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BOUNDED FUNCTIONS IN THE LITTLE BLOCH SPACE

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We give a characterization of the bounded functions in the little Bloch space. In particular, we characterize the Blaschke products in the little Bloch space in terms of the distribution of their zeros, and give an explicit example of such a Blaschke product.

1. Introduction. Let $D = \{|z| < 1\}$ denote the unit disk. The little Bloch space, $\mathscr{B}_0$, consists of the holomorphic functions f on D such that

$$\lim_{|z|\to 1} |f'(z)|(1 - |z|^2) = 0.$$

Then $H^\infty(D) \cap \mathscr{B}_0$ is a subalgebra of $H^\infty(D)$ (the bounded holomorphic functions on D), but which functions does it contain? In this note we shall give a characterization of $H^\infty(D) \cap \mathscr{B}_0$ in terms of the measures which arise in the canonical factorization theorem. We should also mention that the space $H^\infty(D) \cap \mathscr{B}_0$ is sometimes called COP ("constant on parts") because it consists of the functions in $H^\infty(D)$ which are constant on each Gleason part (except D) of the maximal ideal space of $H^\infty(D)$. For this, and other facts about $\mathscr{B}_0$, see **[2]** or **[3]**.

A special case which has received some attention is the question of which Blaschke products are in $\mathscr{B}_0$. For example, any finite Blaschke product is in $\mathscr{B}_0$, but the existence of infinite products in $\mathscr{B}_0$ is not obvious. In **[18]** Donald Sarason constructed such a product (answering a question from **[17]**) from the singular inner function associated to a measure μ whose indefinite integral is in λ_*, the Zygmund class of uniformly smooth functions (see **[1]**, **[10]**, **[14]**, **[15]**, **[19]** and **[22]**). However, his construction does not tell us where the zeros of the product are. Sarason poses the question of characterizing the Blaschke products in $\mathscr{B}_0$ in terms of the distribution of their zeros and, as a first step, the problem of explicitly constructing the zeros of some Blaschke product in $\mathscr{B}_0$. Our characterization of $H^\infty(D) \cap \mathscr{B}_0$, when restricted to Blaschke products, answers Sarason's question and in §4 we will use it to give an explicit example of an infinite Blaschke product in the little Bloch space. Our construction will be quite reminiscent of Kahane's construction of a function in λ_* ([13]).

Another construction of infinite Blaschke products in $\mathcal{B}_0$ was given independently by Ken Stephenson [21] and the author [6] using "cut and paste" techniques. The idea here is to construct the image surface of the function by identifying copies of the unit disk along certain slits. However, as in Sarason's example, the zeros of this Blaschke product cannot be explicitly computed. Other constructions of inner functions in $\mathcal{B}_0$ have been given by Carmona, Cufí and Pommerenke in [8] and Sundberg (personal communication). The little Bloch space arises in several contexts, for example, in the theory of conformal mapping (see [16]). Another example is [4], where Axler proves that the Hankel operator $H_{\bar{f}}$ on the Bergman space $L^2_a(D, dxdy)$ is compact iff $f \in \mathcal{B}_0$ (also see [5]).

To state our result we need some notation. For an interval $I \subset \mathbb{T}$ we call

$$Q(I) = \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r \leq 1\}$$

the Carleson square with base I. Also,

$$T(Q) = \{re^{i\theta} : e^{i\theta} \in I, 1 - |I| \leq r < 1 - |I|/2\}$$

denotes the "top half" of Q and $l(Q) = |I|$ denotes its side length. Given an interval I and $\lambda > 0$ we let λI denote the concentric interval of length $\lambda|I|$. Similarly for squares Q and λQ. Also recall the Poisson kernel on D is given by

$$P_z(w) = \frac{1 - |z|^2}{|1 - \bar{w}z|^2}.$$

Given a Carleson square Q we let P_Q denote the Poisson kernel for the point z at the center of the top edge of Q. Given a sequence $\{z_n\}$ in D we define a positive measure ν by

$$\nu = \sum \delta_n(1 - |z_n|)$$

where δ_n is the unit point mass at z_n. Note that ν is a finite measure iff $\{z_n\}$ are the zeros of some Blaschke product.

If $F \in H^\infty(D)$ the canonical factorization theorem [11, Theorem II.5.5] says we can write

$$F(z) = AB(z)G(z)S(z)$$

where A is a constant of modulus 1,

$$B(z) = \prod_{n=1}^{\infty} \frac{z_n - z}{1 - \bar{z}_n z} \frac{|z_n|}{z_n}$$

is a Blaschke product,

$$G(z) = \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|F(e^{i\theta})| \frac{d\theta}{2\pi}\right)$$

is outer and

$$S(z) = \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\lambda(e^{i\theta})\right)$$

is a singular inner function. Thus we can associate to any bounded function F a measure μ on $\overline{D}$ given by

$$d\mu = \sum \delta_{z_n}(1-|z_n|) + d\lambda - \log|F|\frac{d\theta}{2\pi} = d\nu + d\sigma$$

with ν supported on the interior of the disk and σ supported on the boundary. If $\|F\|_\infty \leq 1$ then μ is positive. Also note that

$$|F(z)| = |B(z)| \exp\left(\int P_z(w) d\sigma(w)\right). \tag{1.1}$$

Our characterization is in terms of the measure μ. It says F is in the little Bloch space iff adjacent Carleson squares get about the same mass from μ. More precisely, if Q is a Carleson square let $Q' \subset Q$ be a square of half the size. Then,

THEOREM 1. *Suppose F is in the unit ball of $H^\infty(D)$. Then $F \in \mathcal{B}_0$ iff for every $\varepsilon > 0$ there exist $N, \delta > 0$ such that $l(Q) < \delta$ implies either*

$$\frac{\mu(Q)}{l(Q)} > 1/\varepsilon \tag{1}$$

or

$$\left|\frac{\mu(Q)}{l(Q)} - \frac{\mu(Q')}{l(Q')}\right| < \varepsilon \tag{2a}$$

$$\int_{(NQ)^c} P_Q(w)\, d\mu(w) < \varepsilon \tag{2b}$$

holds.

The result says that when we look at a square one of two things can happen. Either the box is "heavy" (gets a lot of mass from μ) or it is "light" in which case all the nearby squares get approximately the same mass and the far away squares do not contribute much to the value of F on Q. If F is a Blaschke product, then $\mu = \nu$, so our characterization in terms on μ becomes a characterization in terms of the distribution of zeros. In §4 we will construct a sequence of points $\{z_n\}$ so that the corresponding ν satisfies

$$|\nu(Q) - 2\nu(Q')| < \varepsilon\nu(Q) \tag{3}$$

for all sufficiently small squares, and this will give a Blaschke product in $\mathscr{B}_0$. Also note that a measure on $\mathbb{T}$ satisfies (2a) for all small squares iff its indefinite integral is in λ_*. Thus the functions considered by Sarason in **[18]** are not too different from "typical" functions in $\mathscr{B}_0$. For a given square, (2b) will always hold for a large enough N, so the point here is that it hold uniformly for all small enough boxes. In the course of the proof we shall see explicitly how N and δ depend on ε.

In the next section we prove the sufficiency of our conditions and in §3 we prove necesssity. In §4 we give an explicit example of a Blaschke product in $\mathscr{B}_0$. In §5 we conclude with some remarks and questions. I would like to thank John Garnett, Peter Jones, Don Sarason, Ken Stephenson, Carl Sundberg, Tom Wolff and the referee for many helpful remarks and suggestions. The referee's efforts, in particular, greatly improved the clarity and accuracy of the original manuscript. This paper was written during my visit to the Mathematical Sciences Research Institute for the program on classical analysis, and it is a pleasure to thank MSRI and the organizers for a very pleasant and exciting year.

2. Proof of sufficiency. Suppose F is holomorphic on D. We claim that F is in the little Bloch space iff

$$\lim_{l(Q)\to 0} \left(\max_{T(Q)} |F| - \min_{T(Q)} |F| \right) = 0. \tag{2.1}$$

Clearly $F \in \mathscr{B}_0$ implies this condition. On the other hand, Bloch's theorem (e.g., **[9**, Theorem XII.1.4]) says that if g is holomorphic on the unit disk and $|g'(0)| = 1$ then $g(D)$ contains a disk of radius $1/72$. Applying this to F and a disk in $T(Q)$ shows the above condition implies $F \in \mathscr{B}_0$. Thus it suffices to show that the hypothesis of Theorem 1 implies (2.1) for F.

Fix $\eta > 0$. We will show that for every small enough square Q

$$\left| |F(z)| - \exp\left(-2\pi \frac{\mu(Q)}{l(Q)}\right) \right| \leq \eta$$

for every $z \in T(Q)$. Let ε be a positive number which we will choose later depending only on η and F.

First we consider the case when Q satisfies condition (1).

LEMMA 1. *There is a universal $A > 0$ such that for $z \in T(Q)$,*

$$|F(z)| \le \exp\left(-A\frac{\mu(Q)}{l(Q)}\right).$$

Let

$$G(z,w) = -\log\left|\frac{z-w}{1-\bar{w}z}\right|$$

be the Green's function for the unit disk and recall that

$$1 - \left|\frac{z-w}{1-\bar{w}z}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2} = (1-|w|^2)P_z(w).$$

Also, if $z \in T(Q)$ and $w \in Q$ then

$$G(z,w) \ge \frac{1}{2}\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2} \ge C\frac{1-|w|}{1-|z|}.$$

Thus,

$$\begin{aligned}\log|F(z)| &= -\sum_n G(z,z_n) - \int P_z(w)d\sigma \\ &\le -\sum_{z_n\in Q} G(z,z_n) - \int_Q P_z(w)d\sigma \\ &\le -C\sum_{z_n\in Q}\frac{1-|z_n|}{1-|z|} - C\int_Q \frac{1}{1-|z|}d\sigma \le -A\frac{\mu(Q)}{l(Q)}.\end{aligned}$$

This is the lemma. So if $\frac{\mu(Q)}{l(Q)} \ge 1/(2\varepsilon)$ and ε is small enough, we have

$$\exp\left(-2\pi\frac{\mu(Q)}{l(Q)}\right) \le \exp(-\pi/\varepsilon) \le \eta,$$
$$|F(z)| \le \exp(-A/2\varepsilon) \le \eta.$$

This implies the desired inequality for "heavy" squares. Now suppose Q is a "light" square, i.e., $\frac{\mu(Q)}{l(Q)} \le 1/(2\varepsilon)$.

LEMMA 2. *If $\frac{\mu(Q)}{l(Q)} \le 1/\varepsilon$ and Q is small enough then $\mu(T(Q)) = 0$ (i.e., $T(Q)$ contains no zeros of F).*

Suppose not. Let $Q_1, Q_2 \subset Q$ be the two disjoint Carleson squares with side length $l(Q)/2$. Then by (2a)

$$\frac{1}{2}l(Q) \le \mu(T(Q)) = \mu(Q) - \mu(Q_1) - \mu(Q_2) \le \varepsilon l(Q).$$

This is a contradiction, proving Lemma 2.

Let k be a large integer (to be chosen later depending on η). Fix a point $z \in T(Q)$. If Q is small enough, the zeros of F can be split into two subsets, $\{z_n\} = \{z_n^1\} \cup \{z_n^2\}$ which satisfy

$$|z - z_n^1| > 2^k(1 - |z|)$$

$$|z - z_n^2| \leq 2^k(1 - |z|) \quad \text{and} \quad 1 - |z_n^2| \leq 2^{-k}(1 - |z_n^2|)$$

for all n. This is because if $\tilde{Q}$ is a square with $T(\tilde{Q})$ adjacent to $T(Q)$ then (2a) implies

$$\frac{\mu(\tilde{Q})}{l(\tilde{Q})} \leq \frac{\mu(Q)}{l(Q)} + \varepsilon < \frac{1}{2\varepsilon} + \varepsilon < \frac{1}{\varepsilon}.$$

Thus Lemma 2 applies to $\tilde{Q}$ as well as Q. If $k\varepsilon < 1/(2\varepsilon)$ we can iterate this argument k times to obtain the above splitting. (If necessary, choose $\varepsilon_1 < 1/(2\sqrt{k})$ and assume $2^k l(Q)$ is less than the corresponding δ_1 given by Theorem 1.)

We let $\mathscr{C}$ denote the 2^{2k} disjoint Carleson squares of side length $2^{-k}l(Q)$ and lying inside 2^kQ. Let "$a \approx b$" mean that $|a - b|$ is small (i.e., less than $\eta/4$). Then if k is large enough and $l(Q)$ is small enough we have

$$\begin{aligned}
-\log|F(z)| &= \sum G(z, z_n) + \int P_z(w)\, d\sigma \\
&\approx \int_D P_z(w) d\nu(w) + \int P_z(w)\, d\sigma \\
&= \int_D P_z(w)\, d\mu(w) \approx \int_{2^kQ} P_z(w)\, d\mu \\
&= \sum_{\mathscr{C}} \int_{Q_j} P_z(w)\, d\mu(w) \approx \sum_{\mathscr{C}} \int_{Q_j \cap \mathbb{T}} P_z(e^{i\theta}) \frac{\mu(Q)}{l(Q)}\, d\theta \\
&\approx \frac{\mu(Q)}{l(Q)} \int_{\mathbb{T}} P_z(e^{i\theta})\, d\theta = 2\pi \frac{\mu(Q)}{l(Q)}.
\end{aligned}$$

The first "$\approx$" holds because with k large, $l(Q)$ small and z and w as above,

$$\text{(2.2)} \qquad \begin{aligned}
G(z, w) &= -\frac{1}{2} \log\left(1 - \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2}\right) \\
&\sim \frac{1 - |z|^2}{|1 - \bar{w}z|^2}(1 - |w|) = P_z(w)(1 - |w|).
\end{aligned}$$

(Here $a \sim b$ means a/b is close to 1.) The second one holds because of (2b), and the third because the Poisson kernel P_z is almost constant on each Q_j if k is large enough. The fourth holds because the integral of P_z over the complement of $2^k Q$ is small. Thus if $l(Q)$ is small enough (depending only on F and η),

$$\left| |F(z)| - \exp\left(-2\pi \frac{\mu(Q)}{l(Q)}\right) \right| \leq \eta$$

for $z \in T(Q)$. This completes the proof of sufficiency.

3. Proof of necessity. Now we come to the less obvious direction; proving that the conditions in Theorem 1 are necessary. Fix an $F \in H^\infty(D) \cap \mathscr{B}_0$ and assume $\|F\|_\infty \leq 1$. We will need the following lemma.

LEMMA 3. *Given $\varepsilon > 0$ there exits $\beta, \delta > 0$ such that if $l(Q) \leq \delta$ and $|F(z)| \leq \beta$ for some $z \in T(Q)$ then $\frac{\mu(Q)}{l(Q)} > 1/\varepsilon$.*

This lemma is the key step in the proof of Theorem 1. It is like a converse to Lemma 1 of the previous section, but is slightly harder to prove. Note, for example, that it can fail if F is not in little Bloch (e.g., take disjoint Carleson squares $\{Q_j\}$ with $l(Q_j) \to 0$ and place a zero just above each box). To prove the lemma, divide the measure μ into two measures μ_1 and μ_2 supported in $\frac{1}{4}Q$ and $\frac{1}{4}Q^c$ respectively, and let F_1 and F_2 be the corresponding factorization of F. Suppose $z \in T(Q)$. We will show that given $\eta > 0$ there is a $\beta > 0$ such that $|F(z)| \leq \beta$ implies $|F_1(z)| \leq \eta$. Given this, we deduce the lemma as follows. For $z \in T(Q)$ and $w, z_n \in \frac{1}{4}Q$ we have

$$G(z, z_n) \leq C \frac{1 - |z_n|}{1 - |z|}, \qquad P_z(w) \leq \frac{C}{1 - |z|}.$$

Therefore if $\eta \leq \exp(-2C/\varepsilon)$,

$$\begin{aligned} 2C/\varepsilon \leq -\log \eta \leq -\log |F_1(z)| &= \sum_{z_n \in \frac{1}{4}Q} G(z, z_n) + \int_{\frac{1}{4}Q} P_z(w)\, d\sigma \\ &\leq \int \frac{C}{1 - |z|}\, d\mu \leq 2C \frac{\mu(Q)}{l(Q)} \end{aligned}$$

as desired.

We will now prove the claim. Fix $\beta > 0$ and assume $|F(z)| \leq \beta$ for some $z \in T(Q)$. We will show that $|F_1(z)| \leq \eta(\beta) \leq \beta^C$ for some $C > 0$. We start by setting $\tilde{Q} = \frac{1}{16}Q$ and letting $\tilde{I}$ be the base of $\tilde{Q}$. Fix $e^{i\theta} \in \tilde{I}$ and for $n = 0, 1, 2, \ldots$ consider the points

$$w_n = (1 - 2^{-4-n} l(Q)) e^{i\theta}.$$

Clearly $|F_2(w_n)| \to 1$, but how quickly? We claim that if $z \in \frac{1}{4}Q^c$ then

$$\left|\frac{z-w_n}{1-\bar{w}_n z}\right| \geq \left|\frac{z-w_{n-1}}{1-\bar{w}_{n-1} z}\right|^{\alpha}$$

and

$$P_{w_n}(z) \leq \alpha P_{w_{n-1}}(z)$$

for some $0 < \alpha < 1$. To see the first inequality square both sides and use the inequality $x^\alpha \leq 1-\alpha(1-x)$, $0 < x < 1$ to reduce to showing

$$1 - \left|\frac{z-w_n}{1-\bar{w}_n z}\right|^2 \leq \alpha\left(1 - \left|\frac{z-w_{n-1}}{1-\bar{w}_{n-1} z}\right|^2\right).$$

Using the equality following Lemma 1 this reduces to

$$\left(\frac{1-|w_n|^2}{1-|w_{n-1}|^2}\right)\left|\frac{1-\bar{w}_n z}{1-\bar{w}_{n-1} z}\right|^2 \leq \alpha < 1.$$

When $l(Q)$ is small the first term is close to $1/2$ and the second term is less than

$$1 + \frac{|w_n - w_{n-1}|}{|w_{n-1} - z|} \leq 1 + \frac{l(Q)/32}{l(Q)/8} \leq 5/4$$

which gives the desired inequality. The proof for $P_{w_n}(z)$ is similar. Thus using (1.1) we obtain

$$|F_2(w_n)| \geq |F_2(w_{n-1})|^{\alpha}.$$

Let $\tau > 0$ (to be chosen below depending only on β) and let δ be so small that $1-|z| \leq \delta$ implies $|F'(z)|(1-|z|) \ll \tau$. Then $|F(w_n)| \leq \tau + |F(w_{n-1})|$ and if $\tau \ll \beta$ then $|F(w_0)| < 2\beta$. Now suppose $|F_1(w_0)| > 2\sqrt{\beta}$. Then $|F_2(w_0)| < \sqrt{\beta}$. Now choose k so that

$$|F_2(w_{k-1})| < \sqrt{\beta}, \qquad |F_2(w_k)| \geq \sqrt{\beta}.$$

Then for $j = 1, 2, \ldots$, the preceding estimates give

$$|F_2(w_{k+j})| \geq \beta^{\frac{1}{2}\alpha^j}, \qquad |F(w_{k+j})| \leq \sqrt{\beta} + (j+1)\tau.$$

Taking $j \geq (\log 3)/|\log\alpha|$ and $\tau \leq \sqrt{\beta}/(j+1)$ gives

$$|F_1(w_{k+j})| \leq (\sqrt{\beta} + (j+1)\tau)/(\beta^{\frac{1}{2}(\alpha)^j}) \leq 2\beta^{1/3}.$$

Thus we have proved that either $|F_1(w_0)| \leq 2\sqrt{\beta}$ or $|F_1(w_n)| \leq 2\beta^{1/3}$ for some $n > 0$. This argument works for any $e^{i\theta} \in \tilde{I}$ so we have proven there exists a set $E \subset 2^{-4}Q$ whose radial projection is all of $\tilde{I}$ and such that $|F_1(z)| \leq 2\beta^{1/3}$ on E. A variant of Hall's lemma (see [7]) implies the harmonic measure of such an E in $D \backslash E$ with respect to the point z is bounded away from 0 (independently of E and $z \in T(Q)$). Since $\log|F_1|$ is subharmonic this gives

$$\log|F_1(z)| \leq C\left(\max_E \log|F_1|\right) \leq C\log(2\beta^{1/3}).$$

This implies the desired inequality, and completes the proof of Lemma 3.

We can now start the proof of necessity. Consider a Carleson square Q and assume (1) of Theorem 1 does not hold, i.e.,

$$\frac{\mu(Q)}{l(Q)} \leq 1/\varepsilon.$$

By Lemma 3 $|F| \geq \beta$ on $T(Q)$. Given an $N > 0$ we let F_1 and F_2 denote the factorization of F corresponding to NQ and $(NQ)^c$ respectively. Using (2.2) one sees that to prove (2b), it suffices to show $|F_2(z)| > 1 - C\varepsilon$ for some small $C > 0$ and $z \in T(Q)$. Let $\eta = C\varepsilon$. Now choose an integer k and positive number τ such that (with α as above)

$$(1-\eta)^{\alpha^{-k}} \leq \beta/4, \qquad k\tau \leq \beta/2.$$

Let $N = 2^{k+4}$. Now suppose $z_0 \in T(Q)$ satisfies $|F_2(z_0)| \leq 1 - \eta$. We will obtain a contradiction. Choose δ so small that $1 - |z| < 2^k\delta$ implies $|F'(z)|(1-|z|) < \tau$. For $0 \leq j \leq k$ define points $\{w_j\}$ by

$$\arg(w_k) = \arg(z_0), \qquad 1 - |w_k| = 2^k(1 - |z_0|).$$

Then arguing as in Lemma 3 gives

$$|F(w_k)| \geq \beta - k\tau > \beta/2,$$
$$|F_2(w_k)| \leq (1-\eta)^{(1/\alpha)^k} < \beta/4.$$

This implies $|F_1(w_k)| > 2$, a contradiction (since $|F_1| \leq |F| \leq 1$). Thus $|F_2(z_0)| > 1 - \eta$ and so (2b) must hold with $N = 2^{k+4}$ and δ as above.

Next, we will show that

$$\left|2\pi\frac{\mu(Q)}{l(Q)} + \log|F(z)|\right| < \varepsilon$$

for any $z \in T(Q)$ and from this (2a) follows. First we prove

$$2\pi\frac{\mu(Q)}{l(Q)} \leq -\log|F(z)| + \varepsilon.$$

Let n be an integer (to be chosen below depending on ε) and let $\mathscr{C}$ be the collection of 2^n disjoint Carleson squares in Q of side length $r = 2^{-n}l(Q)$. Let R denote the union of these small squares. Since $|F| \geq \beta$ on $T(Q)$, it is larger than $\beta/2$ on $Q \backslash R$ if Q is small enough (since F is in $\mathscr{B}_0$). Thus $\mu(Q) = \mu(R)$. If Q is even smaller we can conclude $|F(z)| \sim |F(z_0)|$ for all $z \in Q\backslash R$ and $z_0 \in T(Q)$ and that F has no zeros in $2^n Q \cap \{|z| \leq 1-r\}$. Thus using (2.2) as we did in §2, we see that by first taking n large enough and then $l(Q)$ small enough we get

$$\left| \int P_z(w)\, d\mu(w) + \log|F(z_0)| \right| < \varepsilon/4$$

for $z_0 \in T(Q)$ and $z \in Q\backslash R$. Set $\hat{I} = (1+2^{-n/2})I$ (I is the base of Q) and note that

$$\int_{\hat{I}} P_{re^{i\theta}}(w) \frac{d\theta}{2\pi} \geq 1 - 2^{-n/2}$$

for $w \in R$. Hence if $2^{-n/2} \leq \varepsilon/(8|\log\beta|)$,

$$\begin{aligned}
\frac{\mu(Q)}{l(Q)} = \frac{\mu(R)}{l(Q)} &\leq \frac{1}{(1-2^{n/2})l(Q)} \int_R \left\{ \int_{\hat{I}} P_{re^{i\theta}}(w) \frac{d\theta}{2\pi} \right\} d\mu(w) \\
&\leq (1+2^{1-n/2}) \frac{|\hat{I}|}{l(Q)2\pi} \int_R P_{re^{i\theta}}(w)\, d\mu \\
&\leq \frac{1+2^{2-n/2}}{2\pi} \int_D P_{re^{i\theta}}(w)\, d\mu \\
&\leq \frac{1+2^{2-n/2}}{2\pi} (-\log|F(z_0)| + \varepsilon/4) \\
&\leq -\frac{1}{2\pi} \log|F(z_0)| + \varepsilon.
\end{aligned}$$

To prove the other direction we take N so that (2b) is satisfied with $\varepsilon/4$, and increase n (if necessary) so that $2^{n/2} \geq N$. Let $\tilde{I} = (1-2^{-n/2})I$. Then if $e^{i\theta} \in \tilde{I}$,

$$\left| \int_R P_{re^{i\theta}}(w) d\mu(w) - \int_D P_{re^{i\theta}}(w) d\mu(w) \right| < \varepsilon/4.$$

Thus for small enough Q,

$$\begin{aligned}\frac{\mu(Q)}{l(Q)} = \frac{\mu(R)}{l(Q)} &\geq \frac{1}{l(Q)}\int_R \left\{\int_{\tilde{I}} P_{re^{i\theta}}(w)\frac{d\theta}{2\pi}\right\} d\mu(w) \\ &= \frac{1}{l(Q)}\int_{\tilde{I}} \left\{\int_R P_{re^{i\theta}}(w) d\mu\right\}\frac{d\theta}{2\pi} \\ &\geq \frac{|\tilde{I}|}{l(Q)2\pi}\left(\int_D P_{re^{i\theta}}(w)d\mu(w) + \varepsilon/4\right) \\ &\geq \frac{1-2^{1-n/2}}{2\pi}(-\log|F(z_0)| - \varepsilon/4 - \varepsilon/4) \\ &\geq -\frac{1}{2\pi}\log|F(z_0)| - \varepsilon.\end{aligned}$$

This completes the proof of Theorem 1.

4. The example. In this section we will use Theorem 1 to explicitly construct a Blaschke product in $\mathscr{B}_0$. Let $K_1 \leq K_2 \leq \cdots$ be a sequence of integers and set

$$\theta_0 = \pi/8, \qquad \theta_{n+1} = \theta_n/K_n.$$

Now consider the Carleson squares $\{Q_j^n\}$ given by

$$Q_j^n = \{re^{i\theta} : 1 - \theta_n \leq r \leq 1,\ j\theta_n \leq \theta \leq (j+1)\theta_n\}.$$

The whole collection is called $\mathscr{C}$ and the squares of size θ_n are denoted by $\mathscr{C}_n$. To each $Q \in \mathscr{C}$ we associate an integer $N(Q)$. For $Q \in \mathscr{C}_0$ we let $N(Q) = 0$. To define it for later generations we assume each K_n is of the form

$$K_n = k_n + (k_n + 1) + 3 = 2(k_n + 2)$$

for integers k_n. For a square $Q \in \mathscr{C}_n$ with $N(Q) = j$ we label the K_n $(n+1)^{\text{st}}$ generation squares contained in Q so that k_n of them get the number $j-1$, $k_n + 1$ get the number $j+1$ and 3 get j . They are arranged so that the two outside boxes get j's and the third j separates the block of $(j-1)$'s from the $(j+1)$'s. See Figure 1. We have done this so that when we are finished, any two adjacent squares have labels differing by at most 1. ($Q_1 \in \mathscr{C}_n$ and $Q_2 \in \mathscr{C}_m$ are called adjacent if $Q_1 \cap Q_2 \neq \varnothing$ and $|n - m| \leq 1$.) We now define $\{z_n\}$ by placing a point on the top edge of any Q such that $N(Q) = 0$.

Claim. There is a sequence $\{K_n\}$ such that the corresponding sequence $\{z_n\}$ forms the zeros of a Blaschke product in $\mathscr{B}_0$.

We will show that if $Q \in \mathscr{C}_n$ and $Q' \in \mathscr{C}_{n+1}$ satisfies $Q' \subset Q$ then

$$\left|\frac{\mu(Q)}{l(Q)} - \frac{\mu(Q')}{l(Q')}\right| < \varepsilon\frac{\mu(Q)}{l(Q)} \tag{4.1}$$

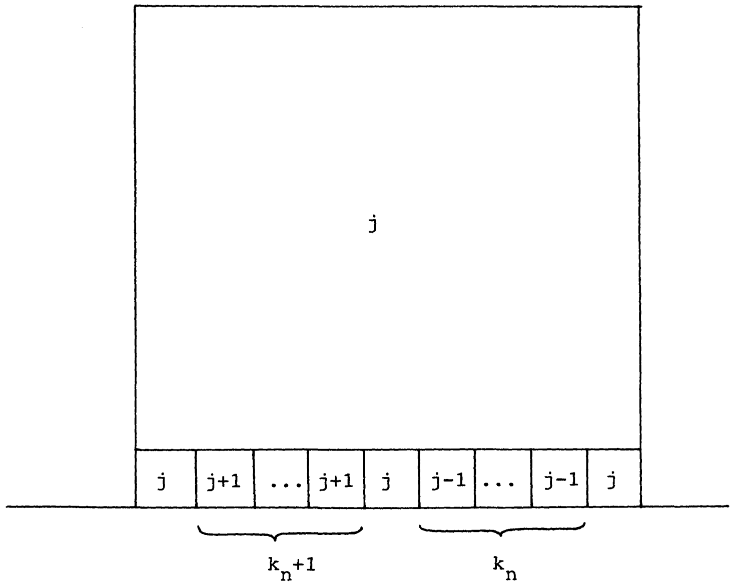

FIGURE 1

if $l(Q)$ is small enough. We then use it to verify the hypotheses of Theorem 1.

We can do this because the ratio $\frac{\mu(Q)}{l(Q)}$ has a very nice interpretation. Let

$$p_n = \frac{k_n + 1}{K_n}, \qquad q_n = \frac{k_n}{K_n} < p_n$$

and consider the random walk on the integers which at time n steps to the right with probability p_n and to the left with probability q_n. A sample path will be denoted ω and its position at time n by $\omega(n)$. Since $p_n > q_n$ this random walk has a drift to the right. Define the "Green's function"

$$G(x, y, j) = \text{expected number of visits to } y \text{ starting at } x \text{ at time } j.$$

Then observe that for $Q \in \mathscr{C}_n$

$$\frac{\mu(Q)}{l(Q)} = c_0 G(N(Q), 0, n)$$

where $c_0 > 0$ is some normalizing constant. We introduce the random walk notation only for convenience. It expresses exactly what we need, and the arguments seem more intuitive when stated about random walks than about Carleson squares.

If the sequence $K_1 = K_2 = \cdots$ is constant then there is a simple, explicit formula for $G(x, y)$ and the desired estimate will follow from this formula and a few simple observations. We also need the following notation:

$$F_k(x,y,j) = \text{probability of first being at } y \text{ at time } j+k, \text{ starting at } x \text{ at time } j,$$

$$F(x,y,j) = \sum_{k\geq 0} F_k(x,y,j)$$
$$= \text{probability of eventually hitting } y, \text{ starting at } x \text{ at time } j.$$

For $j \geq 0$ we let $G^j(x,y)$, $F_k^j(x,y)$ and $F^j(x,y)$ denote the same functions but for the time invariant walk corresponding to $p = p_j$, $q = q_j$. If $p > q$ are fixed and the corresponding Green's function is G then it is known that

$$G(x,y) = \begin{cases} (p-q)^{-1}, & \text{if } x \leq y, \\ (p-q)^{-1}(q/p)^{x-y}, & \text{if } x > y. \end{cases}$$

$$F(x,y) = (p-q)G(x,y).$$

This formula is proved in [**20**, P1.5] under the assumption that $p+q = 1$ (one merely solves the difference equation satisfied by G, i.e., $G(x,y) = pG(x-1,y) + qG(x+1,y)$). For $p+q < 1$ the same proof shows G has the same form except possible for a multiplicative constant and in [**20**, E3.1] that constant is shown to be 1. (Alternatively, one can see that the Green's function for the pair $\{p,q\}$ with $p+q < 1$, must be $\tilde{G}/(p+q)$ where $\tilde{G}$ is the Green's function for the pair $\{p/(p+q), q/(p+q)\}$.)

Next we wish to see that if $\{K_n\}$ grows slowly enough, then $G(0,0,0)$ is finite (and hence that $\{z_n\}$ are the zeros of a Blaschke product). First consider the time invariant walk with $p = 1/3$ and $q = 1/6$, i.e., $6 = K_1 = K_2 = \cdots$. There is a time T_1 at which the probability of finding a sample walk to the left of 100 is less then $1/100$. Now change the walk by taking $K_n = 8$ for $n > T_1$. The number of expected returns to zero now increases, but only slightly. This is because a walk ω with $\omega(T_1) < 100$ is expected to return to zero at most $(3/8 - 1/4)^{-1} = 8$ times and a walk with $\omega(T_1) \geq 100$ is expected to return at most $8(2/3)^{100}$ times. Thus the expected number of returns to zero after time T_1 for the new walk is at most $8/100 + 8(2/3)^{100} < 1/2$. Similarly, we take $K_n = 10$ for $n \geq T_2$, where T_2 is chosen so large that the expected number of returns to zero after time T_2 is less than $1/4$. Continuing in the obvious way we obtain a sequence $\{K_n\}$ tending to infinity, but such that the expected number of returns to zero is finite. Thus $\{z_n\}$ are the zeros of a Blaschke product.

Now we turn to proving (4.1). Using the above formulas and some obvious inequalities,

$$\begin{aligned} G(x-1,0,j) \geq G(x,0,j) &= \sum_{k=1}^{\infty} F_k(x,x-1,j)G(x-1,0,j+k) \\ &\geq G(x-1,0,j)\sum_{k=1}^{\infty} F_k(x,x-1,j) \\ &= G(x-1,0,j)F(x,x-1,j) \\ &\geq G(x-1,0,j)F^j(x,x-1) \\ &= G(x-1,0,j)\frac{q_j}{p_j} = G(x-1,0,j)\frac{k_j}{k_j+1} \\ &\geq G(x-1,0,j)\left(1-\frac{2}{K_j}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} G(x,0,j+1) &\geq G(x,0,j) \\ &= q_jG(x-1,0,j+1) + p_jG(x+1,0,j+1) \\ &\quad + (1-p_j-q_j)G(x,0,j+1) \\ &\geq q_jG(x,0,j+1) + p_j\left(1-\frac{2}{K_{j+1}}\right)G(x,0,j+1) \\ &\quad + (1-p_j-q_j)G(x,0,j+1) \\ &\geq \left(1-\frac{2}{K_j}\right)G(x,0,j+1). \end{aligned}$$

By our earlier remarks, this means if $Q \in \mathscr{C}_n$ and $Q' \subset Q$ is in $\mathscr{C}_{n+1}$ then

$$\begin{aligned} \left|\frac{\mu(Q)}{l(Q)} - \frac{\mu(Q')}{l(Q')}\right| &\leq C|G(N(Q),0,n) - G(N(Q'),0,n+1)| \\ &\leq C|G(N(Q),0,n) - G(N(Q),0,n+1)| \\ &\qquad + C|G(N(Q),0,n+1) - G(N(Q'),0,n+1)| \\ &\leq \frac{4C}{K_n}G(N(Q),0,n) = \frac{4C}{K_n}\frac{\mu(Q)}{l(Q)} \end{aligned}$$

which is the desired inequality, (4.1), if K_n is large enough (i.e., if $l(Q)$ is small enough). We now wish to estimate $\mu(Q)$ for all small squares.

Fix Q and choose n so that $\theta_n \le l(Q) < \theta_{n-1}$. Choose $\hat{Q} \in \mathscr{C}_n$ so that Q overlaps at least half of $\hat{Q}$ and let $N = N(\hat{Q})$ and $K = K_n$. Note that if $\tilde{Q} \in \mathscr{C}_n \cup \mathscr{C}_{n+1}$ hits Q, then $|N - N(\tilde{Q})| \le 5$. Thus,

$$\begin{aligned} \mu(Q) &\ge \sum_{\tilde{Q}\in\mathscr{C}_{n+1}, \tilde{Q}\subset Q} \mu(\tilde{Q}) \\ &\ge \left(1 - \frac{C}{K}\right)^5 \frac{\mu(\hat{Q})}{l(\hat{Q})}(l(Q) - 2l(\hat{Q})/K) \\ &\ge \left(1 - \frac{C}{K}\right) \frac{\mu(\hat{Q})}{l(\hat{Q})} l(Q) \end{aligned}$$

and

$$\begin{aligned} \mu(Q) &\le \sum_{\tilde{Q}\in\mathscr{C}_{n+1}, \tilde{Q}\cap Q\neq\varnothing} \mu(\tilde{Q}) + \sum_{\tilde{Q}\in\mathscr{C}_{n}, \tilde{Q}\cap Q\neq\varnothing} \mu(T(\tilde{Q})) \\ &\le \left(1 + \frac{C}{K}\right)^5 \frac{\mu(\hat{Q})}{l(\hat{Q})}(l(Q) + 2l(\hat{Q})/K) + \sum_{\mathscr{C}_n} \mu(T(\tilde{Q}) \cap Q) \\ &\le \left(1 + \frac{C}{K}\right) \frac{\mu(\hat{Q})}{l(\hat{Q})} l(Q) + \sum_{\mathscr{C}_n} \mu(T(\tilde{Q}) \cap Q). \end{aligned}$$

If $N > 5$ then the sum over $\mathscr{C}_n$ is zero and so we get

$$\left(1 - \frac{C}{K}\right) \frac{\mu(\hat{Q})}{l(\hat{Q})} \le \frac{\mu(Q)}{l(Q)} \le \left(1 + \frac{C}{K}\right) \frac{\mu(\hat{Q})}{l(\hat{Q})}$$

as required. If $N \le 5$ then $\mu(\hat{Q}) = c_0 G(N, 0, n) \ge CKl(\hat{Q})$ and the sum over $\mathscr{C}_n$ is bounded by $3l(Q)$. Thus the inequality above is still true (with a slightly larger C). If we fix ε in the statement of Theorem 1 then taking K large enough (hence $l(Q)$ small enough) in the inequality above shows that either (1) or (2a) must hold. To get (2b), observe that since $K_n \ge K_1 = 6$, arguing as above gives

$$\mu(6Q) \le 2(1 + 2/K_1)^6 \mu(Q) \le 12\mu(Q)$$

and so

$$\begin{aligned} \int_{(6^n Q)^c} P_Q(w)\, d\mu(w) &\le \sum_{k=n}^{\infty} C \int_{6^{k+1}Q \setminus 6^k Q} \operatorname{dist}(w, Q)^{-2}\, d\mu(w) \\ &\le C \sum_{k=n}^{\infty} 6^{-2k} \mu(6^{k+1} Q) \le C \sum_{k=n}^{\infty} 3^{-k} \le C 3^{-n}. \end{aligned}$$

This completes the proof that our example is in the little Bloch space.

5. Remarks. First of all, it might be useful to point out that just because $d\mu$ satisfies the conditions in Theorem 1, the individual measures $d\nu$, $d\lambda$, and $\log|F|d\theta/2\pi$ need not. In other words, $F \in H^\infty(D) \cap \mathcal{B}_0$ does not imply that its Blaschke, singular inner or outer factors are in $\mathcal{B}_0$. This was first observed by Harold Shapiro in **[19]** (also see **[1]**).

Greg Hungerford **[12]** has recently proven the following: if E is the singular set of an infinite Blaschke product in $\mathcal{B}_0$ (the accumulation set of its zeros on $\mathbb{T}$) then the Hausdorff dimension of E is 1. The main idea is to use Lemma 3 to show there exists an $\eta > 0$ so that if Q is small enough and $|B(z)| < \eta$ for some $z \in T(Q)$ then there is a disjoint collection of subsquares $\{Q_j\}$ such that $|B(z_j)| < \eta$ for some $z_j \in T(Q_j)$, $l(Q_j) < \beta$ and $\sum l(Q_j) > \alpha l(Q)$, where $\beta = o(l(Q))$ as $l(Q) \to 0$ and α is independent of Q. From this one can deduce the result. A similar, but simpler, argument shows that if B is a Blaschke product in $\mathcal{B}_0$ then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|B(z)| = 0$ and $1 - |z| < \delta$ imply

$$E \cap \left\{ e^{i\theta} : \left| e^{i\theta} - \frac{z}{|z|} \right| < \varepsilon(1 - |z|) \right\} \neq \emptyset.$$

I do not know if this has been previously observed.

Finally, note that if z_j is a zero of a Blaschke product B, then

$$(1 - |z_j|^2)|B'(z_j)| = \prod_{k \neq j} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|.$$

B is called interpolating if the right hand side is bounded away from zero independently of j, so if B is in the little Bloch space it is definitely not interpolating, since the left hand side goes to zero uniformly as $|z_j| \to 1$. It is conjectured that every Blaschke product can be uniformly approximated by interpolating Blaschke products, but if this were to fail a Blaschke product in $\mathcal{B}_0$ might be a good candidate for a counterexample. Can the Blaschke product constructed in §4 be uniformly approximated by interpolating Blaschke products?

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