $M \geq C\|\varphi\|_{\text{BMO}}$.

$$e^{-C} \leq \frac{\Phi'(w_0)}{\Phi'(w)} \leq e^C,$$

for $w \in \mathcal{D}$, so $\mathcal{D}' = \Phi(\mathcal{D})$ is a rectifiable domain with $\ell(\partial \mathcal{D}') \leq C\text{dist}(z_0, \partial\Omega)$ and $\omega(z_0, \partial D' \cap \partial\Omega, \mathcal{D}') \geq \delta$. It may not be chord-arc however.

(2) $\Rightarrow$ (3): Condition (3) will follow as a special case of the following result.

Lemma 5.1 *Suppose Ω is simply connected, $\Lambda_1(\partial\Omega) = M$, $0 \in \Omega$, $\text{dist}(0, \partial\Omega) = 1$ and $E \subset \partial\Omega$ with $\omega(E) = \epsilon$. Then there exists $\Omega' \subset \Omega$ which is a $C(M, \epsilon)$ chord-arc domain and satisfies $\Lambda_1(\partial\Omega' \cap \partial\Omega \cap E), \omega(\partial\Omega' \cap \partial\Omega \cap E) \geq \delta(M, \epsilon)$.*

To prove this we will use:

Lemma 5.2 ([30]) *Suppose $f : [0,1]^m \to \mathbf{R}^{m+k}$ satisfies $\|f\|_{\text{LIP}} = 1$. Then for every $\delta > 0$ there is a $M(\delta) > 0$ and closed sets $K_1, \ldots, K_M, N \leq N(\delta)$ such that*

$$h(f(Q_0) \setminus \bigcup_{j=1}^{N} K_j)) \leq \delta,$$

and such that

$$|f(x) - f(y)| \geq \frac{\delta}{2}|x - y|, \quad x, y \in K_j, \quad 1 \leq j \leq N.$$

Here h denotes m-dimensional Hausdorff content, $h(E) = \inf \sum_{j=1}^{\infty} r_j^m$, where the infimum is over all coverings of E by balls of radius $\{r_j\}$.

To prove Lemma 5.1 let $\Phi : \mathbf{D} \to \Omega$ be univalent with $\Phi(0) = 0$. Since $|\Phi'(0)| \geq 1$, $\| \log |\Phi'| \|_{L^1(\mathbf{T})} \leq \|\Phi'\|_{L^1(\mathbf{T})} \leq 2M$. Thus the nontangential maximal function of $\varphi = \log|\Phi'|$ is in weak L^1 and so we can choose a $C = C(M, \epsilon)$ and a 2-Lipschitz region $\mathcal{D}$ where $e^{-C} \leq |\Phi'| \leq e^C$ and which hits $\Phi^{-1}(E)$ in length bigger than $\epsilon/2$. We want to apply Lemma 5.2 to the mapping Φ on the set $\partial\mathcal{D}$, but we will want the exceptional set to have small Λ_1 measure, not just small Hausdorff content.

To get this we need to establish that the image curve $\Gamma = \Phi(\partial\mathcal{D})$ is Ahlfors–David regular. Fix $x \in \Gamma$ and $r > 0$ and let $F = \Phi^{-1}(\Gamma \cap D(x, r))$. Define a new set by

$$\tilde{F} = \bigcup_{x \in F} D(x, re^{-C}) \cap \partial\mathcal{D}.$$

To each component interval I of $\tilde{F}$ associate a maximal collection of points $\{z_j\}$ with $z_j \in \mathcal{D}$, $|z_j| \leq 1 - re^{-C}$, $\text{dist}(z_j, \tilde{F}) \leq re^{-C}$ and $|z_j - z_k| = re^{-C}$ for $j \neq k$. To each point we associate a disk $D_j = D(z_j, re^{-C}/2)$. Since $|\Phi'| \leq e^C$ on $\mathcal{D}$ and every z_j is in$\mathcal{D}$, every disk has image in $D(x, 3r)$. On the other hand, the $\{\Phi(D_j)\}$ are all disjoint and each has diameter $\geq re^{-C}$. Thus there are at most Ce^{2C} such disks and so $\Lambda_1(F) \leq \Lambda_1(\tilde{F}) \leq Cre^{2C}$. Thus $\Lambda_1(\Gamma \cap D(x,r)) \leq Ce^{3C}r$. This proves the claim that Γ is regular.

Now we return to the proof of Lemma 5.1. Since Γ is Ahlfors–David regular, for any $E \subset \Gamma$, $\Lambda_1(E) \leq Ce^{3C}h(E)$. Applying Lemma 5.2 to Φ and $\partial\mathcal{D}$ gives sets $K_1, \ldots, K_N$ such that Φ is $Ce^{-C}\delta$-bi-Lipschitz on each and

$$h(f(Q_0)\backslash \cup K_j) \leq Ce^{3C}\delta.$$

By our remark above this implies

$$\Lambda_1(f(Q_0)\backslash \cup K_j) \leq Ce^{6C}\delta.$$

So if we take δ so small that $Ce^{6C}\delta < \epsilon/4$, then one of the sets K_j hits the set $\mathcal{D} \cap \Phi^{-1}(E)$ in measure at least $\epsilon/(4N)$. The proof of Lemma 5.1 now reduces to the following claim: there is a $M > 0$ such that given a sawtooth domain $W \subset \mathbf{D}$ and a univalent mapping Φ on $\mathbf{D}$ which satisfies $e^{-C} \leq |\Phi'| \leq e^{C}$ on W and which is δ-bi-Lipschitz on $K = \partial W \cap \mathbf{T}$ then that Φ is $M\delta$-bi-Lipschitz on all on W.

To prove the claim we need only establish the lower estimate. Choose points $z_1, z_2 \in \partial W$ and let $d_1 = 1 - |z_1|, d_2 = 1 - |z_2|$ and without loss of generality assume $d_2 \leq d_1$. We consider three cases. First assume $|z_1 - z_2| \leq d_1/2$. Then using the Koebe 1/4 theorem,

$$|\Phi(z_1) - \Phi(z_2)| \geq C|z_1 - z_2||\Phi'(z_1)| \geq Ce^{-C}|z_1 - z_2|.$$

For the second case assume $d_1/2 \leq |z_1 - z_2| \leq Ad_1$ (where A will be chosen in case 3). Then the distortion theorem implies

$$|\Phi(z_1) - \Phi(z_2)| \geq Cd_1|\Phi'(z_1)| \geq CA^{-1}e^{-C}|z_1 - z_2|.$$

Finally assume $|z_1 - z_2| \geq Ad_1$. Pick points $\zeta_1, \zeta_2 \in \partial W \cap \mathbf{T}$ which are closest to z_1, z_2 respectively. Then since W is a sawtooth domain, $|\zeta_1 - \zeta_2| \geq Ad_1/2$, so since Φ is bi-Lipschitz on the circle, $|\Phi(\zeta_1) - \Phi(\zeta_2)| \geq \frac{A}{2}e^{-C}d_1$. By the triangle inequality,

$$\begin{aligned}
|\Phi(z_1) - \Phi(z_2)| &\geq |\Phi(\zeta_1) - \Phi(\zeta_2)| - |\Phi(z_1) - \Phi(\zeta_1)| - |\Phi(z_2) - \Phi(\zeta_2)| \\
&\geq \frac{A}{2}e^{-C}d_1 - e^{C}Bd_1 - e^{C}Bd_2 \\
&\geq \frac{A}{4}e^{-C}d_1 \\
&\geq \frac{1}{4}e^{-C}|z_1 - z_2|,
\end{aligned}$$

if A is large enough depending on B, C. This completes the proof of the claim and therefore the proof of (2) $\Rightarrow$ (3). We note that if Ω is a quasidisk, the proof becomes easier by avoiding Lemma 5.2. This is because if $\mathcal{D} \subset \mathbf{D}$ is Lipschitz, then $\Phi(\mathcal{D})$ is also a quasidisk. In this case $|\Phi'|$ being bounded away from zero easily implies Φ is bi-Lipschitz on ∂D.

(3) $\Rightarrow$ (1): Let $\mathcal{D}' = \Phi^{-1}(\mathcal{D})$, $w_0 = \Phi^{-1}(z_0)$. Condition (2) implies that for each w_0 there exists a compact set $E \subset \partial\mathcal{D}' \cap \mathbf{T}$ such that $\omega(w_0, E, \mathbf{D}) \geq \delta$ and $|\varphi(w) - \varphi(w_0)| \leq M$ on E for some uniform M (in fact we may assume this inequality holds on a subdomain of $\mathcal{D}'$ containing E in its boundary, e.g, the region where the nontangential maximal function of χ_E is bounded by some appropriate constant). In particular, we may assume $|E| \geq \delta(1 - |w_0|)$ and $E \subset D(w_0, 1/\delta)$. If I is a complementary interval of E with center ζ let $w_1 = (1 - \delta|I|)\zeta$. Note that $|\varphi(w_1) - \varphi(w_0)| \leq CM$ by Harnack's inequality. We obtain a subset $E_1 \subset I$ with $|E_1| \geq \delta|I|$ and $|\varphi(w) - \varphi(w_0)| \leq 2CM$ on E_1. Applying this argument to every complementary intervals for n generations we obtain a set F with

$$\omega(w_0, F, \mathbf{D}) \geq 1 - (1 - \delta)^n > 3/4$$

on which $|\Phi'| \leq (CM)^n$. This implies $\varphi = \log|\Phi'|$ is in BMO (e.g., [25, Exercise VI.4]).

(1) $\Rightarrow$ (4): This is due to Zinsmeister [50, Lemma 7]. Since $\varphi \in$ BMO both

$$|\varphi'(z)|^2(1 - |z|)dxdy, \quad |\varphi''(z)|^2(1 - |z|)^3 dxdy$$

are well known to be Carleson measures (e.g., [25, Theorem VI.3.4]). Since φ is in Bloch, this implies

$$|\varphi'(z)|^4(1 - |z|)^3 dxdy,$$

is also a Carleson measure. Since $S(\Phi) = \varphi'' - \frac{1}{2}(\varphi')^2$ a simple calculation shows the measure in (3) is dominated by the sum of these three, hence is Carleson.

(4) $\Rightarrow$ (1): This is due to Astala and Zinsmeister [4], but we include a proof for completeness. Fix a square Q_0. We will show

$$\iint_{Q_0} |\varphi'(z)|^2(1 - |z|)dxdy \leq C\ell(Q_0) + C\iint_{Q_0} |S(\Phi)(z)|^2(1 - |z|)^3 dxdy,$$

for absolute $C > 0$. This implies $|\varphi'(z)|^2(1 - |z|)dxdy$ is a Carleson measure and hence $\varphi \in$ BMO.

Fix a $\delta, \epsilon > 0$ to be chosen later. We define three collection of dyadic squares, as in the proof of Theorem 3:

$$\mathcal{L} = \{Q : |S(\Phi)(z)| \geq \delta/(1-|z|^2)^2 \text{ some } z \in T(Q)\},$$
$$\mathcal{B} = \{Q \notin \mathcal{L} : |\varphi'(z)| \geq \epsilon/(1-|z|^2) \text{ some } z \in T(Q)\},$$
$$\mathcal{G} = \{Q \notin \mathcal{L} \cup \mathcal{B}\}.$$

Since φ is Bloch we have for $Q \in \mathcal{L}$,

$$(5.1) \qquad \iint_{T(Q)} |\varphi'(z)|^2(1-|z|)dxdy \leq C\ell(Q) \leq \frac{C}{\delta^2} \iint_{T(Q)} |S(\Phi)(z)|^2(1-|z|)^3 dxdy,$$

for any $Q \in \mathcal{L}$. Thus the estimate is o.k. on the $\mathcal{L}$ squares.

If $Q \in \mathcal{B}$ is maximal define $\mathcal{D} = Q \setminus \cup_{Q' \in \mathcal{L} \cup \mathcal{G}} Q'$. Then Lemma 3.4 implies there are at most C nth generation daughters of Q with $T(Q)$ in $\mathcal{D}$. Thus

$$\sum_{Q' \subset \mathcal{D}} \ell(Q') \leq C\ell(Q).$$

Since Q is maximal, by Lemma 3.5 it must either be Q_0 or the daughter of a $\mathcal{L}$ square Q''. Thus since φ is in Bloch either

$$\iint_S |\varphi'(z)|^2(1-|z|)dxdy \leq C\ell(Q_0),$$

or

$$(5.2) \qquad \iint_S |\varphi'(z)|^2(1-|z|)dxdy \leq C\ell(Q) \leq \frac{C}{\delta^2} \iint_{T(Q'')} |S(\Phi)(z)|^2(1-|z|)^3 dxdy.$$

In either case we have controlled the integral over the tops of all the squares in $\mathcal{B}$.

This leaves only the squares in $\mathcal{G}$. If $Q \in \mathcal{G}$ is maximal (with respect to inclusion) form a region $\mathcal{D} = Q \setminus \cup_{\mathcal{L} \cup \mathcal{B}} Q'$. Then $\mathcal{D}$ is a chord-arc domain and

$$|\varphi'(z)| \leq \frac{\epsilon}{1-|z|^2},$$

everywhere in $\mathcal{D}$. If w is the center of Q then Lemma 3.6 shows

$$\begin{aligned}\iint_{\mathcal{D}} |\varphi'(z)|^2(1-|z|)dxdy &\leq C|\varphi'(w)|^2\ell(Q)^3 + C\iint_{\mathcal{D}} |\varphi''(z)|^2(1-|z|)^3 dxdy \\ &\leq C|\varphi'(w)|^2\ell(Q)^3 + C\iint_{\mathcal{D}} |S(\Phi)(z)|^2(1-|z|)^3 dxdy \\ &\quad + C\epsilon^2 \iint_{\mathcal{D}} |\varphi'(z)|^2(1-|z|)^2 dxdy.\end{aligned}$$

The far right term is small compared to the left hand side and so can be subtracted. Thus is only remains to estimate $\sum |\varphi'(w_j)|^2(1-|w_j|)^3$ where the sum is over all the regions $\{\mathcal{D}_j\}$ arising from maximal $\mathcal{G}$ squares. However, by maximality each such square is the daughter of a $\mathcal{L}$ or a $\mathcal{B}$ square so w_j is within a bounded hyperbolic distance of a square Q in $\mathcal{L} \cup \mathcal{B}$. Thus

$$|\varphi'(w_j)|^2 \ell(Q_j)^3 \leq C \sum_{\mathcal{L} \cup \mathcal{B}} \ell(Q).$$

But we have already noted that the sum on the right is bounded by

$$C\ell(Q_0) + C \iint_{Q_0} |S(\Phi)(z)|^2(1-|z|)^3 dxdy,$$

so the proof is complete.

(1), (4) $\Rightarrow$ (5): Take a Lipschitz region as in proof of (1) $\Rightarrow$ (2) where Φ' is bounded and bounded away from zero and then use (4).

(5) $\Rightarrow$ (1): This follows by applying the proof of Theorem 3 to the Lipschitz domain given by (5). We obtain a region $\mathcal{R} \subset \mathcal{D}$ with $\Phi' \in L^1(\mathcal{R})$ and which hits $\mathbf{T}$ in positive length, each with estimates. This is (2).

This completes the proof of Theorem 4. Corollary 6 follows immediately from the arguments (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). To prove Corollary 7 we need to change the part of the argument of (3) $\Rightarrow$ (1) where we used that φ was in Bloch, i.e., the proof of (5.1) and (5.2). (Also note that on the domains $\mathcal{D}$ constructed in the proof the φ does correspond to a univalent function because the Schwarzian is small there). By rescaling it is enough to show:

Lemma 5.4 *There exists a $C > 0$ such that if φ is analytic on the unit disk and*

$$\iint_{\mathbf{D}} |\varphi''(z) - \frac{1}{2}(\varphi'(z))^2|^2 dxdy = A^2 < \infty,$$

with $A \geq 1$ then

$$\iint_{\frac{1}{2}\mathbf{D}} |\varphi'(z)|^2 dxdy \leq CA.$$

Of course, this must fail for small A because of Möbius transformations. However, it says that if a function is far from linear functions in some sense, then it is also far from the Möbius transformations.

To prove Lemma 5.4 this let $D_1 = \{|z| \leq 1/2\}$, $D_2 = \{|z| \leq 3/4\}$ and $F = \varphi'' - (\varphi')^2/2$. The hypothesis and Cauchy's estimate imply $|F| \leq CA$ on D_2. Suppose the second integral is large, say $\geq 100CA$. Then there is a point $z_0 \in D_1$

where $|\varphi'(z_0)| \geq 100CA \geq 100|F(z_0)|$. Consider the path starting at z_0 obtained by following the path of steepest ascent for $|\varphi'|$ (i.e., follow the vector field $\arg(\varphi') - \arg(\varphi''))$. We obtain a differential inequality for $|\varphi'|$ along this path namely

$$\frac{d}{ds}|\varphi'| \geq \frac{1}{2}(|\varphi'|^2/2 - |F|) \geq |\varphi'|^2/8,$$

(ds is arclength) which is valid as long as the path stays in D_2. But since $|\varphi'(z_0)| \geq 100$, the inequality above implies $|\varphi'|$ blows up within distance $1/8$ of z_0 (and thus inside D_2). This is impossible since φ is assumed to be analytic on all on $\mathbf{D}$. Thus the desired inequality must hold.

6. Proof of Theorem 5

Theorem 5 *Suppose Ω is simply connected and $\Phi : \mathbf{D} \to \Omega$ is conformal. Then except for a set of zero measure the following conditions are equivalent:*

(1) *Φ' has a non-tangential limit at $e^{i\theta} \in \mathbf{T}$.*

(2) $$\int_0^1 \eta^2(\Phi(e^{i\theta}), t)\frac{dt}{t} < \infty.$$

(3) $$\int_{W(\theta)} |S(\Phi)(z)|^2(1 - |z|)^2 dxdy < \infty.$$

(4) $$\int_{W(\theta)} |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|)^2 dxdy < \infty.$$

We will prove (1) $\Rightarrow$ (2), (3), (4); (2) $\Rightarrow$ (1); (3) $\Rightarrow$ (1); (4) $\Rightarrow$ (1). Note that to prove each implication it is enough to show that if one condition holds on a set of positive measure then the next holds on some subset of positive measure.

(1) $\Rightarrow$ (2), (3), (4): If (1) holds on a set of positive measure then it is easy to find a Lipschitz domain $\mathcal{D}$ such that $E = \partial\mathcal{D} \cap \mathbf{T}$ has positive length and $\Phi(\mathcal{D})$ is a Lipschitz domain with boundary Γ. Let $F = \Phi(E)$. Then

$$\int_0^1 \beta_\Gamma^2(x, t)\frac{dt}{t} < \infty,$$

$$\int_0^1 \gamma_F^2(x, t)\frac{dt}{t} < \infty,$$

a.e. on F. Together with part (2) of Lemma 3.1 these imply

$$\int_0^1 \eta^2(x,t)\frac{dt}{t} < \infty,$$

a.e. on F, which shows (2) holds. Since $\Phi(\mathcal{D})$ is rectifiable, Lemma 3.7 implies

$$\iint_{\mathcal{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy < C\ell(\Phi(\mathcal{D})) < \infty,$$

so (4) holds for a.e. cone with vertex in E. Finally, Φ' is bounded and bounded away from zero in $\mathcal{D}$ so (3) holds a.e. on E.

(2) $\Rightarrow$ (1): If (2) holds on a set of positive measure then for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$\int_0^\delta \eta(x,t)^2\frac{dt}{t} < \epsilon,$$

on a subset F of positive measure. Using Lemma 3.3 we can find a Lipschitz domain $D \subset \mathbf{D}$ such that $|\partial D \cap \mathbf{T}| > 0$ and $|S(\Phi)(z)|(1-|z|)^2 \leq \epsilon$ for $z \in D$. By Lemma 3.1 $\Phi(\mathcal{D})$ is a quasicircle, say with boundary Γ, and by part (2) of Lemma 3.2

$$\int_0^1 \beta_\Gamma(x,t)^2\frac{dt}{t} < \infty,$$

for $x \in F$, i.e., points of F are tangents of Γ, thus inner tangents of Ω by Theorem 2.

(3) $\Rightarrow$ (1): By taking a union of small cones where (3) holds we can build a Lipschitz region $\mathcal{D}$ such that $|\partial\mathcal{D} \cap \mathbf{T}| > 0$ and

$$\iint_{\mathcal{D}} |S(\Phi)(z)|^2(1-|z|)^3 dxdy < \infty.$$

The proof of Corollary 3 applies to Lipschitz domains and shows $\log \Phi' \in L^2(\partial\mathcal{D})$, which implies Φ' has nontangential limits a.e.

(4) $\Rightarrow$ (1): By taking a union of cones where the integral in (4) is finite we can build a Lipschitz region $\mathcal{D}$ such that

$$\iint_{\mathcal{D}} |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3 dxdy < \infty.$$

The proof of Theorem 3 applied to $\mathcal{D}$ shows $\Phi' \in L^p$ for some $p > 0$ which is enough to imply Φ' has non-tangential limits a.e. on $\mathcal{D}$. This completes the proof of the theorem.

We should point out that one implication of the theorem holds pointwise, (2) $\Rightarrow$ (3). If Q is a Whitney square in $\mathbf{D}$ then Lemma 3.3 implies

$$\iint_Q |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3dxdy < C\Big(\int_r^1 \eta(x,t)\Big(\frac{t}{r}\Big)^{-\mu}\frac{dt}{t}\Big)^2 r,$$

where $r = \mathrm{dist}(\Phi(Q), \partial\Omega)$. Thus by breaking a Stolz cone into a union of Whitney squares,

$$\int_{W(\theta)} |S(\Phi)(z)|^2(1-|z|)^2dxdy \le \sum_j C\Big(\int_{r_j}^1 \eta(x,t)\Big(\frac{t}{r_j}\Big)^{-\mu}\frac{dt}{t}\Big)^2 r_j,$$

where $r_j = \mathrm{dist}(\Phi(Q_j), \partial\Omega)$. Condition (2), normal families and induction imply (see e.g. [10, Section 3])

$$C^{-1}(1-|w|)^{10/9} \le \mathrm{dist}(\Phi(w), \partial\Omega) \le C(1-|w|)^{9/10},$$

for $w \in W(\theta)$ so the r_j tend to zero geometrically. Now apply Minkowski's inequality and we are done.

We should also point out that one can give simple proofs of (3), (4) $\Rightarrow$ (1) without appealing to Theorems 3 or 4. Given (3) one can take $\mathcal{D}$ not simply with finite integral but (passing to a smaller set if necessary) so that $|S(\Phi)(z)|(1-|z|)^2 \le \delta$ on all of $\mathcal{D}$. Applying Lemma 3.5 one then passes to a subdomain (which still hits $\mathbf{T}$ in positive length) where $|\varphi'(z)|(1-|z|) \le \epsilon$. One can then apply Lemma 3.6 and bootstrap to show $\Phi' \in L^1(\mathcal{D})$.

To show (4) $\Rightarrow$ (1) let $\nu > 0$ (to be chosen later) and choose a Carleson square Q such that

$$\iint_Q |\Phi'(z)||S(\Phi)(z)|^2(1-|z|)^3dxdy < \nu\ell(Q),$$

$$|Q \cap \partial\mathcal{D} \cap \mathbf{T}| \ge \ell(Q)/2.$$

This is possible since these conditions hold all small enough Q at almost every point of $\partial\mathcal{D}\cap\mathbf{T}$. Form the region $\mathcal{D} = Q\backslash \cup_{\mathcal{L}} Q' = Q\backslash \cup Q_j$ by removing the maximal "large" subsquares of Q. We claim $\mathcal{D}$ still hits $\mathbf{T}$ in positive length because

$$\begin{aligned}\sum_j \ell(Q_j) &\le C\delta^{-1}\sum_j \iint_{Q_j} |S(\Phi)(z)(1-|z|)^2dxdy \\ &\le C\delta^{-1}\nu\ell(Q) \\ &\le \ell(Q)/4,\end{aligned}$$

if ν is small enough. Using Lemma 3.5 we can pass to a subdomain $\mathcal{D}'$ which has no bad squares but still hits $\mathbf{T}$ in positive length, and then use Lemma 3.8 to deduce $\Phi(\mathcal{D}')$ is rectifiable. In particular, it has tangents a.e., which implies Ω has inner tangents a.e. on $\mathcal{D} \cap \mathbf{T}$, as required.

The reason for discussing this independent argument is that gives a simpler proof of [10, Theorem 1]: if Ω is simply connected and $E \subset \partial\Omega$ lies on some rectifiable curve then $\Lambda_1(E) = 0$ implies $\omega(E) = 0$. Suppose E is a subset of a rectifiable curve and has positive harmonic measure in some simply connected domain. As in [10, Section 4] we let $\Phi : \mathbf{D} \to \mathbf{C}\backslash E$ be the uniformizing map and let $\mathcal{D} \subset \mathbf{D}$ be a Lipschitz domain on which Φ is 1-1 and such that $|\partial\mathcal{D} \cap \mathbf{T}| > 0$ (see [10, Lemma 4.1]). The argument in [10, Section 4] shows

$$\iint_{\mathcal{D}} |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|)^3 dxdy < \infty,$$

so part (4) of Theorem 5 holds for a.e. point in $\partial\mathcal{D} \cap \mathbf{T}$. Thus $\Phi(\mathcal{D})$ has inner tangents a.e. and E must have positive length, as desired.

We noted following Corollary 1 that the finiteness of those integrals did not imply $\partial\Omega$ was "nice". One can see this by building an open set Ω which is a union of disks $\{D_j\}$ each overlapping slightly with the previous one. This can be done so Ω is simply connected and $\partial\Omega$ has positive area. If the overlaps are chosen small enough then the integrals in Corollary 2 can both be made finite.

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(Received April 28, 1991 and in revised form May 2, 1992)