Annals of Mathematics, 132 (1990), 511–547

Harmonic measure and arclength

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1. Introduction

The purpose of this paper is to prove the following results.

THEOREM 1. *Suppose that* Ω *is a simply connected plane domain and that* Γ *is a rectifiable curve in the plane. If* $E \subset \partial\Omega \cap \Gamma$ *has positive harmonic measure in* Ω *then it has positive length.*

THEOREM 2. *Suppose* Ω, Γ *and* E *are as above and* $z \in \Omega$ *satisfies* $\operatorname{dist}(z, E) \geq 1$. *Let* ω *denote harmonic measure on* Ω *with respect to the point* z *and let* ℓ *denote one-dimensional Hausdorff measure. Then for every* $A < \infty$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\ell(\Gamma) \leq A$ *and* $\ell(E) \leq \delta$ *imply* $\omega(E) \leq \varepsilon$.

THEOREM 3. *Suppose* Γ *is connected. There is a constant* $C_\Gamma < \infty$ *such that*

$$\ell(\Phi^{-1}(\Gamma \cap \Omega)) \leq C_\Gamma$$

for every simply connected domain Ω *and Riemann mapping* $\Phi\colon \mathbf{D} \to \Omega$ *if and only if* Γ *is Ahlfors regular; i.e., there is an* $M > 0$ *such that* $\ell(\Gamma \cap D(x, r)) \leq Mr$ *for every disk* $D(x, r)$.

Theorem 2 is just a quantitative version of Theorem 1, and Theorem 3 turns out to be an easy consequence of Theorem 2. Before describing the proofs of these results, we discuss some of their history. In 1916, F. and M. Riesz proved their famous theorem that a univalent mapping onto a rectifiable domain preserves sets of zero length. More precisely, suppose $\mathbf{D}$ is the unit disk, Ω is a simply connected domain bounded by a rectifiable curve and Φ is a Riemann mapping from $\mathbf{D}$ to Ω. Then if $E \subset \partial\Omega$, $\ell(E) = 0$ if and only if $\omega(E) = 0$. One direction corresponds to the fact that $\Phi' \in H^1$ (the Hardy space) and the other to the observation that since $\log|\Phi'|$ is subharmonic, Φ' cannot vanish on a set of

*Supported in part by an NSF postdoctoral fellowship.
**Supported in part by NSF grant DMS-860 2550.

positive length. In 1936, Lavrentiev observed that by combining the F. and M. Riesz theorem with Jensen's inequality one obtains (with the normalization $\operatorname{dist}(\Phi(0), E) \geq 1$)

$$\omega(E) \leq \frac{C \log \ell(\partial\Omega)}{|\log \ell(E)| + 1}. \tag{1.1}$$

Here C is an absolute constant. Later examples (e.g., [26], [33]) show that this estimate is sharp for rectifiable domains. However, if $\partial\Omega$ has infinite length then the Riesz theorem need not hold; Lavrentiev constructed a domain Ω and a set $E \subset \partial\Omega$ such that E has zero length, but $\omega(E) > 0$. This example was later simplified and strengthened by various authors including McMillan and Piranian [33], Carleson [9] and Kaufman and Wu [26].

In 1985, the relation between harmonic and Hausdorff measures was greatly clarified by a stunning result of Makarov. He showed that if Ω is any simply connected domain and if Λ_h is the Hausdorff measure corresponding to the function

$$h(t) = t \exp\left\{ C_0 \sqrt{\log\frac{1}{t} \log\log\log\frac{1}{t}} \right\},$$

then any subset $E \subset \partial\Omega$ with $\Lambda_h(E) = 0$ must have harmonic measure zero. Conversely, Makarov showed that there is always a set of full measure which has Hausdorff dimension exactly 1. Pommerenke [39] improved this to sigma-finite ℓ measure. In fact, harmonic measure is supported on a set of Hausdorff dimension 1 even for arbitrary plane domains (Jones and Wolff [24]). This has been improved to sigma-finite ℓ measure by

Makarov's theorem also tells us exactly which domains can give positive harmonic measure to a set of zero length. A result of McMillan says we can write $\partial\Omega = A \cup T \cup N$ where the A are the inner tangents of $\partial\Omega$, T is the set of twist points and N has harmonic measure 0 (see [32] or [36] for definitions). McMillan's and Makarov's results imply that harmonic measure is mutually absolutely continuous with ℓ on A, but that when restricted to T it gives full measure to a set of zero length. Thus a Lavrentiev-type example occurs whenever the twist points have positive harmonic measure.

The support of harmonic measure also satisfies other geometric constraints. For example, if L is a straight line in the plane and $E \subset \partial\Omega \cap L$, Øksendal [34] proved that $\ell(E) = 0$ implies $\omega(E) = 0$, and Kaufman and Wu showed this remains true if L is replaced by a chord-arc curve [26]. Theorem 1 generalizes this to rectifiable curves, verifying a conjecture of Øksendal. Thus harmonic measure can be concentrated on a set of length zero, but any such set must be so "dispersed" in the plane that it is impossible to draw a rectifiable curve through it. We should also point out that if Ω is a quasidisk, one can prove this directly using the associated quasiconformal reflection to find a new rectifiable curve Γ' with $\Gamma' \cap \Omega = \varnothing$ and $\Gamma' \cap \partial\Omega = \Gamma \cap \partial\Omega$ (see Ahlfors [1] or [2]). The result then follows from the maximum principle and the F. and M. Riesz theorem.

The problem can also be interpreted in terms of Fuchsian groups and covering maps, and this interpretation is fundamental to our approach. Given a closed subset E of the Riemann sphere, let $\Phi\colon \mathbf{D} \to \mathbf{C} \setminus E$ denote the universal covering of its complement and consider $\mathbf{C} \setminus E \approx \mathbf{D}/\mathscr{G}$, where $\mathscr{G}$ is a Fuchsian group acting on $\mathbf{D}$. Let

$$\mathscr{F} = \{z \in \mathbf{D}\colon |g'(z)| < 1 \text{ for all } g \in \mathscr{G},\ g \neq \mathrm{id}\}.$$

Then $\mathscr{F}$ is the normal fundamental domain for $\mathscr{G}$. It is a simply connected, hyperbolically convex subdomain of $\mathbf{D}$ with the following "extremal" property. If $\mathscr{D} \subset \mathbf{D}$ is a simply connected subdomain which contains $\{0\}$, but contains no two $\mathscr{G}$ equivalent points, then $\omega(0, \partial\mathscr{D} \cap \mathbf{T}, \mathscr{D}) > 0$ implies $\ell(\partial\mathscr{F} \cap \mathbf{T}) > 0$. So to prove the theorem one may assume $\Omega = \Phi(\mathscr{F})$ ([37], [38]). The group $\mathscr{G}$ is called accessible if $\partial\mathscr{F} \cap \partial\mathbf{D}$ has positive length. Thus an equivalent formulation of Theorem 1 is that if E is a subset of a rectifiable curve, then the associated group $\mathscr{G}$ is accessible if and only if E has positive length.

For example, consider the case $E \subset \mathbf{R}$ (Øksendal's theorem). Since ω is a Borel measure we may assume E is closed, and let $\mathbf{R} \setminus E = \bigcup_{j=0}^{\infty} I_j$ be the decomposition of the complement of E into open intervals. Suppose $\Phi(0) \in I_0$. By symmetry, $\partial(\Phi(\mathscr{F})) = \mathbf{R} \setminus I_0$. But on the domain bounded by $\mathbf{R} \setminus I_0$ it is a simple exercise to compute the Riemann mapping and thus deduce that harmonic measure and arclength on $\mathbf{R} \setminus I_0$ are mutually absolutely continuous (this also follows from the F. and M. Riesz theorem). Thus we see Theorem 1 holds in this special case.

The main difficulty in proving Theorem 1 has been to utilize properly the hypothesis that E lies on a rectifiable curve. The main idea of this paper is to use a result of the second author [22] which characterizes subsets of rectifiable curves in terms of geometrically defined "square functions." This result is combined with L^2 estimates and a stopping time construction to show that there exists a certain Lipschitz domain $\mathscr{L} \subset \mathscr{F} \subset \mathbf{D}$ such that $\omega(\Phi(0), E, \Phi(\mathscr{L})) > 0$ and

$$\ell(\Phi(\partial\mathscr{L})) < \infty. \tag{1.2}$$

The F. and M. Riesz theorem now implies Theorem 1. The inequality in (1.2) comes with bounds depending only on the length of the shortest curve containing E, so that a slightly more involved argument allows us to prove Theorem 2, an analogue of Lavrentiev's inequality (1.1). The L^2 techniques used to bound $\ell(\Phi(\partial\mathscr{L}))$ are closely related to those developed during the past decade to study Cauchy integrals on Lipschitz curves. In particular, we need to know L^2 bounds for functions in terms of estimates on area integrals (see e.g. Jerison-Kenig [20] and Kenig [27]), as well as estimates on how far E lies from straight lines on different scales (see [21], [22]).

Theorem 3 was proved in the case where Γ is a straight line by Hayman and Wu [19] and a little later by Garnett, Gehring and Jones [18]. A simple proof has recently been given by Fernández, Heinonen and Martio [14]. Fernandez and Hamilton [13] extended it to chord-arc curves and Fernandez and Zinsmeister [15] proved it for regular curves Γ when $\Gamma \subset \Omega$. It is related to Theorem 1 because a result of [18] reduces the problem to estimates of harmonic measure on a domain Ω' constructed from Ω and Γ. In particular, we must show that certain small subsets of Γ have small harmonic measure in Ω' and this is exactly what Theorem 2 allows us to do.

One can generalize Theorem 1 to finitely connected domains and to some infinitely connected domains (e.g., fully accessible domains; see [37], [38]), but these cases are not fundamentally different from the simply connected case. A more ambitious extension would be to replace the simple connectedness of Ω by some "thickness" condition on $\partial\Omega$, as in [34, Theorem 2]. Unfortunately, one can construct a regular curve Γ and a thick subset E of Γ, such that harmonic measure on $\overline{\mathbf{C}} \setminus E$ gives positive mass to a set of zero length. (Thick means $C_1 r \leq \ell(E \cap D(x, r)) \leq \ell(\Gamma \cap D(x, r)) \leq C_2 r$.) Even a generalization of the theorem to simply connected Riemann surfaces covering planar domains is ruled out by a counterexample given by the first author in [6]. Thus Theorem 1 is something very special about simply connected domains in the plane.

The rest of this paper is organized as follows. In Section 2, we record several needed definitions and well-known lemmas. We will also introduce the geometric square functions mentioned above. In Section 3, we use extremal length to relate the behavior of $S(\Phi)$, the Schwarzian derivative of Φ, to the square function estimates on E and deduce that $|S(\Phi)(z)|(1 - |z|)^2$ is generically small on $\mathscr{L}$. (A less precise version of this estimate, established by normal families, was recorded in the second author's paper [21]. However, that result is far too weak for our present purposes.)

Section 4 uses the estimates of Section 3 and a version of a result of Kenig [27] to prove

$$\ell(\Phi(\partial\mathscr{L})) \leq C\ell(\Gamma) + C \sum_{Q_j \in \mathscr{B}} \ell(Q_j),$$

where Γ is a suitable rectifiable curve passing through E and $\mathscr{B}$ is a collection of "bad" squares in $\mathbf{C} \setminus E$ where the Schwarzian on $\Phi^{-1}(Q_j)$ is very small compared to $|\varphi'|$, where $\varphi(z) \equiv \log \Phi'(z)$. These bad squares are the ones that often show up in the study of Fuchsian groups—they correspond in a certain sense to elements of a Fuchsian group which lie "far" from translations. In Section 5, we bound the sum of the lengths of the bad squares by using Lavrentiev's theorem to associate to each such square certain disjoint subsets of Γ. In Section 6, we refine our estimates and obtain Theorem 2. The changes required are purely technical in nature. In Section 7, we use Theorem 2 to obtain a short proof of Theorem 3. In Section 8, we present the non-simply connected counterexample mentioned above.

2. Some facts and definitions

Before starting the proof of Theorem 1 we will record some well-known definitions and results needed later.

Given a set $E \subset \mathbf{C}$ and an increasing function h with $h(0) = 0$ we let

$$\Lambda_h^\delta(E) = \inf\left\{ \sum h(r_j) : E \subset \bigcup D(x_j, r_j), r_j \le \delta \right\},$$

$$\Lambda_h(E) = \lim_{\delta \to 0} \Lambda_h^\delta(E)$$

denote the Hausdorff measure with respect to h. When $h(t) = t$ we write this as Λ_1 or ℓ (for "length"). Note that Γ is a rectifiable curve if and only if it is connected and satisfies $\ell(\Gamma) < \infty$.

We let $\mathbf{D}$ denote the unit disk $\{|z| < 1\}$ and $\mathbf{T}$ its boundary. $\mathbf{H}$ denotes the upper half-plane. The hyperbolic metric on $\mathbf{D}$ is denoted by ρ. For a domain Ω, a point $z \in \Omega$ and a subset $E \subset \partial\Omega$, we let $\omega(z, E, \Omega)$ denote the harmonic measure of E with respect to the point z. Often the domain and point are understood from context and we merely write $\omega(E)$. (For Lemmas 2.1 to 2.6 see Ahlfors [3] or Pommerenke [36].)

LEMMA 2.1. *Suppose Ω is simply connected and $z \in \Omega$. Let $d = \operatorname{dist}(z, \partial\Omega)$. Then for each $\eta > 0$ there is a $\lambda > 1$ (not depending on z or Ω) such that*

$$\omega(z, \partial\Omega \cap D(z, \lambda d), \Omega) \ge 1 - \eta.$$

In Lemma 2.2 we can take η and λ to satisfy

$$1 - \eta = (2/\pi)\arcsin((\lambda - 1)/(\lambda + 1)).$$

LEMMA 2.2 (Koebe 1/4 Theorem). *If $F\colon \Omega \to \Omega'$ is a univalent mapping of simply connected domains then*

$$|F'(z)| \sim \frac{\operatorname{dist}(F(z), \partial\Omega')}{\operatorname{dist}(z, \partial\Omega)}.$$

LEMMA 2.3. *If F is univalent on* $\mathbf{D}$ *then*

$$|F''(z)| \leq \frac{C|F'(z)|}{1-|z|}$$

for some universal $C > 0$.

LEMMA 2.4. *Suppose* $\mathscr{L} \subset D$ *is star-shaped with respect to the origin and that* Φ *is holomorphic and univalent on* $\{z\colon \rho(z, \mathscr{L}) < \varepsilon\}$ *and that* $\Phi'(0) = 1$. *Then, given* $\delta > 0$, *there exist a number* $K = K(\varepsilon, \delta)$ *and a set* $E \subset \mathbf{T}$ *such that* $|E| < \delta$ *and for* $w \in (\mathbf{T} \cap \partial\mathscr{L}) \setminus E$,

$$\int_0^1 |\Phi'(rw)|\, dr \leq K.$$

If F is a locally univalent function the Schwarzian derivative of F is defined by

$$\begin{aligned} S(F)(z) &= \left[\frac{F''(z)}{F'(z)}\right]' - \frac{1}{2}\left[\frac{F''(z)}{F'(z)}\right]^2 \\ &= \left[\frac{F'''(z)}{F'(z)}\right] - \frac{3}{2}\left[\frac{F''(z)}{F'(z)}\right]^2. \end{aligned}$$

Recall that $S(F) \equiv 0$ if and only if F is a Möbius transformation and that S satisfies the composition law

$$S(F \circ G) = S(F)(G')^2 + S(G).$$

In particular, if G is Möbius then

$$\begin{aligned} S(F \circ G) &= S(F)(G')^2, \\ S(G \circ F) &= S(F). \end{aligned}$$

In addition, given an $\varepsilon > 0$ and a hyperbolic disk D, there is a $\delta > 0$ so that $|S(F)| \leq \delta$ on D implies F uniformly approximates a Möbius tranformation on D to within ε. The final fact we shall need about the Schwarzian is the following result.

LEMMA 2.5. *Suppose F is a univalent mapping defined on* $\mathbf{D}$. *Then*

$$|S(F)(z)| \leq \frac{6}{(1-|z|^2)^2}.$$

Suppose $\mathscr{C}$ is a family of curves in the plane. We say a positive, measurable function σ is admissible for $\mathscr{C}$ (written $\sigma \in \mathscr{A}(\mathscr{C})$) if for every locally rectifiable curve $\gamma \in \mathscr{C}$

$$\int_\gamma \sigma\, ds \geq 1.$$

(ds is arclength on γ.) We define the modulus of $\mathscr{C}$ as

$$M(\mathscr{C}) = \inf\left\{\int\int \sigma^2\, dx\, dy : \sigma \in \mathscr{A}(\mathscr{C})\right\}$$

and the extremal length of $\mathscr{C}$ as

$$\lambda(\mathscr{C}) = M(\mathscr{C})^{-1}.$$

Note that extremal length is a conformal invariant.

LEMMA 2.6 (Beurling's Theorem). *Let I be a subarc of $\mathbf{T} = \{|z| = 1\}$, K a closed disk centered at the origin and $\mathscr{C}$ the family of curves in the unit disk with one endpoint in K and the other on I. Then*

$$\omega(0, I, \mathbf{D}) \sim \exp\{-\pi\lambda(\mathscr{C})\}.$$

The constants for " $\sim$ " depend only on the diameter of K.

One consequence of this we shall use is that $\omega(z, D(x,r) \cap \partial\Omega, \Omega) \leq C\sqrt{r}$ for any simply connected domain Ω with a constant C that depends only on $\operatorname{dist}(z, \partial\Omega)$. The next lemma makes precise a statement from the introduction and is found in Pommerenke's paper [37].

LEMMA 2.7. *Let $\mathscr{F}$ be the normal fundamental domain for $\mathbf{C} \setminus E$ defined in Section 1. Suppose $\Omega \subset \mathbf{D}$ is a simply connected domain containing no two $\mathscr{G}$ equivalent points. Then*

$$|\partial\mathscr{F} \cap \mathbf{T}| \geq \omega(0, \partial\Omega \cap \mathbf{T}, \Omega).$$

Combined with the fact that $\partial\mathscr{F}$ is rectifiable with length at most π^2 (since it is hyperbolically convex [16]) this implies by Lavrentiev's theorem that $\omega(0, \partial\mathscr{F} \cap \mathbf{T}, \mathscr{F})$ is bounded away from zero by a constant depending only on $\omega(0, \partial\Omega \cap \mathbf{T}, \Omega)$.

A Jordan domain $\mathscr{L}$ is called chord-arc with constant M if the shorter arc of $\partial\mathscr{L}$ connecting two points $z, w \in \partial\mathscr{L}$ has length bounded by $M|z - w|$. Given a domain $\mathscr{L}$ we let $d(z) = \operatorname{dist}(z, \partial\mathscr{L})$.

LEMMA 2.8 ([20], [27]). *Suppose $\mathscr{L} \subset \mathbf{D}$ is chord-arc with constant M and suppose F is holomorphic on $\mathbf{D}$ and satisfies*

$$\int\int_{\mathscr{L}} |F'(z)|^2\, d(z)\, dx\, dy < \infty.$$

Then $F \in H^2(\mathscr{L})$ *and*

$$\int_{\partial\mathscr{L}} |F(z)|^2 \, ds(z) \sim |F(0)|^2 + \iint_{\mathscr{L}} |F'(z)|^2 \, d(z) \, dx \, dy$$

with constants depending only on M. Similarly,

$$\int_{\partial\mathscr{L}} |F(z)|^2 \, ds(z) \sim |F(0)|^2 + |F'(0)|^2 + \iint_{\mathscr{L}} |F''(z)|^2 \, d(z)^3 \, dx \, dy.$$

See [10] for a simple proof when $\mathscr{L}$ is a Lipschitz domain.

We also need some facts from [22]. A dyadic square Q is a square of the form

$$Q = [k2^n, (k+1)2^n] \times [j2^n, (j+1)2^n]$$

with $j, k, n \in \mathbf{Z}$. Let $\ell(Q)$ denote the side length of Q and for $\alpha > 0$, let αQ denote the concentric square with side length of $\alpha\ell(Q)$. Given a dyadic square Q there is a unique dyadic square Q' containing Q such that $\ell(Q') = 2^n\ell(Q)$. We denote this square by Q^n. Now suppose E is a compact subset of the plane. To each dyadic square Q we will associate two numbers, $\beta(Q) = \beta_E(Q)$ and $\gamma(Q) = \gamma_E(Q)$. $\beta(Q)$ is defined as the smallest $\beta \geq 0$ such that there exists a line L in $\mathbf{C}$ so that

$$E \cap 10Q \subset S = \left\{ z : \operatorname{dist}(z, L) \leq \frac{\beta}{2}\ell(Q) \right\}.$$

We also let S_Q denote a fixed choice of an infinite strip of width $\beta\ell(Q)$ such that $E \cap 10Q \subset S_Q$.

To define $\gamma(Q)$, let $(E \cap 10Q)^*$ denote the orthogonal projection of $E \cap 10Q$ onto L_Q (the axis of S_Q). Let M be the maximum of the lengths of the intervals in $(L_Q \cap 10Q) \setminus (E \cap 10Q)^*$ and set $\gamma(Q) = M/\ell(Q)$. Thus $\gamma(Q)$ represents the size of the largest "gap" of S_Q not containing a point of E.

The second author proved in [22] that the set E is a subset of a rectifiable curve Γ if and only if

$$\beta^2(E) \equiv \sum_{\text{dyadic } Q} \beta_E^2(Q)\ell(Q) < \infty$$

and that the length of the shortest such curve is comparable to

$$\operatorname{diam}(E) + \beta_E + \beta_E^2.$$

More precisely, we have the following:

LEMMA 2.9. (1) *If* $\Gamma \subset \mathbf{C}$ *is connected then*

$$\sum_Q \beta_\Gamma^2(Q)\ell(Q) \leq C\ell(\Gamma).$$

(2) *If* $E \subset \mathbf{D}$ *satisfies*

$$\beta^2(E) = \sum_Q \beta_E^2(Q)\ell(Q) < \infty,$$

then there exists Γ *connected with* $E \subset \Gamma$ *and*

$$\ell(\Gamma) \leq 4 + C_0\beta^2(E)).$$

Furthermore, Γ *can be taken to satisfy*:

(a) *If* $z_1, z_2 \in \Gamma$, *then* z_1 *and* z_2 *can be connected by a subarc of* Γ *of length less than* $C|z_1 - z_2|$.

(b) *If* $\beta_E(Q) \leq 1/1000$, $\ell(Q) \leq \operatorname{diam}(E)$ *and* $E \cap Q \neq \varnothing$, *then there exists an infinite strip* S_Q *with axis* L_Q *which crosses* $3Q$, *and the orthogonal projection* Γ_Q^* *of* $\Gamma \cap 3Q$ *onto* L_Q *is all of* $L_Q \cap 3Q$ (i.e., *no "gaps"*).

A curve Γ is called regular if there is an $M > 0$ such that

$$\ell(\Gamma \cap D(x,r)) \leq Mr$$

for every disk $D(x,r)$. Using Lemma 2.9 one can show that Γ is a regular curve if and only if there is a constant C such that for all Q,

$$\sum_{\tilde{Q} \subset Q} \beta_\Gamma^2(\tilde{Q})\ell(\tilde{Q}) \leq C\ell(Q).$$

3. An estimate on the Schwarzian derivative

Suppose $E \subset \mathbf{C}$ is compact and let $\Phi\colon \mathbf{D} \to \mathbf{C} \setminus E$ be the uniformizing map. Let $\mathscr{F}$ be the normal fundamental domain for Φ. Set $\varphi(z) = \log \Phi'(z)$ (which makes sense because Φ' is never zero). Fix $w \in \mathscr{F}$ and let $z_0 = \Phi(w)$. Let $Q = Q(z_0)$ be the smallest dyadic square containing z_0 such that $\ell(Q) \geq \operatorname{dist}(z_0, E)$. In this section we shall prove:

LEMMA 3.1. *Suppose* Φ *is univalent on* $D(w, \varepsilon(1 - |w|))$. *Then*

$$|S(\Phi)(w)|(1 - |w|)^2 \leq C\varepsilon^{-2} \sum_{n=0}^{\infty} \delta_n 2^{-n\mu} \tag{3.1}$$

where

$$\delta_n = \max(\beta_E(Q^n), \gamma_E(Q^n)).$$

The number μ *satisfies* $0 < \mu < 1$ *but can be taken as close to* 1 *as wished* (*part of statement of Lemma* 3.1). *The constant C depends only on the choice of* μ.

Actually the estimate is true for $0 < \mu < 2$, but this is harder to prove and we will not need the extra decay. In fact, we will only need the lemma for some $\mu > 1/2$. Also, we will only need the case $\varepsilon = 1/2$ for Theorem 1, but we will consider small ε's when we prove Theorem 2. The lemma is easily verified by hand in many cases. For example, if the right-hand side in (3.1) is 0 then E

must be a straight line. In this case Φ is simply a Möbius transformation and so the left-hand side must also be zero. Next suppose that $E = \mathbf{Z} \subset \mathbf{R} \subset \mathbf{C}$. The universal covering map Ψ from the upper half-plane $\mathbf{H}$ to $\mathbf{C} \setminus E$ can be constructed by letting Ψ map $\hat{\mathscr{F}} = \mathbf{H} \setminus \bigcup_{n \in \mathbf{Z}} \{|z - n + 1/2| \leq 1/2\}$ to $\mathbf{H}$ and extending Ψ to the rest of $\mathbf{H}$ by reflections. Then $(\hat{\mathscr{F}})$ is one half of a fundamental domain $\mathscr{F}$ which can be obtained by reflecting $\hat{\mathscr{F}}$ through any of the circles $\{|z - n + 1/2| \leq 1/2\}$. We also have $|\Psi(z) - z| \leq C$ on $\hat{\mathscr{F}}$. It is not hard to compute that $|S(\Psi)(x + iy)|y^2 \leq Cy^{-1}$ if $|y| \geq 1$. For $z = x + iy$ this corresponds to Lemma 3.1 in the case where $\beta(Q_j^n) = 0$ and $\gamma(Q_j^n) \sim y^{-1}2^{-n}$; i.e., $\sum_n \delta_n(Q_j) \sim y^{-1}$. The covering map for any Denjoy domain can be similarly constructed by a method due to Beurling (see Rubel and Ryff's paper [41]).

To prove Lemma 3.1 first note that by Lemma 2.5

$$|S(\Phi)(w)|(1 - |w|)^2 \leq C_1 \varepsilon^{-2}.$$

Now fix $0 < \eta \leq 1/100$ (to be chosen later depending on μ). We may assume that $\delta_0 < \eta$, for otherwise the inequality would be trivially satisfied by taking $C = C_1/\eta$. Since E is bounded, $\gamma(Q^n)$ is eventually larger than η and so we can define $N = N(Q) < \infty$ to be the smallest integer such that $\delta_N \geq \eta$.

Let S_0 be the strip across $5Q$ collinear with S_Q but of width $\delta_0 > \beta(Q)$. Let H_0 be the half-plane in $\mathbf{C} \setminus S_0$ which contains the point z_0 (note: $z_0 \notin S_0$ since $\delta \leq 1/100$) and let L_0 be the boundary of H_0. Set $V_0 = H_0$. In general, S_n is the strip collinear with S_{Q^n} but of width δ_n. H_n is the half-plane in $\mathbf{C} \setminus S_n$ which intersects $H_{n-1} \cap Q^{n-1}$ (both sides of S_n cannot hit $H_{n-1} \cap Q^{n-1}$ without violating $\gamma(Q^n) \leq \delta_n \leq 1/100$). Let $A_n = 5Q^n/5Q^{n-1}$ and set

$$V_n = (H_0 \cap 5Q) \cup (H_1 \cap A_1) \cup \cdots \cup (H_{n-1} \cap A_{n-1}) \cup (H_n \setminus 5Q^{n-1}).$$

Clearly $\{V_n\}$ is a sequence of planar domains in $\mathbf{C} \setminus E$. For example V_2 is the region above the solid line in Figure 1 (see next page).

Let G_n denote the Green's function on V_n with pole at z_0. When $n = 0$, V_0 is just a half-plane and so there is a Möbius transformation τ mapping $\mathbf{D}$ to V_0 and taking w to z_0. In terms of τ, G_0 is given by

$$G_0(\tau(z)) = -\log\left|\frac{z - w}{1 - \bar{w}z}\right|.$$

We define another function G on V_N by the formula

$$G(\Phi(z)) = -\log\left|\frac{z - w}{1 - \bar{w}z}\right|$$

which is well defined because Φ^{-1} can be defined on V_N. Then G is a positive

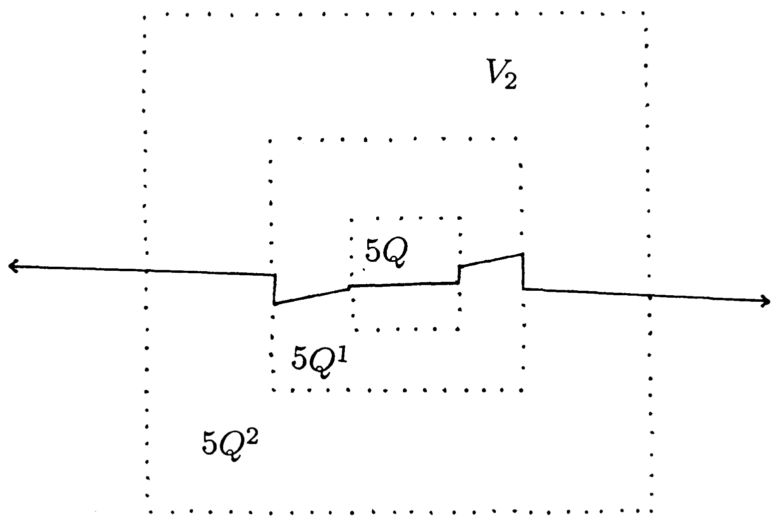

FIGURE 1. The domain V_n

harmonic function on $V_N \setminus \{z_0\}$ and has a logarithmic pole at z_0. (In fact, we may view G as the Green's function of the universal covering surface of $\mathbf{C} \setminus E$ restricted to V_N.) By rescaling we may assume that $0 \in E$ and $\operatorname{dist}(z_0, E) = 1$. Let $A = \{z \colon |z - z_0| = 1/2\}$. We will prove that

$$\sup_{z \in A} |G(z) - G_0(z)| \le C \sum_{n=0}^{\infty} \delta_n 2^{-n\mu}. \tag{3.2}$$

To see why (3.2) implies (3.1) let τ be as above and note that by Koebe's theorem (Lemma 2.2) $|\tau'(w)| \sim (1 - |w|)^{-1}$.

LEMMA 3.2. *Suppose* $z \in V_N$ *and* $z = \Phi(w)$. *Then*

$$|\Phi'(w)| \sim \operatorname{dist}(z, E)/(1 - |w|).$$

One direction follows because $D(z, \operatorname{dist}(z, E)) \subset \Omega$. To prove the other direction, note that by the definition of V_N, if $|z| \in A_n \cap V_N$ then $\operatorname{dist}(z, E) \ge \delta_n$. Since $\gamma_n \le \delta_n$, we can find points $z_1, z_2 \in E$ such that

$$|z - z_1| \sim |z - z_2| \sim |z_1 - z_2| \sim \operatorname{dist}(z, E).$$

The estimate now follows by renormalization and the fact that the functions on the disk which omit the values 0, 1 and ∞ form a normal family.

Thus $|\Phi'(w)| \sim (1 - |w|)^{-1}$. Now let $B = \tau^{-1}(A)$. By our formulas for G and G_0 and the fact that $|\tau'| \sim (1 - |w|)^{-1}$ on B, the inequality $|G(z) - G_0(z)| < \nu$ for $z \in A$ implies

$$\Phi(B) \subset \{1/2 - C\nu \le |z - z_0| \le 1/2 + C\nu\}.$$

The composition law for the Schwarzian now gives

$$|S(\Phi)(w)| \sim \frac{|S(\Psi)(z_0)|}{(1 - |w|)^2}$$

where Ψ is a conformal mapping of A to $\Phi(B)$ which fixes z_0. To see why such a map satisfies $|S(\Psi)(z_0)| \le C\nu$, we use the following simple lemma.

LEMMA 3.3. *Suppose Ω is a simply connected domain satisfying*

$$D(0, 1-\nu) \subset \Omega \subset D(0, 1+\nu).$$

Let Ψ: $\mathbf{D} \to \Omega$ be a conformal mapping such that $\Psi(0) = 0$ and $\Psi'(0) > 0$. Then for all z with $|z| = 1/2$,

$$|\Psi(z) - z| \le C\nu.$$

Thus $||\Psi'(0)| - 1| \le C\nu$, $|\psi''(0)| \le C\nu$ and $|\Psi'''(0)| \le C\nu$.

To prove this note that

$$1 - \nu \le \left|\frac{\Psi(z)}{z}\right| \le 1 + \nu$$

holds for all $z \in \mathbf{T}$ and hence for all $z \in \mathbf{D}$ (since $\Psi(z)/z$ has no zeros). Thus $F(z) = \log(\Psi(z)/z)$ satisfies

$$-\nu \le \operatorname{Re}(F) \le \nu,$$

$$\operatorname{Im}(F(0)) = 0,$$

$$|\nabla \operatorname{Im}(F(z))| = |\nabla \operatorname{Re}(F(z))| \le \frac{C\nu}{1-|z|}.$$

Hence for $|z| = 1/2$

$$|\operatorname{Im}(F(z))| \le |\operatorname{Im}(F(0))| + \left|\int_0^z \frac{C\nu}{1-|z|}\, d|z|\right| \le C\nu.$$

This implies

$$\left|\frac{\Psi(z)}{z} - 1\right| \le C\nu$$

as desired. The estimates on Ψ'' and Ψ''' now follow from the Cauchy estimates applied to $\Psi(z) - z$.

Thus Lemma 3.3, combined with the formula for $S(\Psi)$ in Section 2, allows us to deduce (3.1) from (3.2). To establish (3.2) we will prove the inequalities

$$|G_N(z) - G_0(z)| \le C \sum_{n=0}^{N} \delta_n 2^{-2n\mu}, \tag{3.3}$$

$$|G(z) - G_N(z)| \le C \sum_{n=1}^{N} \delta_n 2^{-n\mu}, \tag{3.4}$$

for $z \in A$. Clearly (3.3) and (3.4) imply (3.2). We will prove (3.3) first. We start with an estimate which shows how G_n decays far away from z_0.

LEMMA 3.4. *For fixed $\eta > 0$ there is a $0 < \mu < 1$ and $C > 0$ so that if $0 \le n \le N$ then for $z \in A$,*

$$\omega(z, A_n \cap \partial V_N, V_N) \le C 2^{-n\mu}.$$

Furthermore, $1 - C'\eta < \mu < 1$.

This is just the Ahlfors distortion theorem [3], but for completeness we will give a proof, using the method of extremal length. We define a metric σ on V_N by

$$\sigma(z) = \begin{cases} \dfrac{1}{|z| n \log 2}, & \text{for } 1 \le |z| \le 2^n \\ 0, & \text{otherwise} \end{cases}.$$

Let $\mathscr{C}$ be the family of curves connecting A to $A_n \cap \partial V_N$ in V_n. Then σ is an admissible metric for $\mathscr{C}$ since $\gamma \in \mathscr{C}$ implies

$$\int_\gamma \sigma\, ds \ge \frac{1}{n \log 2} \int_1^{2^n} \frac{dr}{r} = 1.$$

So to get a lower estimate on $\lambda(\mathscr{C})$ we need an upper bound on $\int\int \sigma^2$. We write

$$\int\int_{V_n} \sigma^2\, dx\, dy = \left(\frac{1}{n \log 2}\right)^2 \int_1^{2^n} \frac{\theta(r)}{r}\, dr$$

where $\theta(r)$ is the angle measure of $\{|z| = r\} \cap V_N$. For $2^k \le r \le 2^{k+1}$, we have $D(0, r) \subset 5Q^k$ and so the definition of $\beta(Q)$ implies

$$|\theta(r) - \pi| \le C\beta(Q^k) \le C\delta_k \le C\eta.$$

Consequently,

$$M(\mathscr{C}) \le \int\int_{V_N} \sigma^2\, dx\, dy \le \frac{(\pi + C\eta)}{n \log 2}.$$

Thus by Beurling's theorem, for $z \in A$,

$$\begin{aligned} \omega(z, A_n \cap \partial V_N, V_N) &\le C \exp\{-\pi\lambda(\mathscr{C})\} \\ &\le C \exp\left\{\frac{-n \log 2}{1 + C\eta}\right\} \\ &\le C 2^{-n\mu}. \end{aligned} \tag{3.5}$$

This completes the proof of Lemma 3.4.

A similar argument shows that for $1 \leq n \leq N$,

$$\omega(z_0, \partial 5Q^n, V_N \cap 5Q^n) \leq C2^{-n\mu}$$

and for $k = 1, 2, \ldots,$

$$\omega(z_0, L_N \cap A_{N+k}, V_N) \leq C2^{-N\mu}2^{-k}$$

since V_N looks like a half-plane outside $5Q^N$. Let z_0^* denote the reflection of z_0 across L_0 and note that

$$G_0(z) = \log\left|\frac{z - z_0^*}{z - z_0}\right|$$

is actually defined and harmonic on $\mathbf{C} \setminus \{z_0, z_0^*\}$ and satisfies

$$|\nabla G_0(z)| \leq C2^{-n}$$

for $|z| \in A_n$. Also note that for $z \in \partial V_N \cap A_n$,

$$\operatorname{dist}(z, L_0) \leq C2^n \sum_{j=0}^{\min(n,N)} \beta_j$$

and hence $|G_0(z)| \leq C\sum_{j=0}^n \beta_j$. Thus for $|z - z_0| = 1/2$,

$$\begin{aligned} |G_N(z) - G_0(z)| &\leq \int_{\partial V_N} |G_N - G_0|\, d\omega_z \leq \sum_{n=0}^{\infty} \int_{\partial V_N \cap A_n} |G_0|\, d\omega_z \\ &\leq C \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \beta_j\right) 2^{-n\mu} \leq C \sum_{j=0}^{\infty} \beta_j \sum_{n=j}^{\infty} 2^{-n\mu} \\ &\leq C \sum_{j=0}^{\infty} \beta_j 2^{-j\mu}. \end{aligned}$$

We now turn to the proof of (3.4). We start by observing that if $z, w \in V_N$ with $|z - w| = \operatorname{dist}(z, E)/2$ then $\rho(z, w) \sim 1$ (ρ is the hyperbolic metric induced by Φ). Clearly $\rho(z, w) \leq 1$ since $D(z, 2|z - w|) \cap E = \emptyset$. (Recall that $z, w \in \Omega \subset \Omega'$ implies $\rho_\Omega(z, w) \geq \rho_{\Omega'}(z, w)$.) The other direction follows from Lemma 3.2.

Now let Γ be a curve in V_N which satisfies

$$\operatorname{dist}(z, S(Q^n)) \sim (\delta_{n-1} + \delta_n + \delta_{n+1})$$

for $z \in \Gamma \cap A_n$. (We will assume $\delta_n > 0$. The case $\delta_n = 0$ is easier and left to the reader.) Also assume that Γ is uniformly smooth on scale 2^n; i.e., if $\Gamma_n = \Gamma \cap A_n$ and if f is an arclength parameterization of Γ_n then

$$|\nabla_2 f| \leq C2^{-n}$$

(∇_2 means all second derivatives of f). Let $\mathscr{D}$ be the subdomain of V_N bounded by an arc Γ_b of Γ and an arc Γ_μ of $\partial 5Q^N$ and containing z_0. Divide

$\partial\mathscr{D}$ into intervals $\{I_k\}$ such that the hyperbolic diameter of each I_k is about 1. By Lemma 3.4 if $I_k \subset \Gamma_b \cap A_n$ then

$$\ell(I_k) \leq C(\delta_{n-1} + \delta_n + \delta_{n+1})2^n.$$

Suppose $z \in \mathscr{D} \cap A_n$ and $\operatorname{dist}(z, E) \sim 2^n$. Because of the smoothness of Γ on A_n, harmonic measure there looks like (normalized) arclength; e.g., if $I_k \subset \Gamma \cap A_n$ then

$$\omega(z, I_k, \mathscr{D}) \leq C\ell(I_k)2^{-n}.$$

But by the remark following the proof of Lemma 3.4,

$$\omega\big(z_0, \partial 5Q^{n-1}, \mathscr{D} \cap 5Q^{n-1}\big) \leq \omega\big(z_0, \partial 5Q^{n-1}, V_N \cap 5Q^{n-1}\big) \leq C2^{-n\mu}$$

and thus

$$\omega(z_0, I_k, \mathscr{D}) \leq C\ell(I_k)2^{-n}2^{-n\mu}.$$

Therefore,

$$\omega(z_0, I_k, \mathscr{D}) \leq C(\delta_{n-1} + \delta_n + \delta_{n+1})2^{-n\mu}.$$

On the other hand, if $I_k \subset \Gamma_u$ then

$$\omega(z_0, I_k, \mathscr{D}) \leq \omega(z_0, \Gamma_u, \mathscr{D}) \leq C2^{-N\mu}.$$

Since N was chosen so that $\delta_N \geq \eta$,

$$\omega(z_0, I_k, \mathscr{D}) \leq C\delta_N 2^{-N\mu}.$$

Therefore, for any I_k,

$$\omega(z_0, I_k, \mathscr{D}) \leq C \sum_{n=0}^{N} \delta_n 2^{-n\mu}.$$

LEMMA 3.5.

$$\sup_{\partial\mathscr{D}} G(z) \leq C \sum_{n=1}^{N} \delta_n 2^{-n\mu} \equiv C\delta(Q).$$

Let $\tilde{\mathscr{D}} = \Phi^{-1}(\mathscr{D})$, and $J_k = \Phi^{-1}(I_k)$. Observe that

$$\sup_{\partial\mathscr{D}} G(z) = \sup_k \sup_{w \in J_k} \log\frac{1}{|w|} \sim \sup_k \sup_{w \in J_k} (1 - |w|).$$

Let k_0 be an index where the last supremum is attained and let w_0 be the point in J_{k_0} where the supremum is attained. Also let k_{-1}, k_1 be the indices of the adjoining intervals. Set $J = J_{k_{-1}} \cup J_{k_0} \cup J_{k_1}$. Then if $w \in \partial\tilde{\mathscr{D}} \setminus J$, $\rho(w, w_0) \geq C_0$ and $|w| \geq |w_0|$. Set

$$\hat{\mathscr{D}} = \tilde{\mathscr{D}} \cap \big(\{|w| \leq |w_0|\} \cup \{\rho(w, w_0) \leq C_0\}\big).$$

(See Figure 2.)

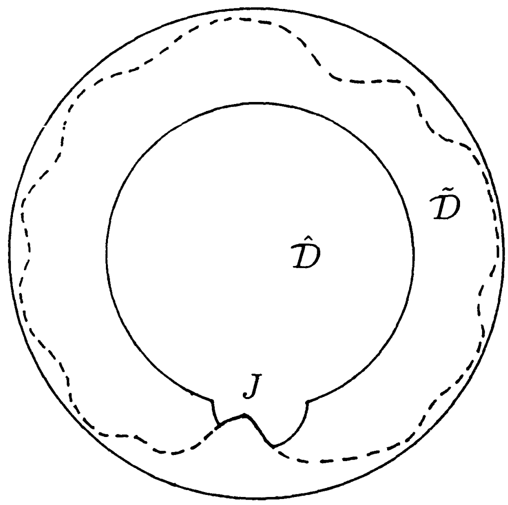

FIGURE 2. $\tilde{\mathscr{D}}$ and $\hat{\mathscr{D}}$

Then clearly

$$\begin{aligned}\omega(0, J, \tilde{\mathscr{D}}) &\geq \omega(0, J, \hat{\mathscr{D}}) \\ &\geq C_1(1 - |w_0|) \geq C \sup_{\partial \mathscr{D}} G(z).\end{aligned}$$

But by the remarks preceding Lemma 3.5,

$$\begin{aligned}\omega(0, J, \tilde{\mathscr{D}}) &= \omega(z_0, \Phi(J), \mathscr{D}) \\ &\leq C\omega(z_0, I_{k_{-1}} \cup I_{k_0} \cup I_{k_1}, \mathscr{D}) \leq C\delta(Q).\end{aligned}$$

Inequality (3.4) now follows easily from Lemma 3.5 because by the maximum principle and the fact $\mathscr{D} \subset V_N \subset \mathbf{C} \setminus E$,

$$\begin{aligned}|G_N(z) - G(z)| &\leq |G_{\mathscr{D}}(z) - G(z)| \\ &\leq \sup_{\partial \mathscr{D}} |G(z)| \leq C\delta(Q).\end{aligned}$$

($G_{\mathscr{D}}$ is, of course, the Green's function for $\mathscr{D}$ with pole at z_0.) This completes the proof of Lemma 3.1.

4. A reduction to the study of bad squares

Suppose Ω is simply connected and $E \subset \partial\Omega$ is closed. As in the introduction let Φ be the universal covering map from $\mathbf{D}$ to $\mathbf{C} \setminus E$ and let $\mathscr{F} \subset \mathbf{D}$ be the normal fundamental domain. Assume $\omega(z, E, \Omega) > 0$. By Lemma 2.9, this implies that $\omega(z, E, \Phi(\mathscr{F})) > 0$. Thus without loss of generality we may assume $\Omega = \Phi(\mathscr{F})$. Since $\omega(z, E, \Omega) > 0$, given any $\eta > 0$ there is a $z_0 \in \Omega$ such that $\omega(z_0, E, \Omega) > 1 - \eta$. We may assume that $\Phi(0) = z_0$. By hypothesis E lies on some rectifiable curve. Let Γ be the special rectifiable curve associated to E by part 2 of Lemma 2.9, satisfying properties (2a) and (2b).

The fundamental domain $\mathscr{F}$ can be written as $\mathscr{F} = \mathbf{D} \setminus \bigcup B_j$ where the $\{B_j\}$ are disks orthogonal to $\mathbf{T}$. If η is small enough, then $B_j \cap \{|z| \le 1/2\} = \emptyset$, for otherwise Lemma 2.1 would imply $\omega(0, \partial\mathscr{F} \setminus \mathbf{T}, \mathscr{F}) \ge \omega(0, \partial B_j, \mathbf{D} \setminus B_j) > \eta$. Let F be the radial projection of $\partial\mathscr{F} \setminus \mathbf{T}$ onto $\mathbf{T}$. Then by the maximum principle,

$$\frac{1}{2\pi}\ell(F) \le \omega(0, \partial\mathscr{F} \setminus \mathbf{T}, \mathscr{F}).$$

LEMMA 4.1. *With notation as above, there exists a Lipschitz domain $\mathscr{L} \subset \mathscr{F}$ such that for all $z \in \mathscr{L}$,*

$$\omega(z, \partial\mathscr{F} \cap \mathbf{T}, \mathscr{F}) = \omega(\Phi(z), E, \Phi(\mathscr{F})) \ge 1 - C\sqrt{\eta}$$

and $\rho(z, \partial\mathscr{F}) \ge 1$.

One proves this by considering the open set where the Hardy-Littlewood maximal function of the characteristic function of $\partial\mathscr{F}$ is larger than $\sqrt{\eta}$. Over each interval I in this set we place a "tent" of the form

$$T_I = \{z \in \mathbf{D}\colon 1 - |z| \le \operatorname{dist}(z/|z|, \mathbf{T} \setminus I)\}.$$

It is easy to check that by taking $\mathscr{L}$ to be the complement of these tents in $\mathbf{D}$, we obtain a domain which satisfies the lemma. Also note, for future reference, that if $z \in \mathscr{L}$, then Φ is univalent on the disk $D(z, (1 - |z|)/2)$ and that the hyperbolic distance from z to $\partial\mathscr{F}$ is very large (depending on η).

By the lemma, E has positive harmonic measure in the domain $\Phi(\mathscr{L}) \subset \Omega$. If $\partial\Phi(\mathscr{L})$ were rectifiable then the F. and M. Riesz theorem would imply that E has positive length. Thus it suffices to prove that for any Lipschitz domain $\mathscr{L}$ satisfying the conclusions of Lemma 4.1,

$$\ell(\partial\Phi(\mathscr{L})) = \int_{\partial\mathscr{L}} |\Phi'(z)|\, ds(z) \le C\big(|\Phi'(0)| + \ell(\Gamma)\big) < \infty.$$

To see why it is necessary to replace $\mathscr{F}$ by the subdomain $\mathscr{L}$, consider the following example. Suppose E consists of N equally spaced intervals of length π/N on $\mathbf{T}$ and let Φ denote the universal covering map which takes 0 to ∞. By symmetry one can show that $\Phi(\partial\mathscr{F})$ consists of E together with the N radial segments which connect the origin to the midpoint of each interval in E. Thus the total length of $\partial\Phi(\mathscr{F})$ is greater than N. However if we consider the corresponding region $\Phi(\mathscr{L})$ (see Figure 3) we see its boundary has length about 1, independent of N because the boundary of this domain must stay near the unit circle.

Let $\{Q_j\}$ be the dyadic squares in the Whitney decomposition of $\mathbf{C} \setminus E$ (see Stein [42, Chapter VI]). Let $\varepsilon, \delta > 0$ (to be chosen later). Define $\mathscr{L}_{\varepsilon,\delta} \subset \mathscr{L}$ to

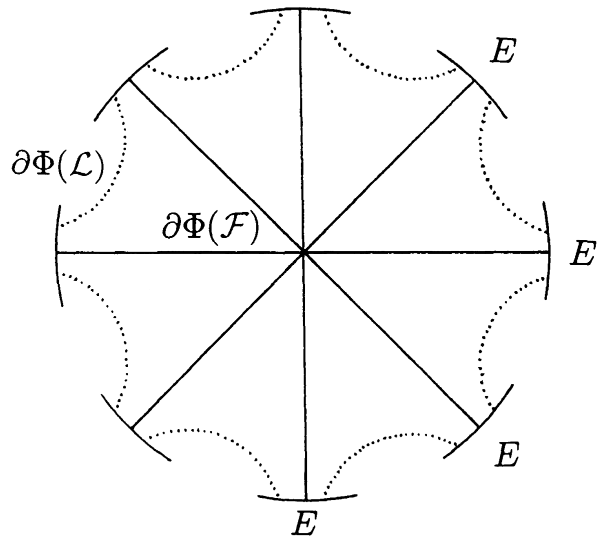

FIGURE 3. $\Phi(\mathscr{F})$ and $\Phi(\mathscr{L})$

be the set of $z \in \mathscr{L}$ such that

$$|\varphi'(z)|(1-|z|) \geq \varepsilon$$

and

$$|S(\Phi)(z)|(1-|z|)^2 \leq \delta.$$

We define a collection of "bad squares" $\mathscr{B}_{\varepsilon,\delta}$ by putting $Q_j \in \mathscr{B}_{\varepsilon,\delta}$ if $\Phi^{-1}(Q_j) \cap \mathscr{L}_{\varepsilon,\delta} \neq \varnothing$. Also let $d(z) = \operatorname{dist}(z, \partial\mathscr{L})$ for $z \in \mathscr{L}$.

LEMMA 4.2. *There are universal constants $\varepsilon_0 > 0$ and $C_0 > 0$ such that whenever $\varepsilon < \varepsilon_0$ and $\delta > 0$ then*

$$\begin{aligned} (4.1) \qquad \int_{\partial\mathscr{L}} |\Phi'(z)|\, ds \leq C_0|\Phi'(0)| &+ C_0(1+\delta^{-2}) \\ &\times \iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z)|^2\, d(z)^3\, dx\, dy \\ &+ C_0 \sum_{Q_j \in \mathscr{B}_{\varepsilon,\delta}} \ell(Q_j). \end{aligned}$$

The proof that we will give also shows that if $\Phi\colon \mathbf{D} \to \mathbf{C}$ is univalent and satisfies

$$|\varphi'(z)| \leq \frac{\varepsilon_0}{1-|z|}$$

(i.e., φ has small Bloch norm) then

$$\ell(\Phi(\mathbf{T})) \leq C_0|\Phi'(0)| + C_0\iint_{\mathbf{D}} |\Phi'(z)|\, |S(\Phi)(z)|^2 (1-|z|)^3\, dx\, dy.$$

This should be compared with Becker's theorem ([4]) which states that $\Phi(\mathbf{D})$ is a quasicircle. Also note that if φ has small Bloch norm on $\mathscr{L}$ then there are no bad squares; so we can simplify the proof. However, in this case $\Phi(\mathscr{L})$ is a quasidisk and we noted in the introduction that this case is easy. If we replace the term $|\Phi'(0)|$ in the inequality above by $\operatorname{diam}(\Phi(\mathbf{D}))$ then the inequality holds whenever $\Phi(\mathbf{D})$ is a quasidisk (the proof will be discussed in a later paper). That the last term in (4.1) is really necessary can be seen by mapping the disk to a half-plane. Zinsmeister [43] has obtained related results.

Now set $F = (\Phi')^{1/2}$ and $\varphi = \log(\Phi')$ and observe

$$F' = \frac{1}{2}(\Phi')^{1/2}(\varphi'),$$

$$F'' = \frac{1}{2}(\Phi')^{1/2}\left(\varphi'' + \frac{1}{2}(\varphi')^2\right)$$

$$= \frac{1}{2}(\Phi')^{1/2}\left(S(\Phi) + (\varphi')^2\right).$$

Normalize so that $|\Phi'(0)| = 1$. The first equality, together with Lemma 2.8, implies

$$\int_{\partial\mathscr{L}} |\Phi'(z)|\, ds \sim 1 + \int_{\mathscr{L}} |\Phi'(z)|\, |\varphi'(z)|^2\, d(z)\, dx\, dy \tag{4.2}$$

and the second gives

$$\begin{aligned}
\int_{\partial\mathscr{L}} |\Phi'(z)|\, ds &\le C_0 + C_0 \iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z) + (\varphi'(z))^2|^2\, d(z)^3\, dx\, dy \\
&\le C_0 + C_1 \iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z)|^2\, d(z)^3\, dx\, dy \\
&\quad + C_1 \iint_{\mathscr{L}_{\varepsilon,\delta}} |\Phi'(z)|\, |\varphi'(z)|^4\, d(z)^3\, dx\, dy \\
&\quad + C_1 \iint_{\mathscr{L}\setminus\mathscr{L}_{\varepsilon,\delta}} |\Phi'(z)|\, |\varphi'(z)|^4\, d(z)^3\, dx\, dy \\
&\le C_0 + C_1 \iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z)|^2\, d(z)^3\, dx\, dy \\
&\quad + C_1 \sum_{Q_j \in \mathscr{B}_{\varepsilon,\delta}} \ell(Q_j) \\
&\quad + C_1 \varepsilon^2 \int_{\mathscr{L}} |\Phi'(z)|\, |\varphi'(z)|^2\, d(z)\, dx\, dy \\
&\quad + C_1 \delta^{-2} \iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z)|^2\, d(z)^3\, dx\, dy.
\end{aligned}$$

In the last inequality the third term is correct because Lemma 2.3 and the fact that Φ is univalent on $D(z,(1-|z|)/2)$ for $z \in \mathscr{L}$ imply $|\varphi'(z)| \leq Cd(z)^{-1}$. So by Lemma 2.2 we have

$$\iint_{\Phi^{-1}(Q_j)} |\Phi'(z)|\, d(z)^{-1}\, dx\, dy \leq C\ell(Q_j).$$

The last two terms bound the integral over $\mathscr{L} \setminus \mathscr{L}_{\varepsilon,\delta}$ since for $z \in \mathscr{L} \setminus \mathscr{L}_{\varepsilon,\delta}$ either $|\varphi'(z)|\, d(z) \leq \varepsilon$ (in which case the first term bounds) or $\varepsilon \leq |\varphi'(z)|\, d(z) \leq C$ and $|S(\Phi)(z)|\, d(z)^2 \geq \delta$ (in which case the second term bounds).

Taking ε small enough and using (4.2) give

$$\int_{\partial\mathscr{L}} |\Phi'(z)|\, ds \leq C_2 + C_2(1+\delta^{-2}) \iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z)|^2\, d(z)^3\, dx\, dy + C_2 \sum_{Q_j \in \mathscr{B}_{\varepsilon,\delta}} \ell(Q_j) + \frac{1}{2}\int_{\partial\mathscr{L}} |\Phi'(z)|\, ds.$$

We can now subtract the term on the far right from the left-hand side to establish an *a priori* estimate; this can be combined with a limiting argument to complete the proof of Lemma 4.1.

Thus to prove that $\int|\Phi'| < \infty$ we must bound both the integral and the sum which appear on the right-hand side of (4.1). In the rest of this section we will prove the integral is finite. In the next section we will deal with the sum.

As above, let $\{Q_j\}$ be the squares of the Whitney decomposition of $\mathbf{C} \setminus E$. Given a square Q_j, let $N(Q_j)$ be the smallest integer $n \geq 0$ such that $\beta(Q_j^n) + \gamma(Q_j^n) \geq 1/1000$. Thus for $z \in \Phi^{-1}(Q_j) \cap \mathscr{L}$ (by Lemma 3.1),

$$|S(\Phi)(z)|(1-|z|)^2 \leq C \sum_{n=1}^{N(Q_j)} \left(\beta(Q_j^n) + \gamma(Q_j^n)\right) 2^{-n\mu} \equiv C\delta(Q_j).$$

We claim that to prove

$$\iint_{\mathscr{L}} |\Phi'(z)|\, |S(\Phi)(z)|^2\, d(z)^3\, dx\, dy < \infty$$

it is sufficient to establish

$$\sum_j \delta^2(Q_j)\ell(Q_j) < \infty. \tag{4.3}$$

To see this, recall that if $z \in \mathscr{L}$, then the hyperbolic distance from z to $\partial\mathscr{F}$ is ≥ 1. Thus Lemma 2.2 implies

$$|\Phi'(z)|(1-|z|) \sim \operatorname{dist}(\Phi(z), E) \sim \ell(Q_j).$$

Hence,

$$\begin{aligned}
&\iint_{\mathscr{L}} |\Phi'(z)|\,|S(\Phi)(z)|^2\, d(z)^3\, dx\, dy \\
&\quad\le \sum_j \iint_{\Phi^{-1}(Q_j)\cap\mathscr{L}} |\Phi'(z)|\,|S(\Phi)(z)|^2 (1-|z|)^3\, dx\, dy \\
&\quad\le C\sum_j \delta(Q_j)^2 \iint_{\Phi^{-1}(Q_j)\cap\mathscr{L}} \frac{|\Phi'(z)|}{(1-|z|)}\, dx\, dy \\
&\quad\le C\sum_j \delta^2(Q_j)\ell(Q_j).
\end{aligned}$$

Therefore it suffices to prove (4.3).

LEMMA 4.3. *Suppose Q is a dyadic square. Then*

$$\sum_{j:\, Q_j^n = Q,\, N(Q_j)\ge n} \ell(Q_j) \le 2^n \ell(Q).$$

This is trivial since there are at most 4^n squares Q_j such that $Q_j^n = Q$ and each has length $2^{-n}\ell(Q)$. The lemma is actually true with 2 replaced by a strictly smaller number, but we will not need this result.

LEMMA 4.4. *For each $n = 0, 1, \dots,$*

$$\sum_{j:\, N(Q_j)\ge n} \beta^2(Q_j^n)\ell(Q_j) \le 2^n \sum_{\text{dyadic } Q} \beta^2(Q)\ell(Q).$$

To prove this just use Lemma 4.3 and observe that

$$\begin{aligned}
\sum_{j:\, N(Q_j)\ge n} \beta^2(Q_j^n)\ell(Q_j) &= \sum_{\text{dyadic } Q} \beta^2(Q) \sum_{j:\, Q_j^n = Q,\, N(Q_j)\ge n} \ell(Q_j) \\
&\le 2^n \sum_{\text{dyadic } Q} \beta^2(Q)\ell(Q).
\end{aligned}$$

COROLLARY 4.5. *If $2^{-\mu}$ is smaller than $1/\sqrt{2}$,*

$$\sum_j \left(\sum_{n=0}^{N(Q_j)} \beta(Q_j^n) 2^{-n\mu} \right)^2 \ell(Q_j) \le C \sum_{\text{dyadic } Q} \beta^2(Q)\ell(Q) \le C\ell(\Gamma).$$

To prove this we merely apply Minkowski's inequality to the left-hand side and then apply Lemma 4.4.

LEMMA 4.6. *For each $n = 0, 1, \dots,$*

$$\sum_{j:\, N(Q_j)\ge n} \gamma^2(Q_j^n)\ell(Q_j) \le C 2^n \ell(\Gamma).$$

First let $\{I_k\}$ be a decomposition of $\Gamma \setminus E$ into disjoint pieces satisfying

$$\operatorname{diam}(I_k) \le \frac{1}{2}\operatorname{dist}(I_k, E),$$

$$\ell(I_k) \ge C \operatorname{dist}(I_k, E).$$

Such a collection can easily be constructed by considering the Whitney decomposition of $\mathbf{C} \setminus E$. Given a dyadic square Q such that $\beta(Q) \le 1/1000$ we can pick an index $k = k(Q)$ such that

$$\gamma(Q) \le C\ell(I_k)/\ell(Q).$$

This is possible because by the definition of $\gamma(Q)$ and Lemma 2.9 (2b), $\mathbf{C} \setminus E$ contains a disk of diameter $\gamma(Q)/\ell(Q)$ centered on Γ. Next, note that if $k(Q) = k$ then $\ell(Q) \ge \ell(I_k)$, and only a bounded number of squares of the same size can hit I_k. Thus grouping the dyadic squares such that $k(Q) = k$ according to size, we get

$$\sum_{Q:\, k(Q)=k} \ell(Q)^{-1} \le C\ell(I_k)^{-1} \sum_{m=0}^{\infty} 2^{-m} \le C\ell(I_k)^{-1}.$$

We split the sum in the lemma into two pieces. In the first we only sum over squares with $\beta(Q) \le 1/1000$ and obtain

$$\begin{aligned}
\sum_{j:\, N(Q_j)\ge n} \gamma^2(Q_j^n)\ell(Q_j) &= \sum_{Q:\, \beta(Q)\le 1/1000} \gamma^2(Q) \sum_{Q_j^n = Q,\, N(Q_j)\ge n} \ell(Q_j) \\
&\le 2^n \sum_Q \gamma^2(Q)\ell(Q) \\
&\le C2^n \sum_Q \ell(I_{k(Q)})^2 \ell(Q)^{-1} \\
&= C2^n \sum_k \ell(I_k)^2 \sum_{Q:\, k(Q)=k} \ell(Q)^{-1} \\
&\le C2^n \sum_k \ell(I_k)^2 \ell(I_k)^{-1} \\
&\le C2^n \ell(\Gamma).
\end{aligned}$$

The last line holds because the $\{I_k\}$ are disjoint. For squares with $\beta(Q) > 1/1000$ we note that $\gamma(Q) \le 1000\beta(Q)$ and then use Lemma 4.4.

Corollary 4.7. *If $2^{-\mu}$ is less than $1/\sqrt{2}$ then*

$$\sum_j \left(\sum_{n=0}^{N(Q_j)} \gamma(Q_j^n) 2^{-n\mu} \right)^2 \ell(Q_j) \le C\ell(\Gamma).$$

COROLLARY 4.8. *For* $\mu > 1/2$, (4.3) *holds*; *i.e.*,

$$\sum_j \delta(Q_j)^2 \ell(Q_j) \leq C\ell(\Gamma).$$

The corollary follows from Corollaries 4.5 and 4.7. We have now shown the integral on the right-hand side of (4.1) is finite.

5. Bounding the bad squares

In this section we will prove

$$\sum_{Q_n \in \mathscr{B}_{\varepsilon,\delta}} \ell(Q_n) \leq C(\varepsilon, \delta)\ell(\Gamma) \tag{5.1}$$

if ε and δ are small enough. Let ε be fixed so small that Lemma 4.2 holds and fix a value of μ in Lemma 3.1 so that Corollary 4.8 holds. Let $\delta_0 > 0$ (to be chosen later). Note that if $\delta(Q_j) \geq \delta_0$ then $\ell(Q_j) \leq (1/\delta_0)^2\delta(Q_j)^2\ell(Q_j)$. By Corollary 4.8 these squares sum, and so we may assume from here on that $\delta(Q_j) \leq \delta_0$. Also note that since the harmonic measure of E is close to 1 on the bad squares we must have $\operatorname{dist}(Q_j, E) \leq C \operatorname{diam}(E)$, because $\Phi(\partial\mathscr{F})$ goes out to ∞. (This is why we use $\mathbf{C} \setminus E$ instead of $\overline{\mathbf{C}} \setminus E$.) Our basic strategy will be to build a new curve $\tilde{\Gamma}$ with $\ell(\tilde{\Gamma}) \sim \ell(\Gamma)$ and to associate to each bad cube Q_n a positive function f_n on $\tilde{\Gamma}$ such that

$$\int_{\tilde{\Gamma}} f_n \, ds \geq C_1\ell(Q_n), \tag{5.2}$$

$$\sum_n f_n(x) \leq 4 \tag{5.3}$$

except for a set of zero length in $\tilde{\Gamma}$. Thus

$$\sum_{Q_n \in \mathscr{B}_{\varepsilon,\delta}} \ell(Q_n) \leq \sum_n \frac{1}{C_1}\int f_n \, ds \leq \frac{1}{C_1}\int_{\tilde{\Gamma}} \sum_n f_n \, ds$$

$$\leq \frac{4}{C_1}\ell(\tilde{\Gamma}).$$

Since $\ell(\tilde{\Gamma}) \leq C\ell(\Gamma)$, this will complete the proof of Theorem 1. This idea was also used by Jones and Marshall in [23] in a related, but much simpler, situation.

We start with the following lemmas which allow us to exploit our assumption that $S(\Phi)$ is small on the "bad" cubes. As before we set $\varphi = \log(\Phi')$.

LEMMA 5.1. *Given $\varepsilon > 0$ and $A > 0$ there are a C and a δ such that if Φ is locally univalent on $\mathbf{D}$ and there is a point z such that*

$$|\varphi'(z)| \geq \frac{\varepsilon}{1-|z|},$$

$$|S(\Phi)(w)| \leq \frac{\delta}{(1-|w|)^2},$$

for all $\rho(w, z) \leq C$, then there exist a hyperbolic disk D centered at z_0 of radius A and a point $w_0 \in D$ such that $\rho(D, z) \leq C$ and $1 - |w_0| \leq 1 - |z|$ and

$$|\varphi'(w_0)| \geq \frac{\varepsilon}{2(1-|w_0|)},$$

$$|\varphi'(w)| \leq \frac{\varepsilon}{1-|w|}, \qquad w \in D.$$

The proof is very easy. We merely observe that if Φ is a Möbius transformation, then the lemma is true for some $C > 0$ (it is instructive to consider $1/z$ on the upper half-plane). If the lemma failed in general, we would take a sequence of functions Φ_n with $S(\Phi)(w)(1-|w|)^2 \to 0$ uniformly on $\rho(w, z) \leq C$, but such that the corresponding $\varphi_n'(w)(1-|w|)$ remained larger than ε. Passing to a subsequence and a limit we obtain a Möbius transformation for which the lemma fails, a contradiction. This proves the second inequality holds. We can easily get the first to hold by moving D closer to z if necessary. Also note that we can get the second inequality to hold just by taking any disk D close enough to $\mathbf{T}$ as long as it stays away from one boundary point (which depends on Φ), because the same is true for Möbius transformations. This observation will be useful when we prove Theorem 2.

LEMMA 5.2. *Given $\varepsilon > 0$ there is a $\delta > 0$ with the following property. Suppose $z_0 = r_0 e^{i\theta} \in \mathbf{D}$ and let $\gamma = [r_0 e^{i\theta}, r_1 e^{i\theta}]$ be a radial line segment with endpoint z_0. Suppose $|\varphi'(z_0)| \leq \varepsilon/(1-|z_0|)$ and for all $z \in \gamma$*

$$|S(\Phi)(z)| \leq \frac{\delta}{(1-|z|)^2}.$$

Then for all $z \in \gamma$,

$$|\varphi'(z)| \leq \frac{\varepsilon}{1-|z|}.$$

Moreover, for $z \in \gamma$ and $\rho(z, z_0)$ large enough (depending only on δ and ε),

$$|\varphi'(z)| \leq \frac{C\sqrt{\delta}}{1-|z|}.$$

To prove this, suppose $|\varphi'(te^{i\theta})| \le \varepsilon(1-t)^{-1}$ and $|S(\Phi)(te^{i\theta})| \le \delta(1-t)^{-2}$ for $r_0 \le t \le r$. Then

$$|\varphi''(te^{i\theta})| \le |S(\Phi)(te^{i\theta})| + \frac{1}{2}|\varphi'(te^{i\theta})|^2$$
$$\le (\delta + \varepsilon^2/2)(1-t)^{-2} < \varepsilon^2(1-t)^{-2}$$

if $\delta < \varepsilon^2/2$. Clearly this implies $|\varphi'(te^{i\theta})| \le \varepsilon(1-t)^{-1}$ for $t \le r_1$. In fact, we get $|\varphi'(te^{i\theta})| \le \varepsilon^2(1-t)^{-1}$ for t close enough to 1. The final claim comes from iterating this argument n times until $\varepsilon^{2^n} < \delta$.

Now consider the sum in (5.1). By definition each "bad" square Q_j contains a point $\Phi(y_j)$ with $y_j \in \mathscr{L}$, $|\varphi'(y_j)| \ge \varepsilon/(1-|y_j|)$ and $|S(\Phi)(y_j)| \le \delta/(1-|y_j|)^2$. Moreover, $S(\Phi)$ remains small on a large hyperbolic neighborhood of y_j because we may assume $\delta(Q_j) \le \delta_0$ is very small (use Lemma 3.1 and our comments at the beginning of the section). Applying Lemma 5.1 to $z = y_j$ with ε as above and $A = 10$, we obtain a point z_j (z_0 of the lemma) which satisfies the conclusions of Lemma 5.1. Also note that z_j is only a bounded hyperbolic distance from $\mathscr{L}$ (and therefore in $\mathscr{F}$).

Arrange the $\{z_j\}$ so that $|z_1| \le |z_2| \le \dots$. By dividing $\mathscr{B}_{\varepsilon,\delta}$ into a finite number of disjoint families, we may also assume

$$(5.4) \qquad \rho(z_j, z_k) \ge C_2 \gg 1, \qquad j \ne k.$$

C_2 is a large positive constant which will be chosen later.

To each point z_j we wish to associate a Carleson box, i.e., a set of the form

$$R_j = \{z : 1 - |I_j| \le |z| \le 1, z/|z| \in I_j\}$$

for some interval $I_j \subset \mathbf{T}$. We also let S_j denote $\partial R_j \cap \mathbf{D} = \partial R_j \setminus I_j$ and let T_j denote the top edge of R_j. Suppose by induction that we have defined subdomains $R_1, \dots, R_{n-1}$ such that

(1) $S_j \subset \mathscr{F}$.

(2) $\ell(\Phi(S_j)) \le C_3 \operatorname{dist}(\Phi(z_j), E) \equiv C_3 d_j$.

(3) $\rho(z_j, T_j) \le 10$.

(4) If $R_j \cap R_k \ne \varnothing$ and $k \ge j$ then $R_k \subset R_j$ and $\operatorname{dist}(R_k, S_j) \ge \ell(R_k)$.

Also note that $d_j \sim |\Phi'(z_j)|(1-|z_j|) \sim \ell(Q_j)$.

To define R_n we consider two cases. The first occurs if $\operatorname{dist}(z_n, S_j) \ge 2(1-|z_n|)$ for every $1 \le j \le n-1$. If so, build R_n as follows. Since z_n is only a bounded distance from $\mathscr{L}$, Harnack's inequality implies $\omega(z_n, \partial\mathscr{F} \cap \mathbf{D}, \mathscr{F}) \le C\sqrt{\eta}$. Let J denote the arc on $\mathbf{T}$ of length $2(1-|z_j|)$ centered at $z_n/|z_n|$. By a localized version of Lemma 2.4 there is a set $F \subset J \cap \mathscr{F}$ with $|F| > |J|/2$ such that for every radial segment γ with an endpoint on F and length $1-|z_n|$, we have $\ell(\Phi(\gamma)) \leq Cd_n$. Choosing the sides of our Carleson square R_n from such segments, we easily satisfy (1), (2) and (3). We may also assume R_n contains z_n in its top half. Then (3) and (5.4) imply $\operatorname{dist}(R_n, S_j) \leq \ell(R_n)$ for every $1 \leq j \leq n-1$ which is (4).

In the second case there is some $1 \leq j \leq n-1$ such that $\operatorname{dist}(z_n, S_j) \leq 2(1-|z_n|)$. Choose k so that this distance is minimized. Then if C_2 is large enough, $\ell(R_k) \geq 50(1-|z_n|)$. Now build R_n satisfying (1)–(3) inside the box R_k and so that

$$1 \leq \frac{\operatorname{dist}(S_n, S_k)}{\ell(R_n)} \leq 2.$$

To see that (4) holds consider $j \leq n-1$ and $j \neq k$. If $\ell(R_j) \geq 50 l(R_n)$ then

$$\begin{aligned}\operatorname{dist}(S_j, S_n) &\geq \operatorname{dist}(S_j, S_k) - 2\operatorname{dist}(S_n, S_k)\\ &\geq \max\big(\ell(R_j), \ell(R_k)\big) - 4\ell(R_n) \geq \ell(R_n).\end{aligned}$$

On the other hand, if $\ell(R_j) \leq 50\ell(R_n)$ then $\operatorname{dist}(S_j, S_n) \geq 10\ell(R_n)$ because $\rho(z_j, z_n) \geq C_2$. Proceeding by induction, we obtain a sequence of Carleson boxes $\{R_n\}$ satisfying (1)–(4). (See Figure 4.)

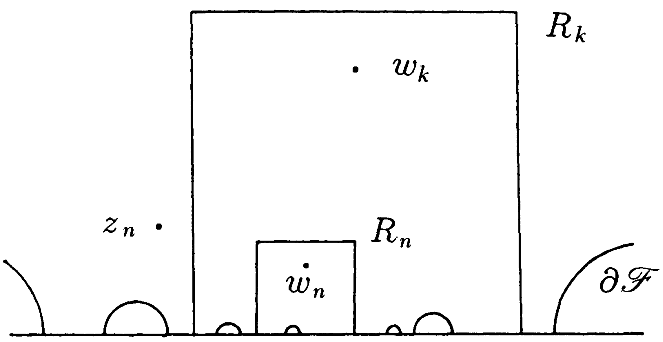

FIGURE 4. Building the Carleson squares

Next we observe that we can ignore the bad squares which are "near" Γ. More precisely, let w_n be the center of the top half of R_n. Suppose $\rho(\Phi(w_n), \Gamma) \leq C_2/10$. Then there exists an arc $I_n \subset \Gamma \cap \{\rho(z, \Phi(w_n)) \leq C_2/2\}$ with $\ell(I_n) \sim d_n$ (if δ_0 is small enough, depending on C_2). Clearly all the arcs arising in this way are disjoint. Thus setting f_n to be the characteristic function of I_n gives (5.2) and (5.3). Therefore we may now assume that $\rho(\Phi(w_n), \Gamma) \geq C_2/10$.

Now let $\mathscr{C}_n = \Phi(S_n)$, and $\mathscr{C}_n^*$ denote the component of $\mathscr{C}_n^* \setminus \Gamma$ which contains $\Phi(T_n)$. (Such a component exists because we are assuming the hyperbolic distance from $\Phi(w_n)$ to Γ is large.) Let W_n denote the component of $\mathbf{C} \setminus (\Gamma \cup \mathscr{C}_n^*)$ containing $\Phi(w_n)$. Observe that $\mathscr{C}_n^*$ has its endpoints on

$\Gamma \cap S_Q$ and by the assumption that $\delta(Q) \leq \delta_0$ we can find a collection of points $\{x_k\} \subset \Gamma \cap S_Q$ with $|x_k - x_{k-1}| \leq \delta_0 d_j$ for each k. By part (a) of Lemma 2.9, Γ connects x_k to x_{k-1} inside a disk of size $C\delta_0 d_n$. Therefore

$$\operatorname{diam}(W_n) \sim d_n. \tag{5.5}$$

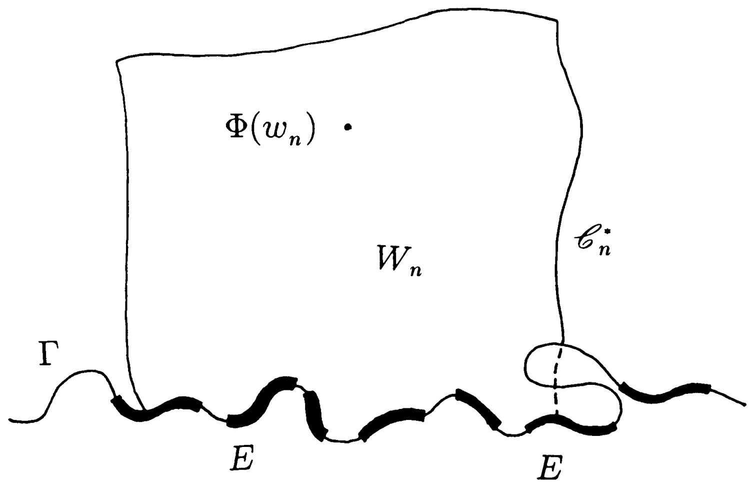

FIGURE 5. The domain W_n

Now let $\mathscr{D}_n$ be the component of $\mathbf{C} \setminus (\Gamma \bigcup_k \mathscr{C}_k^*)$ which contains $\Phi(w_n)$. Since $\mathscr{D}_n \subset W_n$,

$$\operatorname{diam}(\mathscr{D}_n) \leq \operatorname{diam}(W_n) \leq Cd_n.$$

This and the fact that $\rho(w_n, w_k)$ is large imply that for $n \neq k$, $\mathscr{D}_n \cap \mathscr{D}_k = \varnothing$. To see this, assume $d_n \leq d_k$. Then $\Phi(w_k) \in \mathscr{D}_n$ implies $|\Phi(w_n) - \Phi(w_k)| \leq Cd_n$. Combined with $d_k \geq d_n$ and $\delta(Q_n) < \delta_0$, this implies $\rho(w_n, w_k) \leq C_2$ (for C_2 large enough). Furthermore, $n \neq k$ implies $C_n \cap C_k = \varnothing$ since S_n and S_k are disjoint in $\mathscr{F}$ and hence $\partial\mathscr{D}_n$ consists exactly of the arc $\mathscr{C}_n^*$, a subset of Γ and some collection of other arcs $\mathscr{C}_k^*$'s. Furthermore, if $\mathscr{C}_k^* \cap \partial\mathscr{D}_n \neq \varnothing$ then $\mathscr{C}_k^* \subset \partial\mathscr{D}_n$ and $d_k < d_n/2$. Consequently,

$$n, k \neq l \quad \text{and} \quad C_l^* \cap \partial\mathscr{D}_n \neq \varnothing \Rightarrow C_l^* \cap \partial\mathscr{D}_k = \varnothing. \tag{5.6}$$

We now define the new curve $\tilde{\Gamma}$ by adding certain arcs to Γ. For each R_n choose a maximal collection of dyadic subcubes $\{Q_k^n\}$ of R_n with the property that the top half of each Q_k^n contains a point z such that $\Phi(z)$ is in a Whitney cube $\tilde{Q}_k^n$ of $\mathbf{C} \setminus E$ with $1/10 \geq \delta(\tilde{Q}_k^n) \geq \delta_1$ (δ_1 to be chosen later). For each k let F_k^n denote the line segment containing the top edge T_k^n of Q_k^n, but three times as long. Let E_k^n be a curve which connects F_k^n to $\Phi^{-1}(\Gamma)$ and such that $\ell(\Phi(F_k^n \cup E_k^n)) \leq C\ell(\tilde{Q}_k^n)$. We can do this because $\ell(\Phi(F_k^n)) \sim \ell(\tilde{Q}_k^n)$ by the assumption that $\delta(\tilde{Q}_k^n) \leq 1/10$ and the estimates of Section 3 (Lemma 3.2 in particular). E_k^n can be chosen so that $\Phi(E_k^n)$ is a curve of length $\leq C\ell(\tilde{Q}_k^n)$

connecting $\Phi(F_k^n)$ to $E \subset \Gamma$. Finally we set

$$\tilde{\Gamma} = \Gamma \cup \bigcup_{n,k} \Phi(F_k^n \cup E_k^n).$$

To see that $\tilde{\Gamma}$ has finite length, note that each square $\tilde{Q}_k^n$ has a uniformly bounded hyperbolic diameter (because $\delta(\tilde{Q}_k^n) \leq 1/10$); so $\Phi^{-1}(\tilde{Q}_k^n)$ can hit the top half of only finitely many dyadic Carleson squares, and hence can be chosen only finitely often. Moreover, since $\delta(\tilde{Q}_k^n) \geq \delta_1$ the lengths of these squares sum by the remark at the beginning of this section. Hence

$$\ell(\tilde{\Gamma}) \leq \ell(\Gamma) + C \sum_{n,k} \ell\left(\tilde{Q}_k^n\right) \leq C\ell(\Gamma).$$

Define $\tilde{\mathscr{D}}_n$ to be the component of $\mathscr{D}_n \setminus \tilde{\Gamma}$ which contains $\Phi(w_n)$. We claim that

$$\omega\left(\Phi(w_n), \tilde{\Gamma} \cap \partial\tilde{\mathscr{D}}_n, \tilde{\mathscr{D}}_n\right) \geq C_4. \tag{5.7}$$

Let V_n be the component of $\tilde{\mathscr{D}}_n \setminus \mathscr{C}_n)$ containing $\Phi(w_n)$. Then $V_n \subset \tilde{\mathscr{D}}_n$ and $\partial V_n \setminus \partial\tilde{\mathscr{D}}_n = \mathscr{C}_n \setminus \mathscr{C}_n^*$ has large hyperbolic distance from $\Phi(w_n)$ depending only on C_2 (just recall that $\mathscr{C}_n$ is the image of the sides of the Carleson box R_n). This implies $\omega(\Phi(w_n), \partial V_n \setminus \partial\tilde{\mathscr{D}}_n, V_n)$ is small depending only on C_2. Therefore if we can show

$$\omega\left(\Phi(w_n), \tilde{\Gamma} \cap \partial V_n, V_n\right) \geq 2C_4$$

for some absolute $C_4 > 0$ we can deduce

$$\begin{aligned}\omega\left(\Phi(w_n), \tilde{\Gamma} \cap \partial\tilde{\mathscr{D}}_n, \tilde{\mathscr{D}}_n\right) &\geq \omega\left(\Phi(w_n), \tilde{\Gamma} \cap \partial V_n, V_n\right) \\ &\quad - \omega\left(\Phi(w_n), \partial V_n \setminus \partial\tilde{\mathscr{D}}_n, V_n\right) \geq C_4\end{aligned}$$

if C_2 is large enough.

We begin by observing that any radial segment starting at the top edge of R_n must hit $\Phi^{-1}(\tilde{\Gamma})$ before it hits any $R_j \subset R_n$. To see this, fix some $R_j \subset R_n$. By definition there is a point z_j (the point w_0 of Lemma 5.1) within hyperbolic distance A of S_j where $|\varphi'(x_j)| \geq \varepsilon/(2(1-|x_j|))$. Let γ be the radial segment which connects x_j to S_n. Since $\rho(x_j, S_n) \geq C_2 - 2A$ is large, we will contradict Lemma 5.2 unless there is a point $z \in \gamma$ where $|S(\Phi)(z)| \geq \delta/(1-|z|)^2$. So choosing δ_1 to be the δ given by Lemma 5.2, we see that γ must hit one of the squares Q_k^n in the definition of $\tilde{\Gamma}$. Moreover, the hyperbolic distance of the top of this square to R_j is very large if $\delta_0 \ll \delta_1$ (because Lemma 3.1 and δ_0 very small imply $S(\Phi)$ remains very small in a large hyperbolic neighborhood of S_j). Finally, $\rho(R_j, \Gamma) \leq A$, $\rho(R_j, T_k^n) \geq 10$ and $\gamma \cap T_k^n \neq \varnothing$ imply F_k^n (the "triple" of T_k^n) covers R_j.

Consider the domain $U_n = R_n \setminus \bigcup_k \hat{Q}_k^n$, where $\hat{Q}_k^n$ consists of Q_k^n together with the two adjacent Carleson squares of the same size (see Figure 6). Then by the argument above, any $R_j \subset R_n$ lies in some $\hat{Q}_k^n$. Hence if $\tilde{V}_n$ is the component of $\Phi(U_n) \cap V_n$ containing $\Phi(w_n)$ we see $\tilde{V}_n \subset V_n$ and $\partial V_n \setminus \partial \tilde{V}_n \subset \tilde{\Gamma}$.

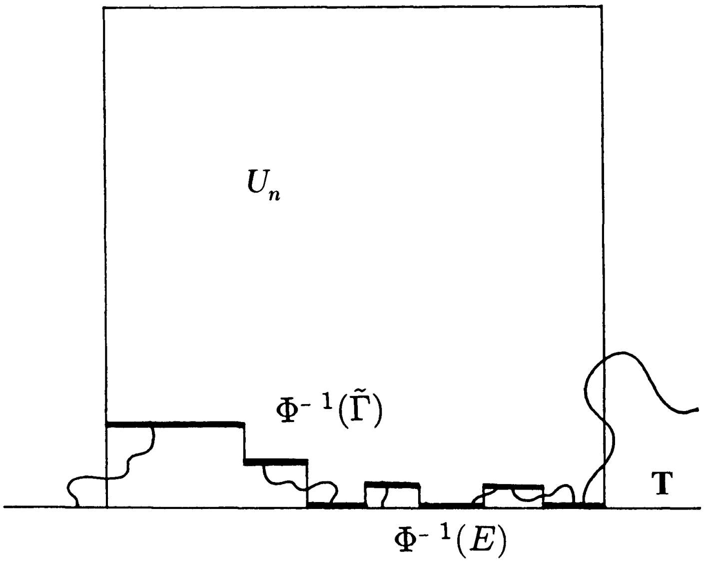

FIGURE 6. U_n and $\Phi^{-1}(\tilde{\Gamma})$

Thus

$$\omega\big(\Phi(w_n), \tilde{\Gamma}, \tilde{V}_n\big) \le \omega\big(\Phi(w_n), \tilde{\Gamma}, V_n\big).$$

Let $F \subset \partial U_n$ consist of $\partial U_n \cap \mathbf{T}$ plus the top edge of each $\hat{Q}_k^n$. By the maximum principle,

$$\omega\big(\Phi(w_n), \tilde{\Gamma}, \tilde{V}_n\big) \ge \omega(w_n, F, U_n)$$

because $F \subset \Phi^{-1}(\tilde{\Gamma})$. Note that U_n is a chord-arc domain with $\ell(\partial U_n) \sim 1 - |w_n|$ and that $\ell(F) \ge 1 - |w_n|$. (Any domain formed by removing dyadic Carleson squares from a larger Carleson square is chord-arc with constant ≤ 6.) Thus by the A_∞ condition for harmonic measure on a chord-arc domain (Jerison and Kenig [20]),

$$\omega(w_n, F, U_n) \ge 2C_4$$

as required.

We are now ready to begin the construction of the f_n's. Let us first note that except for a set of zero length each $x \in \tilde{\Gamma}$ is contained in the boundaries of at most two of the domains $\tilde{\mathscr{D}}_n$. This is because these domains are disjoint and (by rectifiability) contain cones with angle near π at ℓ-a.e. boundary point. Therefore we can consider each point on $\tilde{\Gamma}$ as a pair of points, each corresponding to a "side" of the curve. With this convention it makes sense to claim that each point of $\tilde{\Gamma}$ is in at most one of the sets $\tilde{\Gamma} \cap \partial \tilde{\mathscr{D}}_n$.

It is enough to bound every finite subsum of (5.1), and so we may assume our collection of domains $\{\tilde{\mathscr{D}}_n\}$ is finite. Relabel them so that $d_1 \le d_2 \le \dots$. Define f_1 to be the characteristic function of $\tilde{\Gamma} \cap \partial \tilde{\mathscr{D}}_1$. This satisfies (5.2) since $\partial \tilde{\mathscr{D}}_1$ consists only of C_1^* and a piece of $\tilde{\Gamma}$ of diameter $\sim d_1$. (The upper estimate follows from Lemma 2.9 (2a) and $\operatorname{diam}(\tilde{\mathscr{D}}_1) \sim d_1$. The other direction follows from the lower bound on the harmonic measure of $\tilde{\Gamma}$ in $\tilde{\mathscr{D}}_1$ and the usual estimates on harmonic measure.) Now suppose $f_1, \dots, f_{n-1}$ have already been constructed satisfying (5.2). To construct f_n we consider two cases.

Let $C_5 > 0$ be a small constant to be chosen below. If $\ell(\tilde{\Gamma} \cap \partial \tilde{\mathscr{D}}_n) \ge C_5 d_n$, then let f_n be the characteristic function of the set $\tilde{\Gamma} \cap \partial \tilde{\mathscr{D}}_n$. Certainly (5.2) is satisfied if we take $C_1 \le C_5$.

Now suppose $\ell(\tilde{\Gamma} \cap \tilde{\mathscr{D}}_n) \le C_5 d_n$. Suppose C_0 is the constant in Lavrentiev's inequality (1.1). Equation (5.7) and Lavrentiev's theorem imply that $\ell(\partial \tilde{\mathscr{D}}_n) \ge C_4 |\log C_5| d_n / C_0$. Since we know that $\ell(C_n^*) \le C_3 d_n$ and are assuming $\ell(\tilde{\Gamma} \cap \tilde{\mathscr{D}}_n) \le C_5 d_n$, we must have

$$\ell\Bigg(\bigcup_{\mathscr{C}_k^* \subset \partial \tilde{\mathscr{D}}_n} \mathscr{C}_k^*\Bigg) \ge \left(\left(\frac{C_4 |\log C_5|}{C_0}\right) - (C_3 + C_5)\right) d_n$$
$$\ge \frac{1}{2} \frac{C_4 |\log C_5|}{C_0} d_n \ge 2 C_3 d_n$$

if C_5 is small enough. Define

$$f_n = \frac{1}{2} \sum_{k:\, \mathscr{C}_k^* \subset \partial \mathscr{D}_n} f_k$$

and observe by the induction hypothesis and (5.6) that

$$\int f_n \, ds \ge \frac{1}{2} \sum_{k:\, \mathscr{C}_k^* \subset \partial \mathscr{D}_n} \int f_k \, ds$$
$$\ge \frac{1}{2} \sum_{k:\, \mathscr{C}_k^* \subset \partial \mathscr{D}_n} C_1 d_k \ge \frac{1}{2} \frac{C_1}{C_3} \sum \ell(\mathscr{C}_k^*)$$
$$\ge \frac{1}{4} \frac{C_1}{C_3} \frac{C_0 |\log C_5|}{C_4} d_n \ge C_1 d_n.$$

We now turn to the proof of (5.3). Fix $x \in \tilde{\Gamma}$ (not in the exceptional set and with the "two sides" convention discussed above), and let k_0 be the first index such that $f_{k_1}(x) \ne 0$. Then it must be the case that $x \in \tilde{\Gamma} \cap \tilde{\mathscr{D}}_{k_0}$ and $f_{k_0}(x) = 1$.

Let $k_1 > k_0$ be the next index such that $f_{k_1}(x) \neq 0$ and similarly for $k_2, k_3, \ldots$. Since $x \in \tilde{\Gamma} \cap \tilde{\mathscr{D}}_{k_1}$ we have $C^*_{k_0} \subset \partial \tilde{\mathscr{D}}_{k_1}$ and hence $f_{k_1}(x) = 1/2$. Now consider k_2. Since by (5.6) $\mathscr{C}^*_{k_0} \subset \partial \tilde{\mathscr{D}}_1$ implies $\mathscr{C}^*_{k_0} \not\subset \partial \tilde{\mathscr{D}}_2$, we must have $\mathscr{C}^*_{k_1} \subset \partial \tilde{\mathscr{D}}_2$. Therefore $f_{k_2}(x) = f_{k_1}(x)/2 = 1/4$. In general, $\mathscr{C}^*_{k_j} \cap \tilde{\mathscr{D}}_{k_n} = \varnothing$ if $0 < j < n-1$ and $\mathscr{C}^*_{k_{n-1}} \subset \partial \tilde{\mathscr{D}}_{k_n}$. Thus $f_{k_n}(x) = 2^{-n}$ for all n and so $\sum f_k(x) \leq 2$. This proves (5.3) and finishes the proof of Theorem 1.

6. Proof of Theorem 2

We now turn to the proof of Theorem 2. All this really requires is a slight technical modification of the argument of the previous section, and so we will only sketch these changes.

Suppose $\omega(z, E, \Omega) \geq \varepsilon$. We replace Lemma 4.1 with a construction that gives a Lipschitz domain $\mathscr{L} \subset \mathscr{F}$ such that $\omega(z, \partial \mathscr{F} \cap \mathbf{T}, \mathscr{F}) \geq \nu(\varepsilon)$ for $z \in \mathscr{L}$. Furthermore, $\rho(z, \partial \mathscr{F}) \geq C\nu$ for $z \in \mathscr{L}$ and the chord-arc constant of $\partial \mathscr{L}$ depends only on ε. Therefore, Lemmas 3.1 and 4.2 still hold, but with constants C, ε' that depend on ε. Thus we can prove that $\Phi(\partial \mathscr{L})$ has finite length (depending only on ε and $\ell(\Gamma)$), if we can modify the argument in Section 5 to bound the bad squares.

The statements of Lemmas 5.1 and 5.2 still hold since they did not involve either $\mathscr{L}$ or $\mathscr{F}$ or the assumption that the harmonic measure of E was close to 1 in $\Phi(\mathscr{L})$. However, we can no longer choose Carleson squares $\{R_n\}$ which satisfy conditions (1)–(4). In particular, condition (1) may be impossible to satisfy. However, we can construct a sequence $\{w_j\}$ of points associated to the bad squares $\{Q_j\}$ which satisfy

$$|\varphi'(w)|(1 - |w|) \leq \varepsilon', \qquad \rho(w, w_j) < 10,$$

$$|\varphi'(z)|(1 - |z|) \geq \varepsilon'/10, \qquad \text{for some } \rho(z, w_j) \leq C(\varepsilon).$$

To each w_j we will associate an arc I_j centered at $w_j/|w_j|$ so that $\ell(I_j) \geq \eta(\varepsilon)(1 - |w_j|)$. We define

$$R_n = \{z : z/|z| \in I_n, 2(1 - |w_n|) \leq 1 - |z| \leq 1\}.$$

Instead of being "squares" these are tall, narrow rectangles, but with eccentricity bounded by $1/\eta$. We can choose $\{w_j\}$ and $\{I_j\}$ so that we also have

(1′) $S_j \subset \mathscr{F}$.

(2′) $\ell(\Phi(S_j)) \leq C_1(\varepsilon) d_j$.

(3′) $\rho(z_j, T_j) \leq C_2(\varepsilon)$.

(4′) If $R_j \cap R_k \neq \varnothing$ and $k \geq j$, then $R_k \subset R_j$ and $\operatorname{dist}(R_k, S_j) \geq C_3(\varepsilon) \ell(R_k)$.

(As before, S_j and T_j denote the "sides" and "top" of R_j.) Now the construction proceeds exactly as before except the values of the constants change.

To construct $\{w_n\}$ and $\{I_n\}$ we proceed as follows. To each bad square Q_j we associate a point $y_j \in \mathscr{L}_{\varepsilon', \delta}$ just as in Section 5. Then apply Lemma 5.1 to the point y_j with $A = 10/\nu$. This gives us a sequence of points z_j satisfying the inequalities in Lemma 5.1. By dividing this sequence into a finite number of subsequences we may assume the z_j's are well separated in the hyperbolic metric, say $\rho(z_j, z_k) \geq 100/\nu$ for $j \neq k$. We may also choose the $z_j \in \mathscr{F}$ so that they are at least a fixed hyperbolic distance from $\partial\mathscr{F}$. Since $\mathscr{F}$ is hyperbolically convex this implies we can connect z_j to $\mathscr{L}$ by a geodesic of bounded length which stays a fixed distance from $\partial\mathscr{F}$. Applying Harnack's inequality along the geodesic gives $\omega(z_j, \partial\mathscr{F} \cap \mathbf{T}, \mathscr{F}) \geq \nu/C$. Replacing ν by ν/C we will assume this inequality holds with $C = 1$. Let $l_j = 1 - |z_j|$ and let J be the arc on $\mathbf{T}$ which is centered at $z_j/|z_j|$ but has length $10l_j/\nu$. Now apply Lemma 2.4 to obtain $F \subset \partial\mathscr{F} \cap \mathbf{T}$ such that $|F| \geq |\partial\mathscr{F} \cap J|/2$ and so that every radial segment γ with endpoint in F and of length l_j satisfies $\ell(\Phi(\gamma)) \leq C(\varepsilon)d_j$. Then there exist intervals $J_1, J_2 \subset J$ such that $\operatorname{dist}(J_1, J_2) \geq \nu l_j/4$ and $|J_1 \cap F|, |J_2 \cap F| \geq \nu l_j/4$. There exist points $x_1, x_2 \in J_1 \cap F$ such that $|x_1 - x_2| \geq \nu l_j/16$. (To see this, consider the convolution of the characteristic function of $F \cap J_1$ with itself. Since it is a function bounded by $\nu l_j/4$ but having L^1 norm $(\nu l_j/4)^2$, it must be nonzero at some point at least distance $\nu l_j/16$ from zero, which is equivalent to our claim.) Now let K_1 be the arc between x_1 and x_2. Using the same argument on J_2, we define an interval $K_2 \subset J_2$ with endpoints in F.

We now mimic the construction of the Carleson squares in Section 5, assuming we have ordered the points $\{z_j\}$ so that $|z_1| \leq |z_2| \leq \cdots$ and that we have already constructed the points $\{w_j\}$ and intervals $\{I_j\}$ up to $n - 1$ satisfying (1′)–(4′). If the sequence $\{z_j\}$ is sufficiently separated in the hyperbolic metric, so will $w_1, \ldots, w_{n-1}$ be since $\rho(w_j, z_j) \leq C(\varepsilon)$. Therefore, we can have $\rho(z_n, S_k) \leq 10/\nu$ for at most one index $k < n$. Moreover, at most one of the radial edges in S_k can hit J (the interval from the last paragraph). Therefore, at least one of the two intervals K_1, K_2 constructed above must be at distance at least $\nu(1 - |z_n|)/8$ from S_k. We now take I_n to be this interval, and define w_n so that $w_n/|w_n|$ is the center of I_n and $|w_n| = \min(|z_n|, 1 - |I_n|)$. It is now easy to check that we have satisfied all the required conditions, (1′)–(4′).

7. Proof of Theorem 3

In this section we will prove Theorem 3. If Γ is not Ahlfors regular, it is clear that the supremum over Φ and Ω of $\ell(\Phi^{-1}(\Gamma \cap \Omega))$ is infinite. Therefore we may assume that

$$\ell(\Gamma \cap D(x,r)) \leq Mr$$

for all disks. We will use the following result which is due to Garnett, Gehring and Jones [18].

LEMMA 7.1. *Let $\mathscr{S}_1, \ldots, \mathscr{S}_N$ be collections of points in $\Gamma \cap \Omega$ and let $\mathscr{S} = \bigcup_{m=1}^N \mathscr{S}_m$. Let $K_j = \{|z - z_j| \leq \operatorname{dist}(z_j, \partial\Omega)/2\}$ for $z_j \in \mathscr{S}$, and for $z_j \in \mathscr{S}_n$ let*

$$\Omega_j = \Omega \setminus \bigcup_{z_k \in \mathscr{S}_n,\, k \neq j} K_k.$$

Suppose the collections $\mathscr{S}_n$ and $\mathscr{S}$ satisfy

$$\inf_{z_j \in \mathscr{S}} \rho(z, z_j) \leq M, \qquad \text{for all } z \in \Gamma,$$

and

$$\omega(z_j, \partial\Omega \cap \partial\Omega_j, \Omega_j) \geq \varepsilon, \qquad z_j \in \mathscr{S}.$$

Then

$$\ell(\Phi^{-1}(\Gamma \cap \Omega)) \leq C(M, N, \varepsilon)$$

for any univalent mapping $\Phi\colon \mathbf{D} \to \Omega$.

Let $A > 0$. It is an exercise to show that for any $A < \infty$ there are points $\{z_j\}$ in $\Gamma \cap \Omega$ which satisfy

$$\rho(z_k, z_j) \geq A, \qquad z_k \neq z_j; \tag{7.1}$$

$$\inf_j \rho(z, z_j) \leq 4A, \qquad z \in \Gamma; \tag{7.2}$$

$$d(z_j) \leq \frac{32}{A} \ell(\Gamma_j), \tag{7.3}$$

where

$$d(z) = \operatorname{dist}(z, \partial\Omega), \qquad \Gamma_j = \{z \in \Gamma : \rho(z, z_j) < A/4\}.$$

Because of (7.1) we can divide the points $\{z_j\}$ into $N(A)$ disjoint subsets $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_{N(A)}$ such that for any $z_j, z_k \in S_n$, $j \neq k$,

$$|z_j - z_k| \geq A \inf(d(z_j), d(z_k)). \tag{7.4}$$

Now let $K_j = \{z : |z - z_j| \leq \frac{1}{2} d(z_j)\}$ and for $z_j \in \mathscr{S}_n$ set

$$\Omega_j = \Omega \setminus \bigcup_{z_k \in \mathscr{S}_n,\, k \neq j} K_k.$$

By Lemma 7.1 it suffices to prove

$$\omega(z_j, \partial\Omega \cap \partial\Omega_j, \Omega_j) \geq 1/2. \tag{7.5}$$

We now normalize so that $z_j = 0$, $d(z_j) = 1$. Since (by Lemma 2.1)

$$\omega(0, \partial\Omega \cap \{|z| \geq 10\}, \Omega_j) \leq 1/4,$$

we see by the maximum principle that (7.5) follows from

$$\omega(0, (\partial\Omega_j \setminus \partial\Omega) \cap \{|z| \leq 10\}, \tilde{\Omega}_j) \leq 1/4, \tag{7.6}$$

where $\tilde{\Omega}_j$ is the component of $\Omega \cap \{|z| \leq 10\}$ which contains the origin. Let

$$\mathscr{C} = \{z_k \in \mathscr{S}_n : |z_k| \leq 10, \quad k \neq j\}.$$

Condition (7.1) implies the sets Γ_k are disjoint and condition (7.4) implies $\Gamma_k \subset \{|z| \leq 20\}$ for $k \in \mathscr{C}$. Therefore condition (7.3) and the regularity of Γ give:

$$\sum_{\mathscr{C}} d(z_k) \leq \frac{C}{A} \sum_{\mathscr{C}} \ell(\Gamma_k) \leq CA^{-1}\ell(\Gamma \cap \{|z| \leq 20\}) \leq C_0 A^{-1}. \tag{7.7}$$

Let L_k be a line segment connecting K_k to $\partial\Omega$ such that $\ell(L_k) = d(z_k)/2$, set $E = \bigcup_{\mathscr{C}}\{K_k \cup L_k\}$, and let $\hat{\Omega}$ be the component of $\tilde{\Omega}_j \setminus \bigcup_{\mathscr{C}} L_k$ which contains the origin. Then $\hat{\Omega}$ is simply connected and by (7.7), $\ell(E) \leq C_1 A^{-1}$. The set E is contained in the rectifiable curve

$$\tilde{\Gamma} = (\Gamma \cap \{|z| \leq 10\}) \cup (\{|z| = 10\}) \cup E$$

and

$$\ell(\tilde{\Gamma}) \leq 10M + 20\pi + C_1 A^{-1}.$$

Thus by Theorem 2,

$$\omega(0, E, \hat{\Omega}) \leq 1/4$$

if A is large enough. The desired estimate (7.6) follows from this last estimate and the maximum principle.

We should also mention that Theorem 3 actually follows from the estimate for Theorem 1. This is because it is possible to choose the line segments $\{L_k\}$ in such a way that adding them to Γ gives a regular curve. The arguments in Sections 4 and 5 together with the remark following Lemma 2.9 therefore give uniform estimates for harmonic measure in $\hat{\Omega}$ which is all we need.

8. The counterexample

If Γ is a straight line then Theorem 1 is still true for domains which are not simply connected, but which satisfy a "uniform thickness" estimate in terms of ℓ

(see [34, Theorem 2]). However, this is not the case when Γ is merely rectifiable, or even regular.

LEMMA 8.1. *There exist a rectifiable curve Γ and subsets $K \subset E \subset \Gamma$ such that for all $x \in \Gamma$, $y \in E$ and $0 < r < \operatorname{diam}(\Gamma)$*

$$\ell(\Gamma \cap D(x, r)) \leq C_1 r,$$

$$\ell(E \cap D(y, r)) \geq C_2 r$$

and such that $\ell(K) = 0$. However, if ω denotes harmonic measure on $\overline{\mathbf{C}} \setminus E$ then $\omega(K) > 0$.

To prove this, suppose $N_1 \leq N_2 \leq \ldots$ is an increasing sequence of positive integers (to be chosen later). Let $\{x_k^1\}$ be a collection of N_1 equally spaced points on $\mathbf{T}$ and let $E_1 = \bigcup I_k^1$ be the union of N_1 subarcs of $\mathbf{T}$ of length $2\pi/10N_1$ centered at the points $\{x_k^1\}$. Let $\Gamma_1 = \mathbf{T}$.

Now suppose we have defined the $N_1 \cdots N_2$ points $\{x_k^n\}$, the set E_n composed of the intervals $\{I_k^n\}$ and a curve Γ_n. For each point x_k^n we let $D_k^n = D(x_k^n, \ell(I_k^n))$ and choose N_{n+1} equally spaced points $\{x_j^{n+1}\}$ on the circle ∂D_k^n. Let $\{I_j^{n+1}\}$ be the subarcs of ∂D_k^n centered at the points $\{x_j^{n+1}\}$ and of length $\ell(\partial D_k^n)/10N_{n+1}$. Let E_{n+1} be the union of the $N_1 \cdots N_{n+1}$ arcs. Define $\Gamma_{n+1} = \Gamma_n \bigcup_k \partial D_k^n$. Clearly

$$\ell(\Gamma_n) = 2\pi + \sum_{j=1}^{n} \sum_k \ell\left(\partial D_k^j\right) = 2\pi + \sum_{j=1}^{n} \left(\frac{2\pi}{10}\right)^j \leq 20.$$

Thus taking $n \to \infty$ we obtain a rectifiable curve Γ as the limit of the Γ_n's. Furthermore it is easy to show

$$\ell(\Gamma \cap D_k^n) \leq \sum_{j=1}^{n} \sum_k \ell\left(\partial D_k^j\right) \leq 20\ell(D_k^n).$$

Let $r_n = \operatorname{diam}(D_k^n) = (N_1 \cdots N_n)^{-1}(2\pi/10)^n$. Given $r > 0$, choose $n = n(r)$ so that $10r_n \leq r < 10r_{n-1}$. Then a disk $D(x, r)$ centered at a point of Γ, contains $\sim r/r_n$ disks D_k^n, so that

$$\ell(\Gamma \cap D(x, r)) \leq Cr.$$

Set $E = \overline{\bigcup E_n}$ and $K = E \setminus \bigcup E_n$. Then clearly $E \subset \Gamma$ and the observation above implies

$$\ell(E \cap D(x, r)) \geq Cr.$$

Also, since K can be covered by $N_1 \cdots N_n$ disks of radius $2r_n$, we have $\ell(K) = 0$. Finally, to show $\omega(K) > 0$ we need only choose the $\{N_n\}$ growing so fast that

$$\omega(z_0, E_n, \Omega) \le 4^{-n}.$$

But this is easy since if we set $\Omega_n = \overline{\mathbf{C}} \setminus \bigcup_{j=1}^{n+1} E_j$, fix $N_1, \dots, N_n$ and take $N_{n+1} \to \infty$ then $\omega(z_0, E_{n+1}, \Omega_n) \to 1$. Thus for N_{n+1} large enough

$$\omega(z_0, E_n, \Omega) \le \omega(z_0, E_n, \Omega_n) \le 4^{-n}.$$

The domain Ω is not a domain of Widom type. However, if E is a homogeneous subset of a chord-arc curve, Ω must be of Widom type (proven by Jones and Marshall [23] when Γ is a line). This is because when Γ is chord-arc, one can estimate the sum of the Green's function over its critical points, $\Sigma G(z_j)$, by noting $G(\tilde{z}_j) = G(z_j)$ for some $\tilde{z}_j \in \Gamma \setminus E$ and so that each component of $\Gamma \setminus E$ contains at most one $\tilde{z}_j$. The result then follows from the A_∞ condition for harmonic measure on chord-arc curves plus the results of [20].

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(Received April 28, 1989)