NON-RECTIFIABLE LIMIT SETS OF DIMENSION ONE

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ABSTRACT. We construct quasiconformal deformations of convergence groups such that the resulting limit set is a Jordan curve of Hausdorff dimension 1, but having tangents almost nowhere. It is known that no divergence type group has such a deformation. The main tools in proving this are (1) a characterization of tangent points in terms of Peter Jones' β's, (2) a result of Steven Semmes that gives a Carleson type condition on a Beltrami coefficient which implies rectifiability and (3) a construction of quasiconformal deformations of a surface which shrink a given geodesic and whose dilatations satisfy an exponential decay estimate away from the geodesic.

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1. Introduction

In [9] Bowen proved that any deformation of a cocompact Fuchsian group gives a quasi-Fuchsian Kleinian group whose limit set is either a circle or has Hausdorff dimension > 1. This was extended to all divergence type groups by the author in [3] and was shown to be false for all convergence groups by Astala and Zinsmeister in [1]. They showed that all such groups have a deformation such that the limit set is a non-circular rectifiable curve. Zinsmeister asked if Bowen's property could fail in a different way, namely, are there quasi-Fuchsian groups whose limit sets are not locally rectifiable, but still have dimension 1? We will show there are many such groups.

Theorem 1.1. *If $R = \mathbb{D}/G$ is a Riemann surface with Greens function and with bounded injectivity radius, then there is a quasiconformal deformation of G whose limit set has Hausdorff dimension 1 and has tangents on a set of at most 1-dimensional measure zero (and hence it contains no rectifiable subarcs). In fact, given any function $\varphi(t) = o(t)$, we can choose a deformation f_μ so that $\Lambda = f_\mu(\mathbb{T})$ has zero φ-Hausdorff measure.*

First we recall a few relevant definitions and then we describe the proof.

A Fuchsian group G is called convergence type if

$$\sum_{g \in G} 1 - |g(0)| < \infty,$$

and otherwise it is called divergence type. It is well known that G is divergence type iff $R = \mathbb{D}/G$ has a Green's function. A conformal mapping $f : \mathbb{D} \to \Omega$ is called a deformation of the Fuchsian group G if for every $g \in G$, $f \circ g \circ f^{-1}$ is Möbius transformation restricted to Ω. It is called a quasiconformal deformation if f has a quasiconformal extension to the whole plane. If μ is a G invariant Beltrami coefficient on $\mathbb{D}^* = \{|z| > 1\} \cup = \{\infty\}$, i.e., satisfies

$$\mu(g(z)) = \mu(z) g'(z)/|g'(z)|,$$

for all $g \in G$ then there is a corresponding quasiconformal deformation which we will denote f_μ (unique up to Möbius transformations). The injectivity radius of G at $z \in \mathbb{D}$ is $\text{inj}(z) = \frac{1}{2}\rho(z, G^*(z))$, where $G^*(z)$ denotes all the images of z under non-identity elements of G. The hypothesis on the injectivity radius in Theorem 1.1

is not essential; we will see in the course of the proof how it may be weakened and it is likely that the conclusion remains true without any such hypothesis.

If Ω is simply connected, a point $x \in \partial\Omega$ is called an inner tangent point if for every $\theta < \pi$, x is the vertex of a cone in Ω of angle θ and this is not true for any $\theta > \pi$. If $\partial\Omega = \Gamma$ is a Jordan curve then we say x is a tangent point iff it is an inner tangent for both sides of Γ. If Γ is a quasicircle then x being an inner tangent (for either side) is equivalent up to sets of zero linear measure to x being a tangent point.

The support of our Beltrami coefficient μ cannot be compact modulo G. If it were, then results of [8] imply that $\Lambda = f_\mu(\mathbb{T})$ has tangents on a set of positive linear measure (indeed, on full harmonic measure).

Next we describe the proof of Theorem 1.1. The goal is to construct a G-invariant Beltrami coefficient μ on $\mathbb{D}^*$, so that $f_\mu(\mathbb{T})$ has Hausdorff dimension 1, but has tangents almost nowhere (with respect to linear measure).

Suppose R is a Riemann surface and $R = \mathbb{D}/G$, where G is a Fuchsian group. We assume that the injectivity radius of R is bounded above by some $M < \infty$. If we also assumed that there was a positive lower bound for the injectivity radius, then the argument would be easier; much of our effort will be devoted to understanding what happens near very short closed geodesics. Certain other cases (e.g., some infinite covers of a compact surface) are even simpler and could be used if one only wanted a single example as in Theorem 1.1.

A generalized Y-piece in R is a region bounded by three closed geodesics (or punctures) which is homeomorphic to a 2-sphere minus three disks (or points). An X-piece in R is the union of two (not necessarily distinct) Y-pieces which share a common boundary geodesic γ, which is called the central curve of X. Given such a X-piece X_1, γ and any $0 < \epsilon < 1/2$, we will show that there is quasiconformal $f : X_1 \to X_2$ so that $f(\gamma)$ is the central geodesic for X_2 and has length $1 - \epsilon$ times that of γ. Moreover, f is an isometry on the boundary of X_1 and the corresponding Beltrami coefficient $\mu_{\gamma,\epsilon}$ satisfies $\|\mu_{\gamma,\epsilon}\|_\infty < C\epsilon$ and

$$|\mu_{\gamma,\epsilon}(z)| \leq C\epsilon \exp(-2\text{dist}(z,\gamma)). \tag{1}$$

The proof of this estimate takes up the first half of the paper and is only difficult in the case when γ is extremely short, in which case the X-piece has large diameter. Such $\mu_{\gamma,\epsilon}$'s will be the building blocks of the desired Beltrami coefficient μ.

The next step is to show that such a X-piece can be found near any point we want. For example, we will show

Lemma 1.2. *Suppose R is a Riemann surface such that the injectivity radius is bounded above by $M < \infty$ at every point. Then there is a $L = L(M) < \infty$ so that for every point $z \in R$ there is a Y-piece $Y \subset R$ such that each boundary arc of Y has length $\leq L$ and either $z \in Y$ or the distance from z to the thick part of Y is $\leq L$.*

Using this, we can make collections of pairwise disjoint X-pieces which have large harmonic measure with respect to any given point of R. More precisely, given a point $z_n \in R$ and compact set K_n, we will show that we can find a disjoint collection of X-pieces $\{X_{n,j}\}$ with central curves $\{\gamma_{n,j}\}$, which all lie outside K_n and which have large harmonic measure with respect to z_n, i.e., there is a $\delta > 0$ (depending only on M, the injectivity upper bound) so that

$$u_n(z_n) \equiv \omega(z_n, \cup_j \gamma_{n,j}, R \setminus \cup_j \gamma_{n,j}) \geq \delta.$$

Given such a collection, we set $\epsilon_{n,k} = \omega(z_n, \gamma_{n,k}, R \setminus \cup_j \gamma_{n,j})$ and let $\mu_{n,j}$ be the Beltrami coefficient supported on X_j which shrinks the length of $\gamma_{n,j}$ by a factor of $\sqrt{\epsilon \epsilon_{n,j}}$. We then define $\mu_n = \sum_j \mu_{n,j}$ and $\mu = \sum_n \mu_n$.

This definition may look a bit mysterious, but it is quite easy to motivate. In order to show that $f_\mu(\mathbb{T})$ has dimension one, we are going to choose each μ_n so that $f_{\mu_n}(\mathbb{T})$ is rectifiable. To prove that $f_{\mu_n}(\mathbb{T})$ is rectifiable, we will use a theorem of Steven Semmes [15] that this happens if

$$|\mu_n(z)|^2 \frac{dxdy}{1-|z|^2},$$

is a Carleson measure of small norm. This is a Carleson measure if

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |\mu_n(z)|^2 \tilde{g}(z,w) \frac{dxdy}{(1-|z|^2)^2} < \infty,$$

where $\tilde{g}(z,w) = \log|\frac{1-\bar{w}z}{w-z}|$ is the Greens function for the disk. Since $|\mu_n|$ is G-invariant, this integral is the same as

$$\sup_{w \in R} \int_R |\mu_n(z)|^2 g(z,w) dA(z),$$

where g is the Greens function for R and dA denotes hyperbolic area on R. However, this integral (as a function of w) gives a superharmonic function on R whose Laplacian is $|\mu_n|^2$. Thus to make it bounded we want $|\mu_n|^2$ to look like the Laplacian of a bounded subharmonic function on R. This is exactly what our definition of μ_n above does (it is an approximation of Δu_n, where u_n is the harmonic measure of $R \setminus \cup_j \gamma_{n,j}$).

Taking $\{z_n\}$ to be a dense sequence in R and K_n to be a closed r_n-neighborhood of $\cup_{k \leq n} \cup_j X_{j,k}$ we inductively define a sequence of Beltrami coefficients $\{\mu_n\}$ whose supports are far apart and leave every compact set. Using the estimates for the building block $\mu_{n,j}$'s and the definition of $\epsilon_{n,j}$, we can show that

$$\sup_w \int_R \mu_n^2(z) g(z,w) dA(z) < C\epsilon,$$

and hence that

$$\frac{|\mu_n(z)|^2 dxdy}{1-|z|^2},$$

is a Carleson measure on the unit disk of norm at most $C\epsilon$. Thus by Semmes' theorem, for any fixed n, $f_{\mu_n}(\mathbb{T})$ is rectifiable. Combined with the fact that the supports of $\{\mu_n\}$ can be chosen as far apart as we wish, we can then deduce that $f_\mu(\mathbb{T})$ has dimension 1.

On the other hand, we will show that shrinking a closed geodesic γ by a factor of $1-\epsilon$ implies that the corresponding limit set must "wiggle" by a factor of at least $C\epsilon$ when viewed from a point on the geodesic. We will make this idea precise in terms of Peter Jones' β's which measure the deviation of a set from straight line. Jones defines

$$\beta_\Gamma(x,t) = \inf_L \frac{1}{t} \sup_{z \in \Gamma \cap B(x,t)} \operatorname{dist}(z, L),$$

where L is the set of all lines hitting x. It follows from results in [7] that if Γ is a quasicircle, then up to sets of zero 1-dimensional measure, x is a tangent point of Γ iff

$$\int_0^\infty \beta_\Gamma(x,t)^2 \frac{dt}{t} < \infty. \tag{2}$$

For our purposes, it is more convenient to introduce a Möbius invariant version of the β's, namely,

$$\eta_\Gamma(z) = \min(1, \inf_\tau \sup_{x,y \in \Gamma} \log \frac{|\tau(x)|}{|\tau(y)|}),$$

where the infimum is over all Möbius transformations τ such that $\tau(\Phi(z)) = 0$. In other words, η measures the smallest r such that $\partial\Omega$ can be mapped in $\{1 \leq |z| \leq r\}$ with z going to the origin. It is easy to bound η in terms of a sum of β's over different scales and to give an almost everywhere criteria for tangent points in terms of η.

The advantage of η is that it is clearly invariant under Möbius transformations, so if Γ is the limit set of a Kleinian group, then $\eta(z)$ is a well defined function on the quotient Riemann surface. Thus one can think of η as a geometric version of the Schwarzian derivative. We shall prove that if Γ is the limit set of a quasi-Fuchsian group corresponding to a Riemann surface R_1 on one side and its quasiconformal image R_2 on the other and if γ_1 is a closed geodesic in R_1 whose image is closed geodesic γ_2 in R_2 of length $(1+\epsilon)\ell(\gamma_1)$, then $\eta \geq C\epsilon$ at all points in a unit neighborhood of γ_1. In other words, changing the length of a closed geodesic by a definite amount causes a "wiggle" in the limit set of comparable size.

The calculation that shows μ_n corresponds to a Carleson measure of norm a most $C\epsilon$ also shows that its Carleson norm is comparable to ϵ. Thus μ is a sum of measures whose Carleson norms are uniformly bounded away from zero. Moreover, the closed geodesics $\{\gamma_{n,j}\}$ chosen to have large harmonic measure from z_n and $\{z_n\}$ was chosen to be dense in R. Using these facts, we will show that if $\Gamma = f_\mu(\mathbb{T})$, then

$$\int_0^1 \eta_\Gamma(f_\mu(re^{i\theta}))^2 \frac{dr}{1-r} = \infty,$$

for almost every θ. This and the comparison between and η and β allows us to deduce that the integral in (2) diverges almost everywhere and hence that almost no point of $\Gamma = f_\mu(\mathbb{T})$ is a tangent point.

Next we describe the structure of the rest of the paper. It splits roughly into four pieces: Sections 2-7 which construct quasiconformal maps which shrink the length of a given geodesic in R, Sections 8-10 which show there are X-pieces near any point and collections of X-pieces with large harmonic measure with respect to any point and culminate in the definition of μ; Sections 11-14 which prove that $f_\mu(\mathbb{T})$ has tangents almost nowhere; and Sections 15-17 which prove that $f_\mu(\mathbb{T})$ has dimension 1. More precisely,

Section 2: We review some hyperbolic trigonometry.

Section 3: We show that two hyperbolic triangles are quasiconformally equivalent with small constant if the ratio of their side lengths is close to 1.

Section 4: We estimate the dilatation for quasiconformal maps between certain hyperbolic quadrilaterals.

Section 5: We show that certain hyperbolic hexagons can be written a union of a bounded piece and at most three Lambert quadrilaterals.

Section 6: We estimate the dilatation of a quasiconformal maps between Y-pieces in terms of the boundary lengths and the distance to the boundary.

Section 7: We give a few estimates for Greens function on R.

Section 8: We prove that if R has injectivity radius bounded above by M, then every point $z \in R$ is either in a Y-piece with all boundary lengths $\leq L(M)$ or is a bounded hyperbolic distance from the thick part of such a Y-piece.

Section 9: Given a point $z \in R$ and a compact $K \subset R$ we construct a collection of disjoint X-pieces which all lie outside K, put which have large harmonic measure with respect to z.

Section 10: We use the deformations of X-pieces described earlier to define the desired dilatation μ.

Section 11: We prove a lemma about Stolz integrals of Carleson measures.

Section 12: We recall Peter Jones' β's and introduce a Möbius invariant version η. The divergence of a certain integral involving η implies a.e. that a point is not a tangent point.

Section 13: We show that a deformation which shortens a simple closed geodesic by a factor of ϵ forces $\eta(z) > C\epsilon$ at points near the geodesic.

Section 14: We show that $f_\mu(\mathbb{T})$ has tangents almost nowhere.

Section 15: We subdivide the disk into disjoint chord-arc domains, each of which intersects the support of at most one of the μ_n's.

Section 16: We recall a theorem of Steven Semmes about quasiconformal mappings of chord-arc domains and prove that our μ corresponds to a Carleson measure with respect to the stopping time domains of the previous section.

Section 17: We prove that $f_\mu(\mathbb{T})$ has dimension 1.

I thank Michel Zinsmeister for his comments and questions on [4] which led me to consider the questions dealt with here. The results on quasiconformal mappings of Y-pieces are also used in [6] to answer a question raised in [5].

If A, B are quantities that depend on some parameter we write $A \lesssim B$ if the ratio B/A is bounded uniformly independent of the parameter. We will also sometimes write this as $A = O(B)$. Similarly for $\gtrsim$. We write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold and say A and B are comparable.

2. Some hyperbolic trigonometry

In this section we record a few formulas of hyperbolic trigonometry which we will need later.

Recall the sine and cosine rules for hyperbolic geometry (e.g., see page 148 of Beardon's book [2]). Let T denote a hyperbolic triangle with angles α, β, γ and opposite side lengths denoted by a, b, c. See Figure 1. Then we have the Sin Rule,

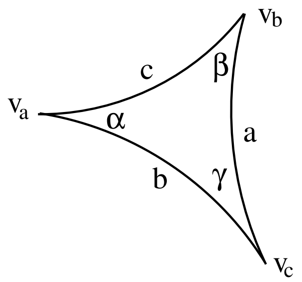

FIGURE 1. Definitions of a, b, c and α, β, γ

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma} \tag{3}$$

the First Cosine Rule,

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma \tag{4}$$

and the Second Cosine Rule

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta} \tag{5}$$

Lemma 2.1. *If $\gamma = \pi/2$ then*

$$\sinh b = \sinh c \sin \beta, \tag{6}$$

$$\tanh b = \sinh a \tan \beta, \tag{7}$$

$$\tanh a = \tanh c \cos \beta. \tag{8}$$

Proof. The first of these is immediate by taking $\gamma = \pi/2$ in (3). To prove (7), note that

$$\begin{aligned} \tanh b = \frac{\sinh b}{\cosh b} &= \frac{\sin\beta}{\sin\alpha}\frac{\sinh a}{\cosh b} \\ &= \frac{\sin\beta}{\sin\alpha}\frac{\sinh a\cosh a}{\cosh c} \\ &= \frac{\sin\beta}{\sin\alpha}\sinh a\cosh a\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\ &= \tan\beta\sinh a[\cosh a\frac{\sin\beta}{\cos\alpha}] \\ &= \tan\beta\sinh a. \end{aligned}$$

where the second, third and fourth equalities hold by (3), (4) and (5) respectively and the last equality holds by (5) (with the roles of a and c reversed). To prove (8), note that

$$\begin{aligned} \tanh a = \frac{\sinh a}{\cosh a} &= \frac{\sinh a\cosh b}{\cosh c} \\ &= \frac{\tanh b\cos\beta\cosh b}{\sin\beta\cosh c} \\ &= \tanh c\cos\beta[\frac{\sinh b}{\sinh c\sin\beta}] \\ &= \tanh c\cos\beta, \end{aligned}$$

where the second, third and last equalities hold by (4), (7) and (3) respectively. □

Formula (5) makes it clear that the side lengths of a hyperbolic triangle are determined by its angles. It is also easy to see that if all three angles are bounded below by some $\theta > 0$, then there is a finite upper bound on the side lengths which only depends on θ. By (5), if all three angles of a hyperbolic triangle are less than δ the the diameter of the triangle is $\geq M(\delta)$ where $M \to \infty$ as $\delta \to 0$.

By (3), if the three sides of a hyperbolic triangle have comparable lengths then the sin of the angles are also comparable. Since at most one of the angles can be close to π we see that if all the angles are bounded away from π then they must all be comparable in size.

3. Quasiconformal mappings of hyperbolic triangles

In this section we record a calculation which we will need later. It simply says that two hyperbolic triangles whose side lengths are pairwise close are quasiconformally equivalent with constant close to 1.

Lemma 3.1. *Suppose $T_1, T_2 \subset \mathbb{D}$ are two triangles in the hyperbolic disk with angles $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ respectively and opposite sides (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively. Suppose that all these angles are strictly positive, say all $\geq \theta > 0$ and let*

$$\epsilon = \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|)$$

Suppose $\epsilon \leq A$. Then there is a constant $C_1 = C_1(\theta, A)$ and a $1+C_1\epsilon$ quasiconformal map from T_1 to T_2 which maps each vertex to the corresponding vertex and which is affine on each edge of T_1 (i.e., it multiplies hyperbolic arclength on each edge by the appropriate factor a_2/a_1, b_2/b_1 or c_2/c_1).

Proof. The angle bisectors of each of the three angles of T_1 meet at a single point ζ_1 in the interior of T_1 by Theorem 7.14.1 of [2]. See Figure 2. This point is equidistant

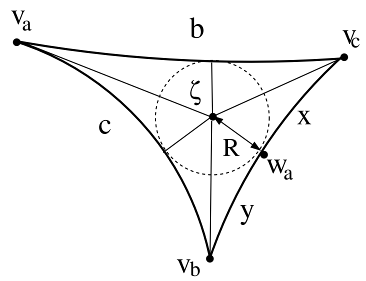

Figure 2. Angle bisectors meet at a point

from each of there three sides of T_1. Let R_1 denote this distance and let w_a^1, w_b^1, w_c^1 denote the closest point to ζ_1 on each of the sides a_1, b_1, c_1 respectively. Divide each edge of T_1 into 2^n equal length segments and connect the endpoints of these segments by geodesics segments to ζ_1. This divides T_1 into $3 \cdot 2^n$ sub-triangles $\{T_{1,k}\}$. See Figure 3. Assume $\zeta_1 = 0$ and for each $k = 1, \dots, 3 \cdot 2^n$ let $\tilde{T}_{1,k}$ be the Euclidean triangle which has the same vertices as $T_{1,k}$ (since $\zeta_1 = 0$, two sides of $\tilde{T}_{1,k}$ will agree exactly with two sides of $T_{1,n}$ but the third side opposite ζ_1 will not). Let ζ_2, R_2, $\{T_{2,k}\}$ and $\{\tilde{T}_{2,k}\}$ be the corresponding objects for T_2 and assume $\zeta_2 = 0$ as well.

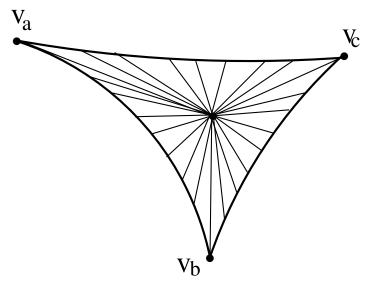

FIGURE 3. The triangles T_k's

We claim that there is a $C = C(\theta) < \infty$ so that each $\tilde{T}_{1,k}$ can be mapped to $\tilde{T}_{2,k}$ by a $1+C\epsilon$ quasiconformal map which is affine in the Euclidean sense. If so, then adjacent triangles have matching boundary values so we get a $1 + C\epsilon$ quasiconformal from $\cup_k \tilde{T}_{1,k}$ to $\cup_k \tilde{T}_{2,k}$. Then taking $n \to \infty$ we obtain in the limit a $1+C\epsilon$ quasiconformal map from T_1 to T_2. By definition this map is affine restricted to the vertices of $E \cap \cup_k \tilde{T}_{1,k}$ for each edge E of T_1, and so by continuity is affine on each edge, as desired. Thus it suffices to verify the claim.

Let L_1 denote an edge of $\tilde{T}_{1,k}$ which connects ζ_1 to an edge E_1 of T_1. Let θ_1 be the angle formed by L_1 and E_1 and let F_1 be the edge of $\tilde{T}_{1,k}$ which is opposite ζ_1. It is clear that θ_1 is minimized when L_1 is one of the angle bisectors of T_1 and hence $\frac{1}{2}\theta \leq \theta_1 \leq \frac{1}{2}\pi$ (where θ is assumed the lower bound on the angles of T_1). Let L_2, E_2, F_2 and θ_2 be the corresponding quantities in T_2. It is enough to check that the Euclidean lengths of these edges satisfy

$$|L_1|/|L_2| = 1 + O(\epsilon), \tag{9}$$

$$|F_1|/|F_2| = 1 + O(\epsilon) \tag{10}$$

$$\theta_1/\theta_2 = 1 + O(\epsilon). \tag{11}$$

(It is easy to check that Euclidean triangles satisfying these conditions are $1 + O(\epsilon)$ quasiconformally equivalent via an affine map.)

We start by showing $R_1/R_2 = 1 + O(\epsilon)$. Let $x_i = \rho(w_a^i, v_c^i)$ and $y_i = \rho(w_a^i, v_b^i)$ so that $x_i + y_i = a_i$ (see Figure 4). Using (7) we see that

$$\tanh R_i = \sinh x_i \tan(\alpha_i/2) = \sinh y_i \tan(\beta_i/2).$$

Since $(x_1 + y_1)/(x_2 + y_2) = 1 + O(\epsilon)$, we must have either $x_1/x_2 \geq 1 - O(\epsilon)$ or $y_1/y_2 \geq 1 - O(\epsilon)$. Without loss of generality assume the first holds. Then

$$\tanh R_1/\tanh R_2 = \sinh x_1 \tan(\alpha_1/2)/\sinh x_2 \tan(\alpha_2/2) \geq 1 - O(\epsilon).$$

Since R_i, x_i are uniformly bounded (in terms of θ) and α_i is bounded away from π (depending only on θ), this implies that $R_1/R_2 \geq 1 - O(\epsilon)$ where the constant depends only on θ. Reversing the roles of R_1 and R_2 shows $R_1/R_2 = 1 + O(\epsilon)$, as desired.

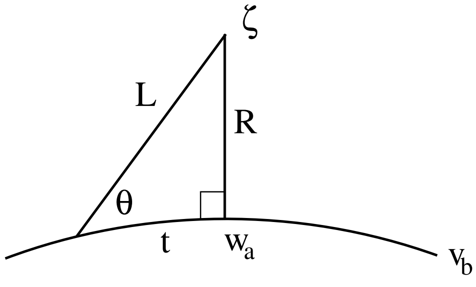

FIGURE 4. Definition of L_i and R_i

Next we prove (9). Suppose L_i connects ζ_i to the edge a_i of T_i and consider the hyperbolic triangle with edge L_i and third vertex w_a^i. See Figure 4. Let t_i be the signed hyperbolic distance from w_a^i to the endpoint of L_i on the edge a_i. Note that $|t_1 - t_2| = O(\epsilon a_1)$. Then by (4), with $\gamma = \pi/2$,

$$\cosh L_i = \cosh t_i \cosh R_i.$$

Now for $0 \leq s \leq 1$ let $t(s) = t_1(1-s) + t_2 s$, $R(s) = R_1(1-s) + R_2 s$ and $\cosh L(s) = \cosh t(s) \cosh R(s)$. Then clearly

$$|t'(s)| = O(\epsilon a_1), \quad |R'(s)| = O(\epsilon a_1),$$

and

$$|L'(s)| \leq \frac{1}{\sinh L(s)}[\sinh t(s) t'(s) \cosh R(s) + \cosh t(s) \sinh R(s) R'(s)] = O(\epsilon a_1).$$

Thus $|L_1 - L_2| = O(\epsilon a_1) = O(\epsilon L_1)$. Thus $|L_1|/|L_2| = 1 + O(\epsilon)$, as desired.

Now we prove (10). The segments F_1, F_2 have hyperbolic diameters which are comparable by a factor of $1 + O(\epsilon)$, since by definition they have hyperbolic length $2^{-n}a_1$ and $2^{-n}a_2$ and $a_1/a_2 = 1 + O(\epsilon)$ by assumption. Moreover, their hyperbolic distances from the origin are L_1 and L_2 which are comparable by the previous paragraph. Since $|dz| = (1-|z|^2)|d\rho|$ this implies their Euclidean lengths are also comparable by a factor of $1+O(\epsilon)$.

Finally we prove (11). By (8),

$$\cos\theta_i \tanh L_i = \tanh t_i,$$

and hence making a parameterized family as above we see

$$\sin\theta(s)\theta'(s)\tanh L(s) + \cos\theta(s)\mathrm{sech}^2 L(s)L'(s) = \mathrm{sech}^2 t(s)t'(s),$$

and since $\theta(s)$ is bounded away from zero in terms of θ, and $|t'(s)|, |L'(s)| = O(\epsilon a)$, we get

$$|\theta'(s)| = O(1)\frac{1}{\tanh L(s)}|\mathrm{sech}^2 t(s) - \cos\theta \mathrm{sech}^2 L(s)|O(\epsilon a) = O(\epsilon).$$

Thus $|\theta_1 - \theta_2| = O(\epsilon)$ and since these numbers are bounded by θ, we get $\theta_1/\theta_2 = 1+O(\epsilon)$, with constant depending only on θ. This ends the proof of Lemma 3.1. $\square$

The preceding lemma can be used to construct quasiconformal maps between polygons by triangulating the polygon and applying the lemma to each piece. Since the maps are affine on the edges, they must agree on the common boundary of any two triangles which share an edge. The following corollaries describe two specific situations where we will use this.

Corollary 3.2. *Suppose Q is a hyperbolic quadrilateral with sides a_1, a_2, a_3, a_4 (in clockwise direction) all bounded by $L < \infty$. Suppose the two angles adjacent to edge a_1 are right angles and the other two angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. Suppose Q' is another quadrilateral with side lengths b_1, b_2, b_3, b_4 whose angles satisfy the same condition. Let $\epsilon = \max_i |\log b_i/a_i|$ and assume $\epsilon < A < \infty$. Then there is a $1 + C(L,\theta,A)\epsilon$ quasiconformal map from Q to Q' which is affine on each of the boundary edges.*

Proof. Connect two opposite corners of Q forming two triangles with edge lengths a_1, a_2, c and a_3, a_4, c. See Figure 5. By (4),

$$\cosh c = \cosh a_1 \cosh a_2. \tag{12}$$

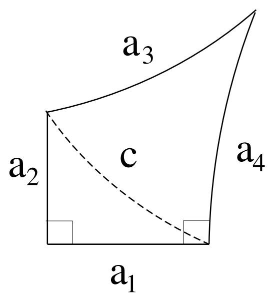

FIGURE 5.

Do the same for Q', obtaining triangles of side length b_1, b_2, d and b_3, b_4, d. Now consider a parameterized family of right triangles on $[0,1]$ with adjacent sides $sa_1 + (1-s)b_1$ and $sa_2 + (1-s)b_2$ and opposite side $c(s)$. Differentiating (12) we get

$$\sinh c \frac{dc}{ds} = \sinh a_1 \frac{da_1}{ds} \cosh a_2 + \cosh a_1 \sinh a_2 \frac{da_2}{ds}.$$

Since c is comparable to a_1 and a_2 we get that $\frac{dc}{da} = O(\epsilon a_1)$. The argument shows $c/d = 1 + O(\epsilon)$. Thus we can apply Lemma 3.1 once we know the angles of the triangles are bounded below uniformly.

However, one angle of the a_1, a_2, c triangle is $\pi/2$ and the side lengths are comparable. Thus by (3) all three angles are comparable (and hence are uniformly bounded below). A similar argument holds for the a_3, a_4, c triangle since one angle (the one between a_3 and a_4) is assumed to be bigger than θ and less than $\pi - \theta$. □

Corollary 3.3. *Suppose H is a hyperbolic hexagon such that all the edge lengths $a_1, \dots, a_6$ (in clockwise direction) are $\leq B$ and are comparable with constant B. Also assume three alternating angles are $\pi/2$ and the remaining angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. Then there is a $C = C(\theta, B)$ so that the following holds. If H' is another such hexagon with edge lengths $b_1, \dots, b_6$ and $\epsilon = \max_i |\log a_i/b_i| \leq 2$, then there is a $1 + C\epsilon$ quasiconformal map from H to H' which is affine on each of the edges of H.*

Proof. Triangulate H by adding edges $[v_2v_4]$, $[v_4v_6]$ and $[v_6v_2]$, as in Figure 6. As in the previous corollary the new edges have length comparable to the edges of H and if the edges of H change by a factor of $1+\epsilon$ then the lengths of the new edges change by at most $1 + O(\epsilon)$. Each of the three triangles which share edges with H has a

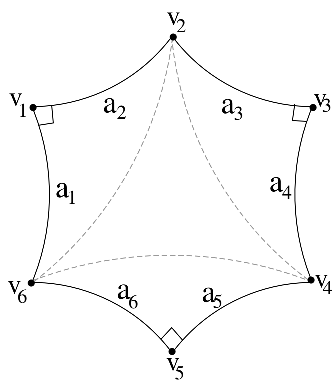

FIGURE 6.

right angle, and hence all their angles are bounded away from zero (since their edges are all comparable).

Thus we need only show the central triangle has angles bounded away from zero. However, all of its angles are smaller than $\pi - \theta$ (since this is true for the hexagon) and since the sides are comparable, so are the angles. Finally, not all three angles can be close to 0 since this would imply the triangle has large diameter (which is impossible since it is contained in H which has diameter bounded by $6B$). $\square$

4. Quasiconformal mappings of Lambert quadrilaterals

In this section we use the result about triangles from the previous section to prove a result about mappings of certain special quadrilaterals. In Section 6 on mappings of Y-pieces these quadrilaterals will correspond to neighborhoods of very short geodesic boundaries and the estimate we prove here will give the exponential decay we need to eventually show our deformations correspond to Carleson measures.

A hyperbolic quadrilateral is called a Lambert quadrilateral if it has three right angles (after J.H. Lambert, 1728-1777, see page 156 of [2]). Following [2] we denote the fourth angle ϕ and the four sides (going clockwise, starting at ϕ) will be labeled b, x, a, y as in Figure 7.

Given such a quadrilateral Q, we will sometimes let the letter a denote the edge of Q and sometimes let it denote the length of the edge. The usage will generally be clear from context, but we will write "a-side" when necessary. Similarly for the other sides of Q.

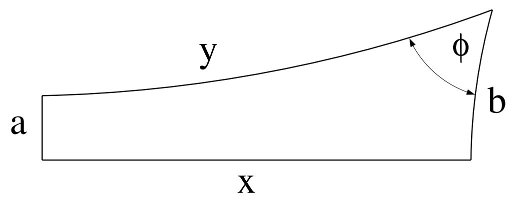

FIGURE 7. A Lambert quadrilateral

The quadrilateral is determined by any two side lengths or one side length and the angle ϕ, as is evident from the following two relations (Theorem 7.17.1 of [2])

$$\sinh a \sinh x = \cos \phi \tag{13}$$

$$\cosh a = \cosh b \sin \phi \tag{14}$$

Using these and standard trigonometric identities it is easy to show

$$\cosh b \sinh x = \sinh y. \tag{15}$$

To prove this use (13) and (14) to get

$$\begin{aligned}
\sin \phi \cosh b &= \cosh a \\
&= (1 + \sinh^2 a)^{1/2} \\
&= (1 + \frac{\cos^2 \phi}{\sinh^2 x})^{1/2} \\
&= (\frac{\cosh^2 x - 1 + \cos^2 \phi}{\sinh^2 x})^{1/2} \\
&= (\frac{\cosh^2 y \sin^2 \phi - \sin^2 \phi}{\sinh^2 x})^{1/2} \\
&= \frac{\sin \phi \sinh y}{\sinh x},
\end{aligned}$$

which proves (15). We will think of b as fixed, but allow a to vary and consider ϕ, x and y as functions of a.

Lemma 4.1. *Suppose Q is a Lambert quadrilateral, labeled as above. If b is fixed and a is allowed to vary then*

$$\frac{b}{a} \simeq \sinh x \tag{16}$$

$$\frac{d\phi}{da} = O(a) \tag{17}$$

$$\frac{dx}{da} = \frac{-1}{\sinh a} + O(a) \tag{18}$$

$$\frac{dy}{da} = \frac{-1}{\sinh a} + O(a) \tag{19}$$

The constants depend only on an upper bound for b.

Proof. We recall an estimate which we will use frequently. Note that if $x > 0$,

$$1 + e^{-2x} \leq \frac{\cosh x}{\sinh x} \leq \frac{1 + e^{-2x}}{1 - e^{-2x}}, \tag{20}$$

and hence $\frac{\cosh x}{\sinh x} = 1 + O(e^{-2x})$ with a constant that depends only on a lower bound for x.

First we prove (16). By (13) and (14),

$$\begin{aligned} \sinh^2 x &= \frac{\cos^2 \phi}{\sinh^2 a} \\ &= \frac{1 - \sin^2 \phi}{\sinh^2 a} \\ &= \frac{\cosh^2 b - \cosh^2 a}{\sinh^2 a \cosh^2 b} \\ &\simeq \frac{b^2}{a^2}, \end{aligned}$$

where we have used $a \leq b/2$ to deduce $\frac{1}{C} b^2 \leq \cosh^2 b - \cosh^2 a \leq C b^2$.

Next we prove (17). By (14)

$$\sin \phi = \frac{\cosh a}{\cosh b} \leq \frac{\cosh b/2}{\cosh b} < 1, \tag{21}$$

with a bound depending only on b. Differentiating the equality gives

$$\frac{d\phi}{da} = \frac{\sinh a}{\cos \phi \cosh b} = O(a). \tag{22}$$

Also note for later use that if $t < s < b/2$ then

$$\phi(s) - \phi(t) = \int_s^t O(a) da = O(|s^2 - t^2|). \tag{23}$$

as desired.

To prove (18), note that by (13) we have

$$\sinh x = \frac{\cos \phi}{\sinh a},$$

and so differentiating and using (13) and (22),

$$\begin{aligned}
\frac{dx}{da} &= -\frac{1}{\cosh x}\frac{-\sin\phi\frac{d\phi}{da}\sinh a - \cos\phi\cosh a}{\sinh^2 a} \\
&= -\frac{1}{\cosh x}\frac{\sinh^2 a/(\cos\phi\cosh b) - \sinh a\sinh x\cosh a}{\sinh^2 a} \\
&= -\frac{1}{\sinh a}[\frac{\sinh a}{\cos\phi\cosh b\cosh x} + \cosh a\frac{\sinh x}{\cosh x}] \\
&= \frac{-1}{\sinh a_2}[O(a^2) + (1+O(a^2))(1+O(e^{-2x}))] \\
&= \frac{-1}{\sinh a_2}(1+O(a^2)),
\end{aligned}$$

where the fourth equality uses the fact (21) that $\cos\phi$ is bounded away from zero (depending only on b) and the fact that (16) implies

$$\frac{1}{C}a \le e^{-x} \le Ca \tag{24}$$

if $a \le b/2$ (for some C depending on b). Also note that

$$|\frac{1}{a} - \frac{1}{\sinh a}| = O(a).$$

Thus integrating we get

$$|x(s) - x(t)| = \int_s^t \frac{1}{a} + O(a)da = |\log\frac{t}{s}| + O(|s^2 - t^2|). \tag{25}$$

Finally, we prove (19). If we differentiate (15) with respect to x we get

$$\cosh b\cosh x = \cosh y\frac{dy}{dx}.$$

Now divide by $\cosh y$ and use (14) to get

$$\frac{dy}{dx} = \cosh a = 1 + O(a^2).$$

The chain rule then gives

$$\frac{dy}{da} = \frac{dy}{dx}\frac{dx}{da} = \frac{-1}{\sinh a}(1+O(a^2))(1+O(a^2)) = \frac{-1}{\sinh a} + O(a),$$

as desired. □

Next we wish to prove

Lemma 4.2. *Assume $b < 1/2$. Suppose Q_s, Q_t are Lambert quadrilaterals with a-sides of length s and t respectively and have equal length opposite sides b in both cases. Assume $t \le s \le b/2$. There is a $C < \infty$ depending only on b so that the following* *holds. There is a quasiconformal mapping $f : Q_s \to Q_t$ which is affine on the edges a and b and whose dilatation satisfies*

$$K(x) \le \begin{cases} Cs/t & \rho(x,a) \le 1 \\ 1 + Cs(s-t)\exp(-2\rho(x,a)), & \rho(x,a) \ge 1. \end{cases}.$$

In particular, if $s/t = 1 - \epsilon$, with ϵ close to 0, the quasiconformal distortion looks like $1 + C\epsilon$ near the edge a and converges to 1 exponentially as we move away from this edge. This is the only case we will need later in this paper.

Proof of Lemma 4.2. We know from (16) that $x = x(s)$ is bounded away from zero uniformly (in terms of b). If $x \le 1$, then the lemma follows from Corollary 3.2. Thus we may assume $x > 1$.

Choose points $x_0, \dots x_n$ along the x-side of Q_s inductively as follows. The point x_0 is the corner of Q_s where the x and b sides meet. The point y_0 is the other endpoint of the b-side. The point x_1 is the point on the x-side which is distance b from x_0. Consider the geodesic segment which is perpendicular to the x-side at x_1 and let y_1 be the point where it hits the y-side of Q_s and let d_1 be its length. Let x_2 be the point on the x-side of Q_s which is distance d_1 from x_1 (and so that x_1 separates x_2 and x_0). In general, given x_{n-1}, consider the geodesic segment to x through x_{n-1}, let y_{n-1} be the point of intersection with the y-side of Q_s and let d_{n-1} be the distance from x_{n-1} and y_{n-1}. Then let x_n be the point on the x-side of Q_s which is distance d_{n-1} from x_{n-1} and which is separated from x_0 by x_{n-1}. See Figure 8.

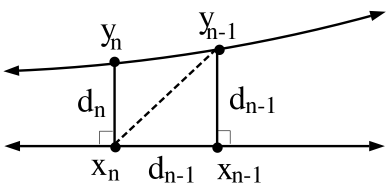

FIGURE 8. Definition of S_{n-1}

Continue until the first time that the distance from x_n to the a-side of Q_s is < 1. Let N denote the index when this occurs. For $n = 1, \dots N$, let $S_n \subset Q_s$ denote the quadrilateral with corners $x_{n-1}, x_n, y_n, y_{n-1}$. Let $T = Q_s \setminus \cup_1^N Q_n$. See Figure 9. Now divide the quadrilateral Q_t in a similar way. Choose points $\tilde{x}_n$ on its x-side so that $\tilde{x}_0$ is the corner of its b and x sides and $\rho(\tilde{x}_n, \tilde{x}_0) = \rho(x_n, x_0)$ for $n = 1, \dots N$. Take the perpendiculars to the x-side of Q_t though these points and let $\tilde{y}_n$ be the points of intersection with the y-side of Q_t and $\tilde{d}$ be their lengths. Let $\tilde{S}_n$ be the corresponding quadrilaterals and let $\tilde{T} = Q_t \setminus \cup_1^N \tilde{S}_n$. See Figure 9

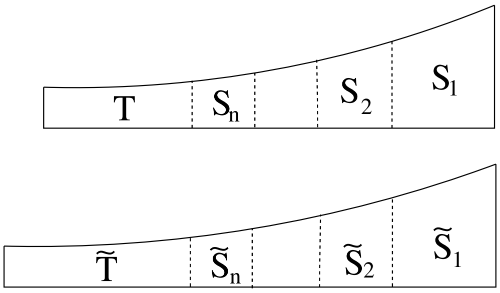

FIGURE 9. Decomposing Q_s and $Q - t$

By Lemma 4.1 we know $|\phi(s) - \phi(t)| \leq C(s^2 - t^2) = O(s(s-t))$. The segment d_n meets the y edges of Q_s and Q_t at angles between ϕ and $\pi/2$. Thus the Sin rule (3) and (16) imply

$$d_n - \tilde{d}_n \leq Cs(s-t)e^{\rho(x_n,x_0)} = C(1 - \frac{t}{s})e^{-2x}e^{\rho(x_n,x_0)} = C(1 - \frac{t}{s})e^{-x-\rho(x_n,a)}.$$

See Figure 10 (in this figure, the vertex of ϕ is at the origin and the b-side for the two quadrilaterals coincide). Using (16) and the fact that d_n is the b-side of a Lambert quadrilateral itself, we see

$$d_n \geq Ca \sinh \rho(x_n, a) \geq Ce^{-x}e^{\rho(x_n,a)} = Ce^{-\rho(x_n,x_0)}.$$

Thus

$$d_n/\tilde{d}_n = 1 + O(e^{-2\rho(x_n,a)}).$$

Thus by Corollary 3.2 there is a $1+O(e^{-2\rho(x_n,a)})$-quasiconformal map from S_n to $\tilde{S}_n$. This gives the lemma on the S_n's.

To prove the lemma on T we consider two cases. First suppose $t \simeq s$, say $s/10 \leq t \leq s$ (this is the only case we need in this paper). Then subdivide the T (which by our choice of N has x-side of length between $1/2$ and 1) into subquadrilaterials with equal length x-sides $\simeq a$. The corresponding side of $\tilde{T}$ is only longer by a factor of $\log \frac{s}{t} + O(a(s-t)) = O(s-t)$ (by Lemma 4.1) and so we can apply Corollary 3.2 and get maps on each sub-quadrilateral with quasiconformal constant $1 + O(s-t)$.

If $t < s/10$ then because T and $\tilde{T}$ are both Lambert quadrilateral themselves, we may place them in the hyperbolic upper half-plane so that the a-sides lie on the

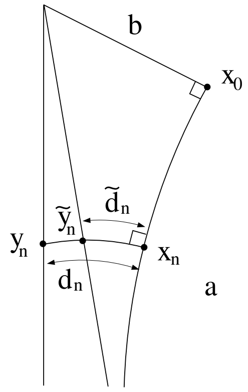

FIGURE 10. Estimating $d_n - \tilde{d}_n$

imaginary axis, the x-side lies on the circle $|z| = 1$ and the y-sides line on circles of the form $|z| = r$ for $r = e^s$ and $r = e^t$. The remaining b sides lie on circles centered on the real axis. See Figure 11. Apply logarithims to conformally may to horizontal strips.

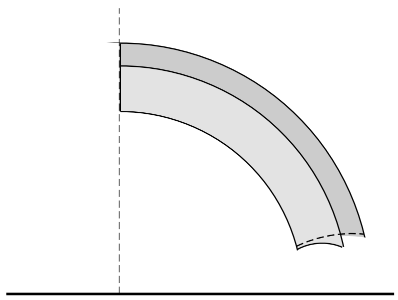

FIGURE 11. T and $\tilde{T}$

We now map T to $\tilde{T}$ in a series of steps as illustrated in Figure 12.

First we map t to $W_1 \subset P_1$ with the b side going to a radial line segment and hyperbolic arclength on this edge being mapped affinely onto arclength of the image. This can be done with quasiconformal constant C_1 bounded only depending on b. Next we map $W_1 \to W_2 \subset \tilde{T}$ by the map $re^{i\theta} \to r^{s/t}e^{i(\frac{p}{i}2-a(\frac{\pi}{2}-\theta))}$ where a is chosen so that $\partial W_1 \cap \{|z| = 1\}$ is mapped onto the a_2 edge of P_λ. Since this edge of W_1 has (Euclidean) length uniformly bounded below, and the (Euclidean) length of the image is at most $\pi/2$, we see that a is uniformly bounded. Thus this map has quasiconformal constant $\leq C_2\frac{s}{t}$, where C_2 is absolute. Finally we map W_2 to P_λ by mapping the radial edge onto the b edge of $\tilde{T}$ with arclength mapping affinely. As

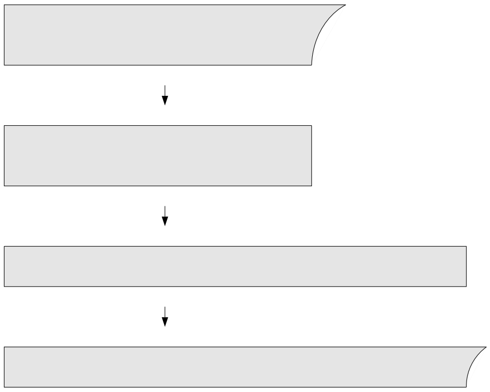

FIGURE 12. The map from T to $\tilde{T}$

above, this can be done with constant C_1 which depends only on b. Composing these steps gives the desired map with a quasiconformal constant of $C_1^2 C_2 \frac{s}{t}$. This finishes the proof of the lemma. $\square$

If $t > s$, then it is easy to formulate and prove the corresponding estimate by considering the inverse of the map constructed above.

5. Decomposing a L-bounded hexagon

Let H be a hyperbolic hexagon with all right angles and suppose the sides have lengths $a_1, b_3, a_2, b_1, a_3, b_2$, ordered counter-clockwise around the boundary (so a_j is opposite b_j). See Figure 13.

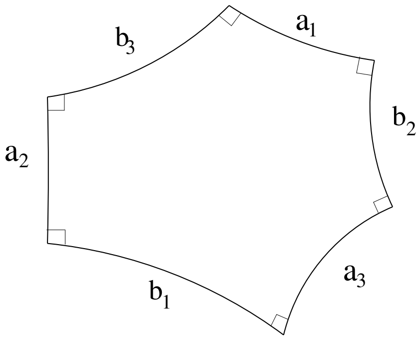

FIGURE 13. Labeling sides of a hexagon

The following two relations are Theorems 7.19.1 and 7.19.2 of [2],

$$\frac{\sinh a_1}{\sinh b_1} = \frac{\sinh a_2}{\sinh b_2} = \frac{\sinh a_3}{\sinh b_3}, \tag{26}$$

$$\cosh b_1 \sinh a_2 \sinh a_3 = \cosh a_1 + \cosh a_2 \cosh a_3. \tag{27}$$

Throughout this section we will assume that the three sides a_1, a_2, a_3 are all bounded by some $L < \infty$. Such a hexagon will be called L-bounded. However, if any of these sides is very small then some of the remaining b-edges will be long. In this section we want to show that any L-bounded hexagon H can be divided into a central hexagon H_0, (all of whose edges are bounded and bounded away from zero in terms of L) and at most three Lambert quadrilaterals.

To accomplish this, suppose we have two numbers a, b (to be chosen later) with $a < \frac{1}{2}b < 1/4$. If $a_j \leq a$, $j = 1, 2, 3$, then form a Lambert quadrilateral, Q_j, with a_j as one side, two other sides lying along the edges b_{j-1} and b_{j+1} (here and below indices are taken mod 3 in $\{1, 2, 3\}$) and the fourth edge has length b and makes a right angle with b_{j-1}. See Figure 14. The unique non-right angle in Q_j will be denotes ϕ_j. Let x_{j-1} be the length of the edge of Q_j which lies in b_{j-1} and let y_{j+1} be the length of the edge of Q_j which lies on b_{j+1}. We shall see below that Q_j is a subset of H if a and b are chosen correctly. If $a_j > a$ then we set $Q_j = \emptyset$ and $x_j = y_j = 0$. Let $H_0 = H \setminus \cup_{j=1}^{3} Q_j$. The six sides of H_0 are in an obvious correspondence with those of H and will say "the side of H_0 in position a_j (or b_j)".

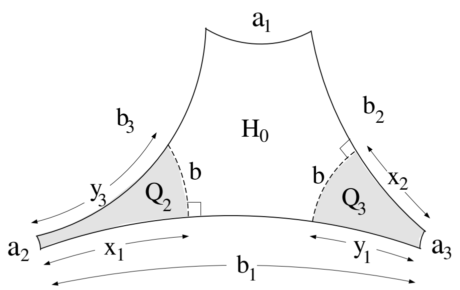

FIGURE 14. A hexagon with two short sides

Lemma 5.1. *There is a choice of a, b so that $H_0 \subset H$ and the side lengths of H_0 are bounded and bounded away from zero depending only on L.*

Proof. Since the sides of H_0 in positions a_1, a_2, a_3 are all $\leq L$ and by construction have length $\geq a$, we only have to check the sides in positions b_1, b_2, b_3. Without loss of generality it suffices to consider just b_1. There are four case depending on whether $a_2 > a$ and $a_3 > a$.

Case 1: First suppose both $a_2, a_3 > a$, so that the b_1-edge of H_0 is the same as the corresponding edge of H, i.e.,has length b_1. By (27),

$$\cosh b_1 = \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \leq \frac{2e^L + 2e^{2L}}{a^2},$$

and so b_1 is bounded above depending only on a and L. On the other hand, by (27) and (20)

$$\begin{aligned}\cosh b_1 = \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} &\geq e^{-2L} + (1 + e^{-2a_2})(1 + e^{-2a_3}) \\ &\geq 1 + 2e^{-2a},\end{aligned}$$

which implies b_1 is bounded below depending only on a.

Case 2: Next suppose $a_2 \leq a$ and $a_3 > a$. Then the side of H_0 in position b_1 has length $b_1 - x_1$. By (13)

$$\sinh x_1 = \frac{\cos \phi_2}{\sinh a_2},$$

and by (27),

$$\begin{aligned}\cosh b_1 &= \frac{1}{\sinh a_2} \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_3} \\ &\geq \frac{1}{\sinh a_2} \frac{\cosh a_3}{\sinh a_3} \\ &\geq \frac{1}{\sinh a_2}.\end{aligned} \tag{28}$$

This implies

$$2e^{b_1} \geq e^{b_1} + e^{-b_1} = 2\cosh b_1 \geq \frac{2}{\sinh a_2} = \frac{4}{e^{a_2} - a^{-a_2}} \geq 4e^{-a_2}$$

and hence

$$b_1 \geq \log \frac{1}{a_2} + \log 2 \geq \log \frac{1}{b} + \log 2, \tag{29}$$

independent of a or L. By (14)

$$\sin \phi_2 = \frac{\cosh a_2}{\cosh b} \geq \frac{1}{\cosh b},$$

so if b is small enough then ϕ_2 is a close as we wish to $\pi/2$ (independent of a). Assume we have fixed b so small that $\cos \phi_2 \leq \frac{1}{2}$ and $b_1 \geq 1$. By (13) and (28)

$$\sinh x_1 = \frac{\cos \phi_2}{\sinh a_2} \leq \frac{1/2}{\sinh a_2} \leq \frac{1}{2} \cosh b_1, \tag{30}$$

and since $b_1 \geq 1$ this implies $b_1 - x_1 \geq C_1$ for some absolute C_1 which does not depend on a, b or L (as long as b satisfies the choices made above).

To prove the upper bound, note that if $a_1, a_2 \leq L$ and $a_3 \geq a$, then

$$\begin{aligned} \cosh b_1 &= \frac{1}{\sinh a_2}\frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_3} \\ &\leq \frac{1}{\sinh a_2}(\frac{\cosh a_1}{\sinh a_3} + \cosh a_2 \frac{\cosh a_3}{\sinh a_3}) \\ &\leq \frac{1}{\sinh a_2}(\frac{e^L}{a} + e^L(1+e^{-2a})) \\ &\leq \frac{A(a,L)}{\sinh a_2}. \end{aligned} \tag{31}$$

Thus

$$\sinh x_1 = \frac{\cos \phi_2}{\sinh a_2} \geq \frac{\cos \phi_2}{A(a,L)} \cosh b_1,$$

which implies $b_1 - x_1 \leq C_2$ for some absolute C_2 which depends on a and L. This proves that $b_1 - x_1$ is bounded above and below in terms of a, b and L.

Case 3: Next assume $a_2 > a$, but $a_3 \leq a$. Since a_2 and a_3 play symmetric roles in (27), the argument above shows that $b_1 - x_2$ is bounded above and below uniformly. We shall now show that $x_2 - y_1$ is small, and hence that $b_1 - y_1$ is also bounded above and below. By (14)

$$\cosh y_1 \sin \phi_3 = \cosh x_2.$$

Since $\sin \phi_3 < 1$, this implies

$$y_1 > x_2. \tag{32}$$

Thus $b_1 - y_1$ is bounded above since $b_1 - x_2$ is. On the other hand,

$$\begin{aligned} \sin \phi_3 &= \frac{\cosh x_2}{\cosh y_1} \\ &= \frac{e^{x_2} + e^{-x_2}}{e^{y_1} + e^{-y_1}} \\ &= e^{x_2 - y_1} \frac{1 + e^{-2x_2}}{1 + e^{-2y_1}}, \end{aligned}$$

and hence

$$\begin{aligned} y_1 &= x_2 + \log\frac{1}{\sin\phi_3} + \log(1+e^{-2x_2}) - \log(1+e^{-2y_1}) \\ &\leq x_2 + \log\frac{1}{\sin\phi_3} + \log(1+e^{-2x_2}) \\ &\leq x_2 + C, \end{aligned}$$

where C may be as small as we wish if b is small enough. Taking it to be less than $\frac{1}{2}C_1$ (which did not depend on b) we get

$$b_1 - y_1 \geq \frac{1}{2}C_1 \tag{33}$$

as our lower bound if b is small enough (independent of L).

Note that the proof of the previous two cases shows that $x_j \leq b_j$ and $y_j \leq b_j$ and hence the quadrilaterals Q_j are really subsets of H, as claimed in the lemma.

Case 4: Finally assume both $a_2, a_3 \leq a$. By (27),

$$\cosh b_1 = \frac{\cosh a_1 + \cosh a_1 \cosh a_3}{\sinh a_2 \sinh a_3} \geq \frac{2}{\sinh a_2 \sinh a_3},$$

whereas by (30),

$$\begin{aligned} \sinh x_1 &\leq \frac{1}{2}\frac{1}{\sinh a_2}, \\ \sinh x_2 &\leq \frac{1}{2}\frac{1}{\sinh a_3}. \end{aligned}$$

Thus

$$\sinh x_1 \sinh x_2 \leq \frac{1}{8}\cosh b_1,$$

and because $b_1 \geq 1$ this implies $b_1 - (x_1+x_2) \geq C_3$, with C_3 independent of b. Choose b so small that $|x_2 - y_1| \leq C_3/2$ and this gives the desired lower bound. On the other hand,

$$\cosh b_1 = \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \geq \frac{e^L + e^{2L}}{\sinh a_2 \sinh a_3},$$

which implies

$$\sinh x_1 \sinh x_3 \geq \frac{1}{A(L,a,b)}\cosh b_1,$$

which implies $b_1 - (x_1 + x_2)$ is bounded above, depending only on L, a, b. Since $y_1 \geq x_2$, this gives the desired upper bound.

Note that this shows the quadrilaterals Q_1, Q_2 and Q_3 are disjoint. This completes all the cases and hence finishes the proof of the lemma. $\square$

Next we record a calculation which we will need later. We will think of the a_1 and a_3 sides of our right hexagon as having fixed length, but as a_2 as varying and we wish to compute the rate of change of the b-sides. Since b_1 and b_3 play symmetric roles we need only compute the derivatives for b_1 and b_2.

Lemma 5.2. *With notation as above*

$$\frac{db_1}{da_2} = \frac{-1}{\sinh a_2}(1 + O(e^{-2b_3})), \tag{34}$$

$$\frac{db_2}{da_2} = O(a_2), \tag{35}$$

where the constants depend only on L, the upper bound for a_1, a_2, a_3.

Proof. If we think of (27) as giving b_1 as a function of a_2 with a_1 and a_3 fixed, and differentiate, we get

$$\begin{aligned}
\frac{db_1}{da_2} &= \frac{\sinh^2 a_2 \cosh a_3 - (\cosh a_1 + \cosh a_2 \cosh a_3)\cosh a_2}{\sinh b_1 \sinh^2 a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{\sinh a_2^2 \cosh a_3 - \cosh a_1 \cosh a_2 - \cosh^2 a_2 \cosh a_3}{\sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{(\sinh a_2^2 - \cosh a_2^2)\cosh a_3 - \cosh a_1 \cosh a_2}{\sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{-\cosh a_3 - \cosh a_1 \cosh a_2}{\sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{-\cosh b_3 \sinh a_1 \sinh a_2}{\sinh b_1 \sinh a_2 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{-\cosh b_3 \sinh a_1}{\sinh b_1 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{-\cosh b_3 \sinh a_3}{\sinh b_3 \sinh a_3} \\
&= \frac{1}{\sinh a_2} \frac{-\cosh b_3}{\sinh b_3} \\
&= \frac{-1}{\sinh a_2}(1 + O(e^{-2b_3})).
\end{aligned}$$

where the fifth equality holds by (27) and the seventh holds by (26) and the constant in the last line only depends on a lower bound for b_3. We know by (27) that

$$\begin{aligned}\cosh b_3 &= \frac{\cosh a_1 + \cosh a_2 \cosh a_3}{\sinh a_2 \sinh a_3} \\ &\geq \frac{1}{\sinh a_2 \sinh a_3} + (\frac{\cosh a_2}{\sinh a_2}\frac{\cosh a_3}{\sinh a_3}) \\ &\geq e^{-2L} + 1,\end{aligned}$$

since $a_2, a_3 \leq L$. Thus b_3 is bounded away from zero depending only on L and so our estimate of the derivative above depends only on L.

To see what happens to the side corresponding to b_2 think of (27) as giving b_2 as a function of a_2 with a_1, a_3 fixed. Differentiating, we get

$$\frac{db_2}{da_2} = \frac{\sinh a_2}{\sinh b_2 \sinh a_1 \sinh a_3}.$$

By Theorem 17.18.1 of [2],

$$\cosh t = \sinh b_2 \sinh a_3,$$

where t is the length of the common orthogonal between sides a_1 and b_1. See Figure 15.

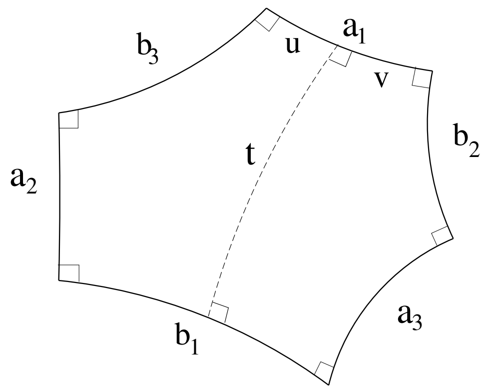

FIGURE 15. Definition of t

Let $u + v = a_1$ be the lengths of the subintervals of a_1 on either side of t as in Figure 15. On page 161 of [2] it is shown that

$$\sinh a_1 \sinh t \geq \sinh u \sinh t = \cosh a_2 \geq 1,$$

and hence

$$\frac{db_2}{da_2} = \frac{\sinh a_2}{\cosh t \sinh a_1} \le \sinh a_2 = O(a_2). \tag{36}$$

□

Lemma 5.3. *Given $L < \infty$, there is a $C = C(L) < \infty$ so that the following holds. Suppose H_1 is right, hyperbolic hexagon and alternate side lengths $a_1, a_2 = s, a_3$ and suppose $s \le L$ (but we do not assume any bound for a_1, a_3). Suppose H_2 is another right hexagon with alternate side lengths $a_1, a_2 = t, a_3$. Assume s and t are both less than a. Then the central hexagons $H_{0,1}$ $H_{0,2}$ of H_1 and H_2 as described above have corresponding sides and angles which differ by less than $C|s^2 - t^2|$.*

Proof. Assume $a_2 \le a$ (if $a_2 \ge a$, the proof simplifies and is left to the reader). Five of the angles in $H_{0,1}$ agree exactly with the corresponding angles in $H_{0,2}$. The only one that may disagree is the angle between the sides in the a_2 and b_3 positions. This angle is complementary to ϕ_2. So by 23 this angle in the two hexagons differs by less than $O(|s^2 - t^2|)$ as desired.

The sides of $H_{0,1}$ and $H_{0,2}$ which correspond to positions a_1, a_2, a_3 are the same length. Thus we need only consider the other three sides b_1, b_2, b_3 (and since b_1 and b_3 play symmetric roles, it is enough to consider only b_1 and b_2). By Lemma 5.2, $\frac{db_2}{da_2} = O(a_2)$. Moreover, the side lengths of the quadrilaterals Q_1 and Q_3 (if they occur) don't depend on a_2 and hence, integrating as in (23), we see that the side lengths of the central hexagons for H_1 and H_2 in position b_2 differ by at most $O(|s^2 - t^2|)$.

Next we consider the sides in position b_1. By the proof of Lemma 5.1 we know $b_3 \ge y_3 \ge x_1$. Thus by (24) we get

$$e^{-2b_3} \le e^{-2x_1} \le a_2^2.$$

Hence, from (18) and (34) we deduce

$$\frac{dx_1}{da_2} - \frac{db_1}{da_2} = \frac{-1}{\sinh a_2}(1 + O(a_2^2)) - \frac{-1}{\sinh a_2}(1 + O(a_2^2)) = O(a_2).$$

Since y_1 does not depend on a_2, this implies the sides of the central hexagons of H_1 and H_2 corresponding to b_1 differ by $O(|s^2 - t^2|)$.

Finally we have to consider the sides in position b_3. By (19),

$$\frac{dy_3}{da_2} = \frac{dy_3}{dx_1}\frac{dx_1}{da_2} = \frac{1}{\sinh a_2}(1 + O(a_2^2)),$$

with constant depending only on b. Since the roles of b_1 and b_3 are symmetric, we also have $\frac{db_3}{da_2} = (1 - O(a_2^2))/\sinh a_2$ and so we can complete the proof just as in the previous case. □

Lemma 5.4. *Suppose H_1 and H_2 are two L-bounded right hexagons with a-sides (a_1^1, a_2^1, a_3^1) and (a_1^2, a_2^2, a_3^2) respectively. Assume $a_1^1 = a_1^2$, $a_3^1 = a_3^2$ and $a_2^1 \leq a_2^2 = (1-\epsilon)a_2^1 \leq a_2^1$. Then there is a quasiconformal map $f : H_1 \to H_2$ which is affine on the a-sides, is isometric on the boundary quadrilaterals associated to a_1^1 and a_3^1 and has dilatation satisfying*

$$K_f(z) \leq 1 + C\epsilon \exp(-2\rho(z, a_1^1)).$$

Proof. Since the quadrilaterals corresponding to sides a_1 and a_3 are identical for both hexagons, we may clearly take the map to be isometric in them. If $a_1^1 \geq a$ (where a is as in Lemma 5.1) then apply Lemma 3.3 to $H_1 \setminus (Q_1 \cup Q_3)$ using the estimates of Lemma 5.3. This gives the desired estimate since this region has bounded diameter. If $a_1^1 < a$ then apply Lemma 3.3 to $H_1 \setminus (Q_1 \cup Q_2 \cup Q_3)$ to get the estimate in the central hexagon and apply Lemma 4.2 to the quadrilateral Q_1. □

6. Quasiconformal mappings of Y-pieces

A Y-piece is a bordered Riemann surface which is topologically a sphere with three disks removed and in which each of the three boundary components is a hyperbolic geodesic. It is well known that a Y-piece is determined up to isometry by the lengths $(a, b, c) \in (0, \infty)^3$ of the three boundary components. A generalized Y-piece is similarly defined, except that we also allow boundaries of length zero, i.e., instead of removing a disk we may remove a point. As before, a generalized Y-piece is determined by the three side lengths $(a, b, c) \in [0, \infty)^3$. We will say a Y-piece is L-bounded if all three boundary components have length $\leq L$ (a puncture being considered as length zero).

We will think of two Y-pieces as being close to each other if their boundary lengths are close, i.e., we define

$$d(Y_1, Y_2) = \max(|\log \frac{a_1}{a_2}|, (|\log \frac{b_1}{b_2}|, (|\log \frac{c_1}{c_2}|).$$

For generalized Y-pieces we interpret $|\log \frac{a_1}{a_2}|$ as zero if $a_1 = a_2 = 0$ and as $+\infty$ if one is zero and the other is not. Similarly for the b and c terms.

A Y-piece can always be realized as a hyperbolic octagon with three side pairings. Moreover, there is a line of symmetry which divides the octagon into two isometric right hexagons. The alternate sides of these hexagons are given by $a_1 = a/2$, $a_2 = b/2$ and $a_3 = c/3$. See Figure 16. The part of Y corresponding to the central hexagons

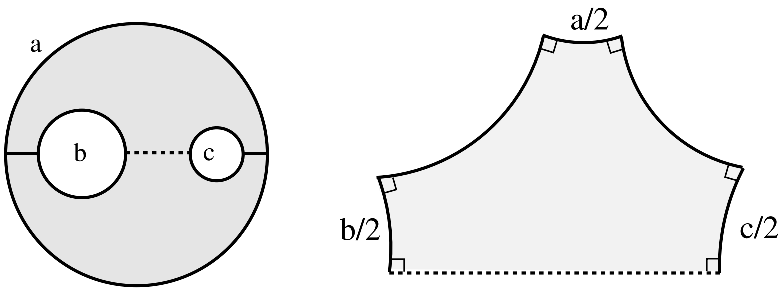

FIGURE 16. Parameterizing a Y-piece

will be called the central piece of Y. Note that the diameter of the central part is bounded depending only on L. The parts of Y corresponding to the quadrilaterals Q_j will be called the boundary pieces of Y. Given $\epsilon > 0$, the thin part of R consists of the points where the injectivity radius of is $\leq \epsilon$ and the thick part is where the injectivity radius is $> \epsilon$. The following is easy and its proof is left to the reader.

Lemma 6.1. *Given $L < \infty$, there is an $\epsilon > 0$ in the definition of the thick and thin parts of R and a $C < \infty$, so that for any L-bounded Y-piece, the ϵ-thin part of $R \cap Y$ is contained in the boundary pieces of Y and the ϵ-thick part of $R \cap Y$ is contained in a C-neighborhood of the central piece.*

For the remainder of this paper, assume that ϵ in the definition of thin parts has been fixed so the conclusion of the lemma holds.

Lemma 6.2. *Suppose Y_1 and Y_2 are two L-bounded generalized Y-pieces with boundary lengths (a_1, b_1, c_1) and (a_2, b_1, c_1) respectively. Assume that $d(Y_1, Y_2) = \epsilon =$*

$|\log a_1/a_2| \leq 2$. *Then there is a* $C = C(L)$ *and a quasiconformal map* $f : Y_1 \to Y_2$ *with constant* $K = 1 + C\epsilon$ *which is affine on each of the boundary components. Moreover, the Beltrami coefficient* μ *of* f *is supported on the boundary piece associated to the* a_1 *boundary component* γ *and the central part of* Y_1 *and it satisfies*

$$|\mu(z)| \leq C\epsilon \exp(-2\mathrm{dist}(z, \gamma))$$

In other words, if a is small, the corresponding dilatation decays exponentially fast away from the boundary and is zero in the thin parts (if they exist) corresponding to the two other boundary pieces.

Proof. For Y-pieces with three non-degenerate boundary components, the proof is immediate from the results of the previous sections (write it as a union of two symmetric hexagons, apply the previous results and use symmetry to show the map is continuous on the Y-piece). If there are one or more punctures on the boundary, then we obtain the result by taking limits over a sequence of non-degenerate Y-pieces that approach the generalized Y-piece. □

The following is an easy consequence which is used in the paper [6]

Lemma 6.3. *Suppose* Y_1 *and* Y_2 *are two* L*-bounded generalized* Y*-pieces and let* $D = d(Y_1, Y_2)$. *Then there is a quasiconformal map* $f : Y_1 \to Y_2$ *with constant* $K = K(L, D)$ *which is affine on each of the boundary components. Moreover, the dilatation* K_f *of* f *satisfies*

$$|K_f(z)| \leq 1 + C(L, D) \exp(-2\mathrm{dist}(z, \partial Y))$$

7. Estimates for Greens function

We have now finished the first part of the paper, by constructing quasiconformal mappings of Y-pieces which satisfy certain estimates. In the remainder of the paper we will use these mappings to construct the group deformation in Theorem 1.1.

In this section we record some simple estimates related to Greens function and Beltrami coefficients satisfying the exponential decay estimates of Section 6. Most of what we need follows from two easy facts. First, Harnack's inequality, which says that if u is a positive harmonic function on the unit disk then

$$u(x) \leq C e^{\rho(x,y)} u(y).$$

Second, the observation that if $x \in R$ lifts to the origin in the disk, then

$$\alpha(x) \equiv \text{area}(\{z \in R : \rho(x,z) \leq 1\}) \simeq \min(1, \text{inj}(x)) \simeq \frac{1}{N},$$

where N is the number of orbits of 0 in the hyperbolic ball $B_\rho(0,1)$

Lemma 7.1. *Suppose g is the Greens function for a hyperbolic Riemann surface. Then there is C_1, such that for any $x, w \in R$,*

$$\frac{1}{C_1} \leq \frac{\int_{B(x,1)} g(z,w)dA(z)}{\alpha(x)\min_{z\in B(x,1)} g(z,w)} \leq C_1. \tag{37}$$

Proof. This is simply Harnack's inequality if $\rho(x,w) \geq 2$, so we may assume $\rho(x,w) < 2$. To prove (37), lift g to the disk so that the point x lifts to the origin. Then g is a sum over G orbits of the Greens function for the disk

$$g(z,w) = \sum_n \tilde{g}(z, w_n),$$

where $\{w_n\}$ is an enumeration of the orbit of zero and

$$\tilde{g}(z,w) = \log |\frac{1-\bar{w}z}{z-w}|.$$

Spit the sum into two terms, $g_1 + g_2$, corresponding to orbits which are more or less than hyperbolic distance 2 from the origin. The sum g_1 over the far orbits is harmonic on a unit hyperbolic ball around the origin and so (37) holds by Harnack's inequality. The number N of orbits inside the ball of radius 2 is bounded by a constant divided by the injectivity radius at x and it is easy to see that

$$\int_{B(0,1)} g_1 dA \simeq N,$$

$$\min_{B(0,1)} g_1 \simeq N,$$

which proves the desired inequalities since $\alpha(x)$ is the area in R of $B(x,1) \subset R$. $\square$

Lemma 7.2. *Suppose g is the Greens function for a hyperbolic Riemann surface. Then there is C_2, such that for any $x, y, w \in R$, such that $1 \leq \rho(w,x) \leq 4$, $1 \leq \rho(w,y) \leq 4$,*

$$\frac{1}{C_2} \leq \frac{g(x,w)}{g(y,w)} \leq C_2. \tag{38}$$

Proof. The proof is almost the same as the proof of (37). Lift g to the disk so the pole lifts to the origin and divide the sum into the orbits that are further than distance 5 from the origin and those that are closer than distance 5. For the first sum the result holds by Harnack's inequality. For the second sum it holds because the sum is comparable to N (the number of terms in the sum) at both x and y. □

Lemma 7.3. *Suppose g is the Greens function for a hyperbolic Riemann surface . Then there is C_3, such that for any $x, y, w \in R$,*

$$\min_{z \in B(x,1)} g(z,w) \leq C_3 e^{\rho(x,y)} \min_{z \in B(y,1)} g(z,w). \tag{39}$$

Proof. There are a few cases, depending on the location of the pole w with respect to x and y. If $\rho(y,w) \geq \rho(y,x)+1$, then the desired inequality follows directly from Harnack's inequality since g is positive and harmonic on $\{z : \rho(y,z) < \rho(y,w)\}$.

If $\rho(w,y) \leq 1$ then choose a point z_1 on the geodesic from y to x which is distance 2 from y. Then by Harnack's inequality and (38),

$$\min_{z \in B(1,x)} g(z,w) \leq Cg(x,w) \leq Ce^{\rho(x,z_1)} g(z_1,w) \leq Ce^{\rho(x,y)} \min_{z \in B(y,1)} g(z,w).$$

Similarly if w is close to x. Finally, if w is more than distance 1 from both x and y, we can choose two points z_1, z_2 on the geodesic from y to x such that

$$\rho(z_1,z_2) \leq 2,$$
$$\rho(z_1,w) \geq 1, \qquad \rho(z_2,w) \geq 1,$$
$$\rho(y,z_1) \leq \rho(y,w)-1, \qquad \rho(x,z_2) \leq \rho(x,w)-1.$$

This is an easy exercise with the triangle inequality. Just as above we deduce that

$$g(y,w) \geq Ce^{-\rho(y,z_1)} g(z_1,w),$$
$$g(x,w) \leq Ce^{\rho(x,z_2)} g(z_2,w).$$

Thus it suffices to show the values at z_1 and z_2 are comparable, but this follows from (38) and Harnack. □

Lemma 7.4. *Suppose $Y \subset R$ is a L-bounded Y-piece and g is the Greens function for R. Suppose μ is bounded function which satisfies the estimate in Lemma 6.2 with respect to a boundary geodesic $\gamma \subset \partial Y$. Then for any $w \in R$*

$$\int_Y |\mu(z)|^2 g(z,w)\, dA(z) \leq C\epsilon^2 \int_\gamma g(z,w)\, d\sigma(z),$$

where $d\sigma$ denotes hyperbolic arclength measure.

Proof. As functions of w, the integrals are both harmonic off Y and tend to zero at infinity. Thus we need only consider $w \in Y$ by the maximum principle. By Lemma 7.1,

$$\int_{B(x,1)} g(z,w)dA(z) \simeq \mathrm{inj}(x) \min_{B(x,1)} g(z,w).$$

By Lemma 7.3

$$\min_{B(x,1)} g(z,w) \leq Ce^{\rho(x,y)} \min_{B(y,1)} g(z,w)$$

Suppose γ has length $\simeq e^{-N}$. Then the boundary piece of Y corresponding to the boundary component $\gamma \subset \partial Y$ has diameter $\simeq N$. Let $A_n = \{z \in Y : n-1 \leq \rho(z,\gamma) \leq n\}$. Note that area$(A_n) \leq Ce^{n-N}$. Thus

$$\begin{aligned}
\int_Y \exp(-4\rho(z,\partial Y_0))g(z,w)dA(z) &\leq \sum_{n=1}^{N} \int_{A_n} e^{-4n} g(z,w)dA(z) \\
&\leq \sum_{n=1}^{N} e^{-N+n} e^{-4n} e^n \min_{z\in\gamma} g(z,w) \\
&\leq Ce^{-N} \min_{z\in\gamma} g(z,w) \\
&\leq C\int_\gamma g(z,w)d\sigma(w).
\end{aligned}$$

□

8. Finding Y-pieces near a given point

In this section, we will show that we can find L-bounded Y-pieces near any given point of R.

Lemma 8.1. *Suppose R is a Riemann surface such that the injectivity radius is bounded above by $M < \infty$ at every point. Then there is a $L = L(M) < \infty$ so the following holds. Suppose γ is a closed geodesic of length $\leq 2M$ and ν is a normal direction to γ. Then there is a generalized L-bounded Y-piece $Y \subset R$ such that γ is one boundary component of Y, and ν points into Y.*

Proof. Suppose γ has hyperbolic length L and lifts to the imaginary axis in the upper half-plane, $\mathbb{H}$. Without loss of generality assume the normal ν points to the right. The group element corresponding to γ is $g(z) = e^L z$. Consider the segments

$\gamma_\theta = [e^{i(\frac{\pi}{2}-\theta)}, e^{L+i(\frac{\pi}{2}-\theta)})$ for $\theta \in [0, \frac{\pi}{2})$. Clearly γ_0 projects to γ in R and each γ_θ projects to a closed curve $\tilde{\gamma}_\theta$. Since R has injectivity radius bounded above, there is smallest θ when $\tilde{\gamma}_\theta$ is not a simple curve and this can be bounded away from $\pi/2$ depending only on M. Thus we can split $\tilde{\gamma}_\theta$ into two closed curves each with length bounded depending only on M. Shrinking each of these to a closed geodesic (or puncture) gives the desired Y-piece. $\square$

The same proof, with only minor changes, also shows the following.

Lemma 8.2. *Suppose R is a Riemann surface such that the injectivity radius is bounded above by $M < \infty$ at every point. Then there is a $L = L(M) < \infty$ so the following holds. Suppose P is a parabolic thin part of R. Then there is a L-bounded Y-piece $Y \subset R$ such that $P \subset Y$.*

The last two lemmas imply that every point of R is near a Y-piece with bounded boundaries in the following sense.

Lemma 8.3. *Suppose R is a Riemann surface such that the injectivity radius is bounded above by $M < \infty$ at every point. Then there is a $L = L(M) < \infty$ and $C = C(M) < \infty$ so that for every point $z \in R$ there is a L-bounded Y-piece $Y \subset R$ such that either $z \in Y$ or the distance from z to the thick part of Y is $\leq C$.*

Proof. First suppose z is in a parabolic thin part of R. Then by Lemma 8.2 there is a Y-piece with the desired estimates containing z. Next suppose z is in a hyperbolic thin part P and let γ be the central simple closed geodesic of the part. Then γ cuts P into two pieces, one of which contains z. Now apply Lemma 8.1 with ν pointing into the piece containing z to obtain a Y-piece containing z. Finally, assume z is in the thick part of R. Then there is a simple closed loop $\tilde{\gamma}$ passing through z and of length $\leq 2M$. Let γ be the simple closed geodesic or puncture corresponding to $\tilde{\gamma}$ and note that z is either a bounded distance from γ or from the thin part corresponding to γ. Using either Lemma 8.1 or 8.2 we get a Y-piece whose thick part is only a bounded distance from z (and the bound depends only on M). $\square$

Recall that if Ω is a domain, $z \in \Omega$ and $E \subset \partial\Omega$, then $\omega(z, E, \Omega)$ denotes the harmonic measure of E in Ω with respect to z. For a fixed E this is a harmonic function on Ω, and for a fixed z is a probability measure on $\partial\Omega$.

Lemma 8.4. *Suppose Y is a L-bounded Y-piece. Then given $r > 0$ there is a $\delta_1 = \delta_1(r, L)$ and a boundary geodesic $\gamma \subset \partial Y$ such that $\omega(z, \gamma, R \setminus \gamma) \geq \delta_1$ for every $z \in R$ such that $\rho(z, Y) \leq r$.*

Proof. First note that a Brownian motion starting at z has a probability of hitting Y which is bigger than $\delta(r) > 0$, depending only on r. Second, a Brownian motion starting in Y has probability at least a $1/2$ of reaching the thick part of Y (since starting on short geodesic it has equal chance of exiting the thin part in either of the two boundary components and if it starts in the thin part in Y, then it has a $\geq 1/2$ chance of exiting the thin part via the thick part of Y). Since the diameter of the thick part of Y is bounded, if we fix a boundary component of Y its harmonic measure will be comparable at any two points of the thick part (by Harnack's inequality), say with a constant $C = C(L)$. Finally, fix a point of the thick part and note that one of the three boundary components of Y must have harmonic measure at least $1/3$. Thus for our original point z the probability of hitting this boundary curve is at least $\delta_1 = \delta(r)\frac{1}{2}\frac{1}{C}\frac{1}{3}$. □

9. Find Y-pieces with large harmonic measure

In this section we will show that given any point z of R we can find disjoint collection of X-pieces which are far from z, but have large harmonic measure with respect to z.

Lemma 9.1. *Suppose R has injectivity radius bounded by M, $z_0 \in R$ and $K \subset R$ is compact. Then there is an $\delta_2 = \delta_2(M)$ and a finite disjoint collection of simple closed geodesics $\{\gamma_n\}$ of length $\leq L = L(M)$ (the constant from Lemma 8.3) so that*

$$\sum_n \omega(z_0, \cup_n \gamma_n, R \setminus \cup_n \gamma_n) \geq \delta_2.$$

Moreover, if Y_n is a L-bounded Y-piece with γ_n as a boundary curve, then $Y_n \cap K = \emptyset$.

Proof. Fix $s > 0$ so large that $K \subset B(z_0, s)$ and consider all hyperbolic thin parts that hit $B(z_0, s + D(L) + 1)$ (where $D(L)$ is the bound on the diameter of the thick part of a L-bounded Y-piece). This is a finite collection so the union is compact, call it K_1. Now take a $D(L) + 1$ neighborhood of K_1 call it K_2. Then take all the hyperbolic thin parts which hit it and let the union be K_3. This is compact, so there is a ball $B(z_0, t)$ which contains it. We claim that if γ is a simple closed geodesic which is outside $B(z_0, t)$ then any L-bounded Y-piece, of which it is a boundary component, is disjoint from K.

To prove this claim, note that if Y hits K in its thick part, then it is contained in K_1. If it hits K with a thin part, then the thick part of Y is contained in K_2 and so Y is contained in K_3. Thus γ must be in K_3, contrary to assumption. This proves the claim.

Now choose $\epsilon > 0$ so small that $E = \{z : g(z, z_0) = \epsilon\}$ is outside $B(z_0, t)$ (recall g is the Greens function with pole at z_0). We can do this because g is strictly positive on any compact set. Clearly E has harmonic measure 1 with respect to z_0. For each $z \in E$ let Y_z be the Y-piece associated to z by Lemma 8.3. By Lemma 8.4 there is a geodesic boundary component $\gamma = \gamma_z$ which has harmonic measure at least δ_1 with respect to z. Let $\{\gamma_n\}$ be an enumeration of the geodesics which arise in this way (we will show it is a finite collection in Lemma 9.2). Then clearly,

$$\omega(z_0, \cup_n \gamma_n, R \setminus \cup_n \gamma_n) \geq \delta_1 \omega(z_0, E, R \setminus E) = \delta_1,$$

as desired. $\square$

Lemma 9.2. *Given $L < \infty$ and a compact set K, there is an $N_1 = N(L, K)$ such that K can intersect at most N_1 simple closed geodesics of length $\leq L$.*

Proof. By the Collar lemma (e.g., [13], [14]), each simple closed geodesic γ of length $\leq L$ has a $\delta = \delta(L)$ annular neighborhood A with the property that γ is the only simple closed geodesic contained entirely in A. Let $\epsilon > 0$ and suppose that K can be covered by n_1 balls of size ϵ. Let B be one of these balls and suppose more than $2\pi n_2 + 1$ simple closed geodesics hit B. Choose one such, say γ_0. Then either there is a geodesic γ_1 hitting B which is disjoint from γ_0 or there are two which both hit γ_0 and the angles they make with γ_0 differ by less than $1/n_2$. In the first case, since both γ_0 and γ_1 have length $\leq L$, the two geodesics must remain within distance $\epsilon + \epsilon e^L$ of each other. So if ϵ is small enough (depending on L and δ), they lie in each others δ collars and hence are the same geodesic. A similar argument works if there are two geodesics which both hit γ_0 within ϵ of each other and with angles differing by $1/n_2$. Thus there must be fewer than n_2 simple closed geodesics of length $\leq L$ hitting B, and hence fewer than $n_1 n_2$ which hit K. $\square$

Lemma 9.3. *Suppose $K \subset R$ is a set of diameter at most D. Then the number of distinct thin parts which can intersect K is bounded by $N_2 = N_2(D)$.*

Proof. Without loss of generality we may assume K is connected (since we can make it connected without increasing its diameter). If K is contained in a single thin part, then we may take $N_3 = 1$. Otherwise, if K intersects a part, it also intersects a component of the part's boundary. Then such a boundary component is also a boundary of an annulus inside the thin part of at most unit width and a fixed hyperbolic area. Thus for each thin part K intersects, the 1-neighborhood of K intersected with the part contains a set of fixed area. Since this neighborhood has diameter $\leq D + 2$ its hyperbolic area is bounded in terms of D and hence K can only intersect a bounded number of thin parts. □

Lemma 9.4. *Suppose γ is a simple closed geodesic of length $\leq L$. Then γ can be a boundary component of at most $N_3 = N_3(L)$ distinct L-bounded Y-pieces.*

Proof. First suppose γ is in the thick part of R and γ is a boundary component of Y. If γ_1 is one of the other two boundary components (either a geodesic or puncture), then either it hits a $D(L)$ neighborhood of γ (if it is also in the thick part) or its associated thin part hits the $D(L)$-neighborhood of γ. In either case, the number of possible such geodesics is bounded in terms of L by Lemma 9.2 and Lemma 9.3.

If γ is in a thin part (or is a puncture), then a similar argument works except that we have to consider $D(L)$ neighborhoods of the two boundary components of the thin part associated to γ. □

We will say an X-piece is L bounded if it is the union of two L-bounded Y-pieces.

Lemma 9.5. *There is a $N = N(L)$ such that a L-bounded X-piece can intersect at most N distinct L-bounded X-pieces.*

Proof. Since a X-piece is a union of two Y-pieces, it clearly suffices to prove the result for Y-pieces instead. So fix a L-bounded Y-piece Y_0 and suppose Y_1 is another L-bounded Y-piece which intersects it. Since one Y-piece cannot properly contain another, a boundary component of one must intersect a boundary component of the other. By Lemma 9.2 each boundary component of Y_0 can intersect at most N_1 distinct simple closed geodesics of length $\leq L$. Moreover, each such geodesic can be a boundary component of at most N_3 distinct Y-pieces. Thus Y_0 can intersect at most $3N_1N_3$ distinct Y-pieces. □

Lemma 9.6. *Suppose R has injectivity radius bounded by M, $z_0 \in R$ and $K \subset R$ is compact. Then there is an $\delta_3 = \delta_3(M)$ and a finite disjoint collection of simple closed geodesics $\{\gamma_n\}$ of length $\leq L = L(M)$ (the constant from Lemma 8.3) so that*

$$\sum_n \omega(z_0, \cup_n \gamma_n, R \setminus \cup_n \gamma_n) \geq \delta_3.$$

Moreover, each γ_n is contained in an L-bounded X-piece X_n such that that the collection $\{X_n\}$ is pairwise disjoint and all are disjoint from K.

Proof. This is immediate from Lemma 9.1, except for the disjointness of the X_n's. Take the collection of X-pieces given by Lemma 9.1 and order then in decreasing order, according to the harmonic measure $\gamma(z_0, \gamma_n, R \setminus \cup_n \gamma_n)$. Form a disjoint subcollection inductively as follows. Take the first X-piece (i.e., the one with geodesic of largest harmonic measure) and remove from the list all later X-pieces which hit it. Then choose the next remaining element and remove the X-pieces which hit it. Continue in this way. The resulting collection is obviously pairwise disjoint and since at each stage we are removing at most N (as in Lemma 9.5) geodesics whose harmonic measure is less than a geodesic we keep, we see that the harmonic measure of the remaining geodesics is at least $\delta_3 = \delta_2/(N+1)$. This completes the proof. □

10. Defining μ by potential theory

In this section we define the desired Beltrami coefficient μ. We will write $\mu = \sum_n \mu_n$ where $\mu_n = \sum_j \mu_{j,n}$ and each $\mu_{j,n}$ is a Beltrami coefficient supported on pairwise disjoint X-pieces $X_{j,n}$ which shrinks a simple closed geodesic $\gamma_{j,n} \subset X_{j,n}$ by a factor of $1 - \epsilon_{j,n}$. The delicate part is that given $\epsilon > 0$ we want to choose the $\epsilon_{j,n}$'s so that the resulting Beltrami coefficients $\{\mu_n\}$ correspond to Carleson measures $|\mu_n(z)|^2(1-|z|)^{-1}dxdy$, with norms $\simeq \epsilon$. We will make this selection using potential theory as our guide.

Suppose we are given an $\epsilon > 0$, $0 < A < 1$, a dense sequence $\{z_n\} \subset R$ in which each elements appears infinitely often and an increasing sequence $\{r_n\} \subset [0, \infty)$. Define a collection of X-pieces $\{X_{j,1}\}$ with central curves $\{\gamma_{j,1}\}$ by applying Lemma 9.6 to the point z_0 and the compact set $K_1 = \{z : \rho(z, z_0) \leq 1\} \subset R$. In general, if we have already defined a collection $\{X_{j,n-1}\}$, let E_n be a closed r_n-neighborhood of $\cup_{k<n} \cup_j X_{j,k}$, let $\delta_n = \min_{E_n} g(z, z_0)$ (where g denotes the Greens function) and let K_n be the closed r_n neighborhood of $\{z : g(z, z_0) \geq A\delta_n\}$. Now define the collection of X-pieces $\{X_{j,n}\}$ and geodesics $\{\gamma_{j,n}\}$ by applying Lemma 9.6 to the point z_n and the the compact set K_n.

Now fix a n and let $\Gamma_n = \cup_j \gamma_{j,n}$ where $\gamma_{j,n}$ is the simple closed geodesic contained in $X_{j,n}$ and given by Lemma 9.6. Let

$$u_n(z) = \omega(z, \Gamma_n, R \setminus \Gamma_n).$$

This is a superharmonic function on R, is harmonic off Γ_n, takes values between 0 and 1, has boundary values 1 on Γ_n and tends to 0 at infinity (since Γ_n is compact and R is convergence type). In the sense of distributions, the Laplacian Δu_n exists as a measure and is supported on Γ_n. Let ν_n denote this measure on Γ_n. Then

$$u_n(z) = \int_{\Gamma_n} g(z, w) d\nu_n(w),$$

where g denotes the Greens function on R. Since each geodesic $\gamma_{j,n}$ has uniformly bounded length, there is a fixed width neighborhood $A_{j,n}$of the geodesic so that u_n is harmonic in $A_{j,n} \setminus \gamma_{j,n}$. Since u_n is bounded between 0 and 1, this implies it has uniformly bounded normal derivatives. Since each geodesic has uniformly bounded length, this means the mass $\nu_n(\gamma_{j,n})$ is uniformly bounded for all j and n.

Let $\epsilon_{j,n} = \sqrt{\epsilon|\nu_n(\gamma_{j,n})|}$ and let $\mu_{j,n}$ be the Beltrami coefficient of the quasiconformal map given by Lemma 6.2 which is supported in $X_{j,n}$ and shrinks $\gamma_{j,n}$ by a factor of $1-\epsilon_{j,n}$ (by our comments above $\epsilon_{j,n} \leq 1/2$ if ϵ is small enough). Now let $\mu_n = \sum_j \mu_{j,n}$ and $\mu = \sum_n \mu_n$.

As usual, given an arc $I \subset \mathbb{T}$ we define the corresponding Carleson box as

$$Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| \leq |I|\}.$$

The top of the box Q is defined as

$$T(Q) = \{z \in \mathbb{D} : z/|z| \in I, \frac{1}{2}|I| \leq 1 - |z| \leq |I|\}.$$

The side length of a box is defined as $\ell(Q) = |I|$. The dyadic subintervals of $\mathbb{T}$ are those of the form $[\exp(ij2^{-n}), \exp(i(j+1)2^{-n})]$ and a dyadic Carleson box is one with a dyadic base. A positive measure on disk is Carleson iff $\nu(Q) \leq C\ell(Q)$ for every Carleson box.

Lemma 10.1. *Suppose μ_n is as above. Then $|\mu_n(z)|^2(1-|z|^2)^{-1}dxdy$ is a Carleson measure on the disk with norm at most $C\epsilon$ (where C is independent of n and ϵ, but depends on R).*

Proof. The key observation is the following simple result (see e.g., [8]).

Lemma 10.2. *Suppose f is a positive function on the disk, bounded by $A < \infty$ and such that for every $w \in \mathbb{D}$,*

$$\int_{\mathbb{D}} f(z) \log |\frac{1-\bar{w}z}{z-w}| \frac{dxdy}{(1-|z|^2)^2} \leq A < \infty.$$

Then $f(z)(1-|z|^2)^{-1}dxdy$ is a Carleson measure with norm at most CA.

In the lemma, the function multiplying f is the Greens function for the disk and $dxdy/(1-|z|^2)^2$ is hyperbolic area on the disk. Thus if f is invariant under a Fuchsian group G, it is a well defined function on R and the integral in the lemma becomes

$$\int_R f(z) g(z,w) dA(z),$$

where g is the Greens function for w and dA denotes hyperbolic area on $R = \mathbb{D}/G$. Thus to prove the lemma, we need only show

$$\sup_{w \in R} \int_R |\mu(z)|^2 g(z,w) dA(z) \leq C\epsilon.$$

By Lemma 7.4,

$$\int_R |\mu(z)|^2 g(z,w) dA(w) \leq C\epsilon \sum_j \int_{\gamma_{j,n}} g(z,w) d\nu_n(w) = C\epsilon u_n(z) \leq C\epsilon.$$

This completes the proof of Lemma 10.1 $\square$

Choose a $a > 0$ so that the components of Γ are at least hyperbolic a distance apart (we can do this by the collar lemma since all the components are closed geodesic of length $\leq L$).

Lemma 10.3. *Let $\tilde{\Gamma}_n$ be the lift of Γ_n the unit disk and let W be a hyperbolic a-neighborhood of $\tilde{\Gamma}$. On each component of W let $\sigma_n(z)$ be $|\nu_n(\gamma_{j,n})|/\ell(\gamma_{j,n})$ where j is chosen so the component projects onto a neighborhood of $\gamma_{j,n}$. Then for any Carleson square $Q \subset \mathbb{D}$,*

$$\limsup_n \int_{Q \cap W} \sigma_n(z) \frac{dxdy}{1-|z|^2} \geq \delta \ell(Q) > 0,$$

where $\delta > 0$ depends only on M, the injectivity bound for R.

Proof. Fix a Carleson square $Q = Q_I$ and for $0 < \lambda < 1$ let λQ denote the Carleson square whose base is centered at the same point that I is, but whose length is $\lambda|I|$. We will show the desired estimate holds for Q if λ is small enough and $z_n \in T(\lambda Q)$. Since $\{z_n\}$ is dense in R, this implies the lemma.

Note that

$$\ell(Q)^{-1} \int_{Q\cap W} \sigma_n(z) \frac{dxdy}{1-|z|^2} \;\geq\; \int_{Q\cap W} \sigma_n(z)\tilde{g}(z,z_n)dA(z),$$

where $\tilde{g}$ is the Greens function for the disk and dA denotes hyperbolic area. Thus

$$\int_{Q\cap W} \sigma_n(z) \frac{dxdy}{1-|z|^2} \;\geq\; \int_{W} \sigma_n(z)\tilde{g}(z,z_n)dA(z) - \int_{Q^c\cap W} \sigma_n(z)\tilde{g}(z,z_n)dA(z). \tag{40}$$

The first integral on the right is the same as

$$\int_R \sigma_n(z) g(z,z_n)dA(z) \geq C\int_R g(z,z_n)d\nu_n \geq Cu(z_n) \geq C_1\delta_3.$$

To estimate the second integral on the right side of (40) use a Möbius transformation to move z_n to the origin. Then Q^c is mapped into a Carleson square Q' of diameter $\simeq \lambda$ and so

$$\begin{aligned} \int_{Q^c\cap W} \sigma_n(z)\tilde{g}(z,z_n)dA(z) &\leq \int_{Q'} \sigma_n(z)\log\frac{1}{|z|}dA(z) \\ &\leq \int_{Q'} \sigma_n(z)\frac{dxdy}{1-|z|^2} \\ &\leq C_2\lambda, \end{aligned}$$

where the last inequality holds because the proof of Lemma 10.1 shows that $\sigma_n(z)\frac{dxdy}{1-|z|^2}$ is a Carleson measure with uniformly bounded norm. Thus if we choose $\lambda < C_1\delta_3/(2C_2)$, we get

$$\ell(Q)^{-1}\int_{Q\cap W} \sigma_n(z)\frac{dxdy}{1-|z|^2} \geq \delta_3/2,$$

which proves the lemma. □

The following is now immediate.

Corollary 10.4. *The Carleson norm of $|\mu_n(z)^2|(1-|z|^2)^{-1}dxdy$ is comparable to ϵ with constants that only depend on the injectivity upper bound for R.*

11. Divergence of a Stolz type sum

Suppose ν is a Carleson measure on the $\mathbb{D}$ with constant M and for $x \in \mathbb{T}$, and let

$$I_\nu(x) = \sum_I \nu(T(Q_I))|I|^{-1},$$

where the sum is over all dyadic intervals containing x. This is a dyadic analog of an integral over a Stolz cone. Then it is easy to see that

$$\int_{\mathbb{T}} I_\nu(x)dx = \nu(\mathbb{D}),$$

so that I_ν must be positive on a set of positive measure (if ν is non-trivial). We need a more quantified version of this fact.

Lemma 11.1. *Suppose ν is a Carleson measure on $\mathbb{D}$ with constant $\leq M$ and suppose Q is a dyadic Carleson square with base $I \subset \mathbb{T}$ such that $\nu(Q) \geq \epsilon \ell(Q)$ for some $\epsilon > 0$. Then there is a $B = B(M, \epsilon)$ such that*

$$|\{x \in I : I_\nu(x) > \epsilon/4\}| \geq B\nu(Q_I).$$

Proof. Without loss of generality, we may assume ν is supported in Q_I. The idea is to show that $I(\nu)$ is in dyadic BMO and then apply the John-Nirenberg theorem. To be more precise, recall that a function f on the circle is said to be in dyadic BMO if for every dyadic interval I there is a number c_I such that

$$\|f\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|}\int_I |f - c_I| < \infty,$$

and the supremum is over all dyadic intervals. If f is in dyadic BMO then the John-Nirenberg theorem (e.g., [10]), states that there is a $C_1, C_2 < \infty$ such that

$$|\{x \in I : |f - m_I(f)| \geq \lambda\}| \leq C_1 |I| \exp(-C_2 \|f\|_{\mathrm{BMO}} \lambda).$$

We claim that if ν is a Carleson measure then I_μ is in dyadic BMO. To prove this let $c_I = \sum_J \nu(T(J))$ where the sum is over all dyadic J containing I. Then

$$\frac{1}{|I|}\int |I_\nu - c_I| dx \leq \|\nu\|_C.$$

Thus I_ν is in dyadic BMO, as claimed. Thus if $f = I_\nu - c_I$,

$$\begin{aligned}
\nu(Q_I) &\leq \int_I f\,dx \\
&\leq \int_{f<\epsilon/4} + \int_{\epsilon/4\leq f<\lambda} + \int_{f\geq\lambda} \\
&\leq \frac{\epsilon}{4}|I| + \lambda|\{\frac{\epsilon}{4} \leq f < \lambda\}| + C_1|I|\int_\lambda^\infty \lambda e^{-C_2\lambda}d\lambda \\
&\leq \frac{\epsilon}{4}|I| + \lambda|\{\frac{\epsilon}{4} \leq f < \lambda\}| + C_1|I|e^{-C_3\lambda}.
\end{aligned}$$

Thus taking λ so large that $C_1e^{-C_3\lambda} \leq \epsilon/4$ we get

$$\begin{aligned}
|\{x \in I : I_\nu \geq \epsilon/4\}| &\geq \frac{1}{\lambda}[\nu(Q_I) - \frac{\epsilon}{4}|I| - C_1|I|e^{-C_3\lambda}] \\
&\geq \frac{1}{\lambda}[\epsilon|I|(1 - \frac{1}{4} - \frac{1}{4})] \\
&\geq \frac{\epsilon}{2\lambda}|I|.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 11.2. *Suppose $\{\nu_n\}$ are positive Carleson measures on $\mathbb{D}$, that*

$$\sup_n \|\nu_n\|_C \leq M < \infty,$$

and that for any Carleson square Q,

$$\limsup_n \nu_n(Q) = \epsilon > 0,$$

(for some ϵ independent of Q). Let $\nu = \sum_n \nu_n$. Then $I_\nu(x) = \infty$ almost everywhere on $\mathbb{T}$.

Proof. Fix an I. By Lemma 11.1 we have $I_{\nu_n} > \epsilon/8$ on a set of size $B|I|$ for infinitely many n's. Thus the series $I_\nu = \sum_n I_{\nu_n}$ diverges on a subset of I of size at least $\delta = B|I|$. Since the series diverges on a fixed fraction of every dyadic interval, the Lebesgue density theorem implies it diverges almost everywhere. □

12. How close is Γ to a circle?

In order to show that $\Gamma = f_\mu(\mathbb{T})$ has tangents almost nowhere, we have to show that Γ has oscillation at almost all points and infinitely many scales. To do this we measure its oscillations using Peter Jones' β's, e.g., [12].

Given a closed curve Γ we define

$$\beta_\Gamma(x,t) = \inf_L \frac{1}{t} \sup_{z\in\Gamma\cap B(x,t)} \mathrm{dist}(z,L),$$

where L is the set of all lines hitting x. For a Riemann mapping $f:\mathbb{D}\to\Omega$ let $\beta(z) = \beta_{\partial\Omega}(f(z/|z|), 10\mathrm{dist}(z,\partial\Omega))$. It follows from the results in [7] that if $\Gamma=\partial\Omega$ is a quasicircle, then up to sets of zero 1-dimensional measure, $x=f(e^{i\theta})$ is a tangent point of Γ iff

$$\int_0^1 \beta(re^{i\theta})^2 \frac{dr}{1-r} < \infty. \tag{41}$$

or (equivalently, up to sets of linear measure zero) iff,

$$\int_0^\infty \beta(x,t)^2 \frac{dt}{t} < \infty. \tag{42}$$

We also define

$$\mathfrak{B}_\Gamma(z,t) = \sum_{n=0}^\infty \beta_\Gamma(z,e^n t)e^{-n}.$$

When Γ is clear from context, we will drop it from our notation.

Lemma 12.1. *With notation as above,*

$$\int_0^\infty \beta(z,t)^2\frac{dt}{t} \simeq \int_0^\infty \mathfrak{B}(z,t)^2\frac{dt}{t}.$$

Proof. One direction is obvious since $\beta\le\mathfrak{B}$. To prove the other direction we use Cauchy-Schwarz to get

$$\begin{aligned}
\int_0^\infty \mathfrak{B}(z,t)^2\frac{dt}{t} &= \int_0^\infty \Big(\sum_{n=0}^\infty \beta(z,e^n t)e^{-n}\Big)^2\frac{dt}{t} \\
&\le \int_0^\infty \Big(\sum_{n=0}^\infty \beta^2(z,e^n t)e^{-n}\Big)\Big(\sum_{n=0}^\infty e^{-n}\Big)\frac{dt}{t} \\
&= C\sum_{n=0}^\infty \int_0^\infty \beta^2(z,e^n t)e^{-n}\frac{dt}{t} \\
&= C\sum_{n=0}^\infty \Big[\int_0^\infty \beta^2(z,s)\frac{ds}{s}\Big]e^{-n} \\
&= C^2\int_0^\infty \beta^2(z,s)\frac{ds}{s}.
\end{aligned}$$

□

For our purposes in this paper, it is more convenient to measure the flatness of Γ is a slightly different way. We define

$$\eta_\Gamma(z) = \min(1, \inf_\tau \sup_{x,y\in\Gamma} \log \frac{|\tau(x)|}{|\tau(y)|}),$$

where the infimum is over all Möbius transformations τ such that $\tau(\Phi(z)) = 0$. In other words, η measures the smallest r such that $\partial\Omega$ can be mapped into the annulus $\{1 \leq |z| \leq r\}$ with z going to the origin. The minimum with 1 is included simply to make it easier to compare η to β below. The main advantage of η over β is that η is invariant under Möbius transformations. In particular, if Γ is the limit set of a Kleinian group then η is well defined on the quotient Riemann surface. In fact, we may consider η as a geometric analog of the Schwarzian deriviative.

Lemma 12.2. *There is a $C < \infty$ such that the following holds. If $\Gamma \subset \mathbb{R}^2$, $x \in \Gamma$, $t > 0$, and $t \leq |w - x| \leq 2\mathrm{dist}(w, \Gamma) \leq 4t$ then*

$$\eta_\Gamma(w) \leq C\mathfrak{B}_\Gamma(z, t).$$

Proof. After a linear change of variables we may assume that $w = i$ and $-1, 1 \in \Gamma$. Let B_r be the hyperbolic ball of radius r centered at i in the upper half-plane, $\mathbb{H}$, and let B_r^* denote its reflection across the real line. Let τ be a Möbius transformation which maps $\mathbb{H}$ to the unit circle, taking i to 0. Suppose that $0 < a < 1$ and that $\mathfrak{B}_\Gamma(0, 1) \leq ae^{-n}$. We claim that if a is small enough (independent of n), then $\Gamma \cap (B_{n-c} \cup B_{n-c}^*) = \emptyset$ for some $c > 0$ independent of n. If we can prove the claim then $\tau(\Gamma)$ lies in a Ce^{-n} neighborhood of the unit circle and we will be done.

Thus it suffices to prove the claim. Let $S_0 = \{z = x + iy : |z| \leq 1, |y| \leq e^{-n}\}$ and for $k > 0$, let $A_k = \{z : e^{k-1} \leq |z| \leq e^k\}$, $C_k = \{z : |z| = e^k\}$ and $S_k = \{z = x + iy \in A_k : |y| \leq e^{2k-n}\}$. It is easy to check that the hyperbolic distance from i to S_k in $\mathbb{H}$ is greater than $n - c$ for some c independent of k, so it suffices to prove $\Gamma \subset \cup_k S_k$.

We prove this by induction on k. Since $\beta \leq \mathfrak{B}$, we clearly have $\Gamma \cap \{|z| \leq 1\} \subset S_0$. Now suppose $\Gamma \cap A_{k-1} \subset S_{k-1}$. Since $\mathfrak{B}_\Gamma(0) \leq ae^{-n}$ we must have $\beta_k \equiv \beta_\Gamma(i, e^k) \leq e^{k-n}\frac{e-1}{e(e+1)}$ if a is small enough. Hence $\Gamma \cap A_k$ is contained in strip S of width $2\beta_k e^k = e^{2k-n-1}$. Since $\Gamma \cap A_{k-1} \subset S_{k-1}$, $S \cap S_k \cap C_{k-1} \neq \emptyset$. Thus the maximum height of a point in $S \cap C_{k-1}$ is at most $e^{k-n} + \beta_k e^k$. Thus the central line of S (which passes through the origin) hits C_k at height at most e times this and so

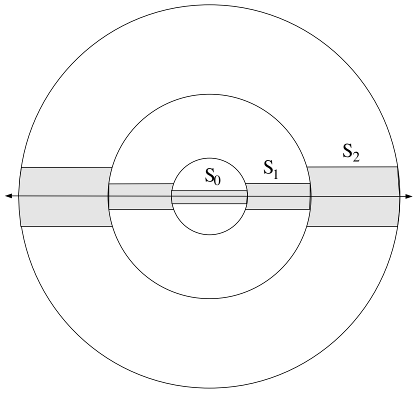

FIGURE 17. Definition of the S_k

$S \cap A_k$ has height at most

$$\begin{aligned} e[e^{2(k-1)-n} + \beta_k e^k] + \beta_k e^k &= e^{2k-1-n} + (e^{2k-n+1} + e^{2k-n})\frac{e-1}{e(e+1)} \\ &= e^{2k-n}(e^{-1} + (e+1)\frac{e-1}{e(e+1)}) \\ &= e^{2k-n}. \end{aligned}$$

Thus $\Gamma \cap A_k \subset S_k$, as desired. This completes the proof of the lemma. □

From the lemma we see that there is a $C < \infty$ so that for any $\Gamma = f(\mathbb{T})$ and any $e^{i\theta} \in \mathbb{T}$

$$\int_0^1 \eta_\Gamma^2(f(re^{i\theta}))\frac{dr}{1-r} \leq C \int_0^1 \mathfrak{B}_\Gamma^2(re^{i\theta})\frac{dr}{1-r}.$$

Hence, except for a set of zero 1-dimensional measure, $f(e^{i\theta})$ is not a tangent point of Γ if

$$\int_0^1 \eta_\Gamma(f(rx))^2 \frac{dr}{1-r} = \infty. \tag{43}$$

13. SHRINKING A GEODESIC MAKES η LARGE

In this section we show that shrinking a closed geodesic forces η to be large. Combined with the results of the previous sections, this will imply that $f_\mu(\mathbb{T})$ has tangents almost nowhere.

Lemma 13.1. *Given $2 \leq M < \infty$, there is a $C = C(M) < \infty$ so that the following is true. Suppose Γ is a closed Jordan curve which is the limit set of a Kleinian group G. Let Ω_1 be the bounded complementary component and Ω_2 the unbounded* *component. Suppose that $g \in G$ fixes $-1, 1 \in \Gamma$ and that g has translation length $r \leq M$ in $R_1 = \Omega_1/G$ and length $(1-\epsilon)r$ in $R_2 = \Omega_2/G$. Then $\eta_\Gamma(0) \geq C\epsilon$.*

Proof. Let ρ_j denote hyperbolic distance on Ω_j for $j = 1, 2$. Let ρ denote the hyperbolic metric on $\mathbb{D}$ and also on $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$. We may assume that η is small, say $\eta < \delta = \delta(M)$ (to be chosen below), because otherwise the lemma is true with $C = 1/\delta$.

Note that if $\Gamma \subset \{\frac{1}{1+\eta} \leq |z| \leq 1 + \eta\}$, then

$$\frac{1}{1 + C\eta} \leq \frac{d\rho_1}{d\rho} \leq 1 + C\eta,$$

for $|z| \leq 1 - \delta$ or $|z| \geq 1 + \delta$ and C depends on δ. The translation length of g in Ω_j for $j = 1, 2$ is equal to

$$\min_{z \in \Omega_j} \rho_j(z, g(z)).$$

By replacing g by a power of itself, we may assume its translation length in Ω_1 is at least 1. If δ is small enough then the axis A of g from -1 to $+1$ must intersect $B_{1/2} = \{|z| \leq 1/2\}$. Choose a point $z \in A \cap B_{1/2}$. Since g has translation length $\leq M$ the image $g(z)$ is still in $\{|w| \leq 1 - \delta\}$ (if δ is small enough depending on M).

The Möbius transformation g can be written as a composition $g = \sigma \circ \tau$ where both τ and σ fix 1 and -1, τ maps the unit circle to itself (and hence $\tau(z^*) = \tau(z)^*$) and σ moves points by less than $C\eta$ in the spherical metric. Thus

$$\begin{aligned}
\rho_1(z, g(z)) &\leq (1+\eta)\rho(z, g(z)) \\
&= (1+\eta)\rho(z^*, g(z)^*) \\
&\leq (1+\eta)[\rho(z^*, g(z^*)) + \rho(g(z^*), g(z)^*)] \\
&\leq (1+\eta)[\rho(z^*, g(z^*)) + \rho(\sigma(\tau(z)^*)), \sigma(\tau(z))^*)] \\
&\leq (1+\eta)[(1+\eta)\rho_2(z^*, g(z^*)) + C\eta] \\
&\leq (1 + C\eta)\rho_2(z^*, g(z^*)).
\end{aligned}$$

Hence the translation length in Ω_1 is at most $(1 + C\eta)$ times the translation length in Ω_2 and hence $\epsilon < C\eta$, as desired. □

14. $f_\mu(\mathbb{T})$ HAS ALMOST NO TANGENTS

Suppose $\mu = \sum_n \mu_n$ is the Beltrami coefficient defined in Section 10 and f_μ is the corresponding quasiconformal map. Each μ_n is a sum of the form $\sum \mu_{n,j}$ where each

$\mu_{n,j}$ is associated to a closed geodesic $\gamma_{n,j}$ and shrinks the length of this geodesic by a factor of $1 - \epsilon_{n,j}$. Let $A_{n,j}$ be a hyperbolic a-neighborhood of $\gamma_{n,j}$ (where a is chosen so these neighborhoods are disjoint). Let $\sigma_{n,j} = \epsilon_{n,j}^2 \chi_{A_{n,j}}$, $\sigma_n = \sum_j \sigma_{n,j}$ and $\sigma = \sum_n \sigma_n$.

Combining Lemmas 10.1, 10.3 and 11.2, we see that $I_\sigma = \infty$ almost everywhere on the circle. By Lemma 13.1 we know that $\eta(z)^2 \geq C\epsilon_{n,j}^2$ on $A_{n,j}$. Thus

$$\int \eta^2(rx)\frac{dr}{1-r} \gtrsim I_\sigma(x) = \infty,$$

for almost every x on the circle. Thus $f_\mu(\mathbb{T})$ has tangents on a set of at most linear measure zero, as desired.

15. A stopping time decomposition

A stopping time region $W \subset \mathbb{D}$, is a region of the form

$$W = Q \setminus \cup_k Q_k,$$

where Q and $\{Q_k\}$ are dyadic Carleson squares. For dyadic intervals, $I \cap J \neq \emptyset$ implies $I \subset J$ or $J \subset I$ and from this is easily follows that the $\{Q_k\}$ in the definition of stopping time region may always be taken to have disjoint interiors.

We will be interested in stopping time regions formed in the following way. Let G be a convergence type Fuchsian group and let $R = \mathbb{D}/G$ be the associated Riemann surface. Since G is convergence type the Greens function for R exists. Fix a base point $z_0 \in R$ and let $g(z, z_0)$ denote the lift to the disk of the Greens function on R with pole at z_0. Note that away from the lifts of z_0, $g(z, z_0)$ is positive harmonic function and hence by Harnack's inequality there is a constant A so that

$$\max_{T(Q)} g(z, z_0) \leq A \min_{T(Q)} g(z, z_0),$$

for any top half of a Carleson box for which $\max_{T(Q)} g(z, z_0) \leq 1$.

Now suppose $1 > \epsilon_0 > \epsilon_1 > \dots$. We may assume that z_0 is the projection of the origin via the quotient map $\mathbb{D} \to R$. Let $\mathcal{C}_{1,0}$ be the collection of dyadic Carleson boxes which are maximal with respect to the property

$$\max_{T(Q)} g(z, z_0) \leq \epsilon_1,$$

and let

$$W_0 = \mathbb{D} \setminus \cup_{Q \in \mathcal{C}_0} Q.$$

Then W_0 is a chord-arc domain and

$$g(z, z_0) \geq \epsilon_1/A, z \in W_0,$$

$$\epsilon_1 \geq g(z, z_0) \geq \epsilon_1/A, z \in \partial W_0.$$

For each $Q \in \mathcal{C}_{1,0}$ define the collection $\mathcal{C}(Q) = \mathcal{C}_{2,1} \cup \mathcal{C}_{0,1}$ where $\mathcal{C}_{2,1}$ is the maximal collection of dyadic sub-boxes such that

$$\max_{T(Q)} g(z, z_0) \leq \epsilon_2,$$

and $\mathcal{C}_{0,1}$ is the maximal collection of dyadic sub-boxes such that

$$\max_{T(Q)} g(z, z_0) \geq \epsilon_0.$$

In general, given a $Q \in \mathcal{C}_{k,n}$ with $k \geq 2$, define $\mathcal{C}_{k+1,n+1}$ as the maximal collection of dyadic sub-boxes Q' of Q such that

$$\max_{T(Q')} g(z, z_0) \leq \epsilon_{k+1},$$

and define $\mathcal{C}_{k-1,n+1}$ as the maximal collection of sub-boxes such that.

$$\max_{T(Q')} g(z, z_0) \geq \epsilon_{k-1}.$$

If $Q \in \mathcal{C}_{k,n}$, define the stopping time region

$$W(Q) = Q \setminus \cup_{Q' \in \mathcal{C}_{k-1,n+1} \cup \mathcal{C}_{k+1,n+1}} Q'.$$

We will say such a W is of type (k, n). Note that

$$\epsilon_{k+1}/A \leq g(z, z_0) \leq \epsilon_{k-1}, z \in W.$$

Furthermore, ∂W is a chord-arc curve, $\partial W \cap \mathbb{T}$ has length zero (since the Greens function g tends radially to 0 almost everywhere on the circle), and

$$\epsilon_k/A \leq g(z, z_0) \leq \epsilon_k$$

on the unique connected component of ∂W which intersects $T(Q)$. Moreover, on every other connected component of $\partial W \setminus \mathbb{T}$ either

$$\epsilon_{k-1}/A \leq g(z, z_0) \leq \epsilon_{k-1}$$

or

$$\epsilon_{k+1}/A \leq g(z, z_0) \leq \epsilon_{k+1}.$$

If $k = 0$ then we have to modify the construction because there is no ϵ_{k-1}. In this case, we just define one collection $\mathcal{C}_{1,n+1}$ as above and let

$$W(Q) = Q \setminus \cup_{Q' \in \mathcal{C}_{1,n+1}} Q',$$

and say W is type $(0, n)$. As above ∂W hits the circle in a set of measure zero,

$$\epsilon_1/A \le g(z, z_0) \le \epsilon_0, z \in W,$$

$$\epsilon_0/A \le g(z, z_0) \le \epsilon_0,$$

for z in the unique component of ∂W hitting $T(Q)$ and

$$\epsilon_1/A \le g(z, z_0) \le \epsilon_1,$$

for z in all the other components of $\partial W \setminus \mathbb{T}$.

If $k = N$ then we also modify the construction to prevent there being a type $N+1$ region. In this case, we just define one collection $\mathcal{C}_{N-1,n+1}$ as above and let

$$W(Q) = Q \setminus \cup_{Q' \in \mathcal{C}_{N-1,n+1}} Q',$$

and say W is type (N, n). As above,

$$g(z, z_0) \le \epsilon_{N-1}, z \in W,$$

and

$$\epsilon_N/A \le g(z, z_0) \le \epsilon_N,$$

for z in the unique component of ∂W hitting $T(Q)$ and

$$\epsilon_{N-1}/A \le g(z, z_0) \le \epsilon_{N-1},$$

for z in all the other components of $\partial W \setminus \mathbb{T}$. However, unlike the previous domains, ∂W hits the circle in a set of positive measure. Let $\mathcal{C} = \cup_{j,n} \mathcal{C}_{j,n}$.

We can think of the decomposition of the disk into these regions as a random walk on $\{0, 1, \dots N+1\}$ which is reflecting at 0 and absorbing at $N+1$: a type k region is followed by either a type $k-1$ or type $k+1$ region except that a type 1 must be followed by type 2 and a type N has a positive probability of having no successor. Later we will choose the parameters $\{\epsilon_k\}$ so that with high probability a type k is followed by a $k+1$ and not a $k-1$.

Lemma 15.1. *There are $\alpha > 0$ and $C < \infty$ so that the following holds. Suppose $Q \in \mathcal{C}_{k,n}$ and W is the corresponding stopping time region constructed above. Suppose $E \subset \partial W$ corresponds to the tops of type $(k-1, n+1)$ squares. Then*

$$\ell(E) \leq C(\frac{\epsilon_k}{\epsilon_{k-1}})^\alpha \ell(\partial W).$$

Proof. Since the Greens function g is harmonic and satisfies $g(z_Q) \geq \epsilon_k/A$ and $g(z) \leq \epsilon_{k-1}$ for $z \in E$, we see that the harmonic measure of E in W with respect to z_Q satisfies

$$\omega(z_Q, E, W) \leq \frac{\epsilon_k}{A\epsilon_{k-1}}.$$

Since harmonic measure and arclength are A_∞ related with constants depending only on the chord-arc constant of W (this is a theorem of Jerison and Kenig, e.g., [11]) there is an α and C (depending only on the chord-arc constant of W) such that $\ell(E) \leq C\omega(E)^\alpha$. Combined with our previous observation, this proves the lemma. □

16. Chord-arc domains and Carleson measures

A locally rectifiable curve γ is called a chord-arc (or Lavrentiev) curve if there is a $C < \infty$ so that for any two points $x, y \in \gamma$, the shorter arc between x, y has length at most $C|x - y|$. A domain W will be called chord-arc if its boundary is a closed chord-arc curve. Clearly, dyadic stopping-time regions as described in the previous section are chord-arc domains (with a uniform constant).

We will need a few well known estimates concerning chord-arc domains. If W is a chord-arc domain then a positive measure ν on $\mathbb{R}^2$ is called a Carleson measure for ∂W if there is a $C < \infty$ so that

$$\nu(B(x, r)) \leq Cr,$$

for every $x \in \partial W$ and $r > 0$. The following is due to Steven Semmes [15].

Theorem 16.1. *Suppose W is a chord-arc domain. There is a ϵ_0, depending only on the chord-arc constant M of W so that the following holds. Suppose $\mu \in L^\infty(\mathbb{R}^2)$ and*

$$d\nu = \frac{\mu^2(z)}{\text{dist}(z, \partial W)} dxdy,$$

is a Carleson measure for γ with norm $\leq \epsilon_0$. Let f_μ be a quasiconformal mapping with dilatation μ. Then $f_\mu(W)$ is chord-arc with constant depending only on $\|\nu\|_C$ and M.

We want to be able to apply Semmes' result to our Beltrami coefficient μ and the stopping-time domains in our collection $\mathcal{C}$. To do this we must prove

Lemma 16.2. *Given $\epsilon > 0$ there is an $r < \infty$ so that the following holds. Let $\Gamma = \cup_{W \in \mathcal{C}} \partial W$ and suppose $f = \sum_n f_n$ is such that the supports of the f_n are pairwise disjoint, $\|f\|_\infty \leq \epsilon$ and for all n,*

$$\frac{f_n}{1-|z|} dxdy,$$

is a Carleson measure for $\mathbb{D}^*$ with norm $\leq \epsilon$ and that $\rho(\mathrm{supp}(\mu), \Gamma^*) \geq r$. Then for any $W \in \mathcal{C}$,

$$\int_{B(x,t)} \frac{f(z)}{\mathrm{dist}(z, \partial W)} dxdy \leq C\epsilon t,$$

for any $x \in \partial W$ and $t > 0$.

Proof. Choose a point $x \in \partial W$ and a $t > 0$. If $t \geq 2\mathrm{diam}(W)$ then

$$\int_{B(x,t)\setminus B(x,\mathrm{diam}(W))} \frac{f(z)}{\mathrm{dist}(z,\partial W)} dxdy \leq \int_{B(x,t)} \frac{f(z)}{1-|z|} dxdy \leq C\epsilon t.$$

Thus it suffices to consider $t \leq 2\mathrm{diam}(W)$.

Choose a Carleson square Q such that $B(x,t) \cap \mathbb{D} \subset Q$ and $\mathrm{diam}(Q) \simeq t$. Clearly it suffices to show that

$$\int_{Q^*} \frac{f(z)}{\mathrm{dist}(z,\partial W)} dxdy \leq C\epsilon \ell(Q)$$

Write $f = g_1 + f_2$ where $g_1 = f\chi_{W^*}$. Then g_1 is the restriction to W^* of some f_n and so

$$\int_{Q^*} \frac{g_1(z)}{\mathrm{dist}(z,\partial W)} dxdy \leq \int_{Q^*} \frac{g_1(z)}{1-|z|} dxdy \leq C\epsilon \ell(Q),$$

by assumption.

To estimate the contribution of g_2, suppose $W \cap Q = Q \setminus \cup Q_j$. We can choose the Q_j's so that $\sum \ell(Q_j) \leq C\ell(Q)$. For each Q_j, g_2 is supported inside a region of width $Ce^{-t}\ell(Q_j)$ along the base of Q_j, and so

$$\int_{Q_j} \frac{g_2(z)}{\mathrm{dist}(z,\partial W)} dxdy \leq e^{-t}\ell(Q_j).$$

Thus

$$\int_Q \frac{g_2(z)}{\mathrm{dist}(z,\partial W)} dxdy \le C\|f\|_\infty e^{-t} \sum_j \ell(Q_j) \le C\epsilon\ell(Q).$$

□

As an immediate consequence we see that $f_\mu(W)$ is chord-arc for any $W \in \mathcal{C}$, with a constant that is independent of W.

Corollary 16.3. *Given $\epsilon > 0$ there is a $\lambda < 1$ so that if $\epsilon_k/\epsilon_{k-1} < \lambda$ and $\{\mu_n\}$ satisfies the conditions of Lemma 16.2, then the following holds. Suppose $Q \in \mathcal{C}_{k,n}$. Then*

$$\sum_{Q' \in \mathcal{C}_{k-1,n+1}} \mathrm{diam}(f_\mu(Q')) \le \epsilon \mathrm{diam}(f_\mu(Q)),$$

and

$$\sum_{Q' \in \mathcal{C}_{k+1,n+1}} \mathrm{diam}(f_\mu(Q')) \le M \mathrm{diam}(f_\mu(Q)).$$

Proof. The second estimate is immediate from the fact that $f_\mu(W(Q))$ is chord-arc. To prove the first, we note that as in the proof of Lemma 15.1, the harmonic measure of $E = \partial W \cap \cup_{\mathcal{C}_{k-1,n+1}} \partial Q$ is bounded by $A\lambda$. Since f_μ is conformal on W (remember that μ is supported outside the unit disk), the same is true for the image domain. Since μ is Carleson with small norm with respect to W, $f_\mu(W)$ is also chord-arc and so harmonic measure is A_∞ related to arclength. Thus E has length as small as we wish if λ is small enough. This proves the result. □

17. $f_\mu(\mathbb{T})$ has dimension 1

We now want to show that $f_\mu(\mathbb{T})$ has dimension 1. To do this we first consider the Beltrami coefficient $\sum_{k=1}^n \mu_k$ and let f_n be the corresponding quasiconformal map fixing $0, 1, \infty$.

Lemma 17.1. *With f_n as above, $f_n(\mathbb{T})$ is rectifiable.*

Proof. To show $f_n(\mathbb{T})$ is rectifiable, we will estimate the length of all the image domains $f_n(W)$, $W \in \mathcal{C}$ and show the total sum is finite. To each such W associate a ball of twice its diameter centered at some point of W. Every point in $f_n(\mathbb{T})$ is either on the boundary of some $f_\mu(W)$ or is covered by infinitely many of the balls. The first set clearly has finite one dimensional measure. The second set has zero length by the following elementary observation.

Lemma 17.2. *If $\sum_n \mathrm{diam}(B_n) < \infty$ and each point of E is in infinitely many of the sets $\{B_n\}$, then E has zero 1-dimensional measure.*

Proof. Clearly E is also covered by $\{B_n\}_N^\infty$ for any N and for N large the sum of diameters is as small as we wish. □

Every domain W in our decomposition of the disk has a label of the form (k, n) where k indicates the size of Greens function on W (approximately between ϵ_{k-1} and ϵ_{k+1}, except for the special cases $k = 1, N$) and n is the number of domains separating it from the origin. If W has type (k, n) then associated to it is a length n string $s = \{j_1, \dots, j_n\}$ where $j_i \in \{1, \dots, N\}$, $j_1 = 1$, $j_n = k$ and $|j_i - j_{i+1}| = 1$ denoting the types of domains which separate 0 from W. Clearly there are at most 2^n such strings of length n.

Lemma 17.3. *Fix a string s of length n as above and let $\mathcal{W}(s)$ be the collection of all domains in our decomposition which have this string. Then*

$$\sum_{W \in \mathcal{W}(s)} \mathrm{diam}(f_\mu(W)) \leq (\frac{M}{\epsilon})^{N/2} (M\epsilon)^{n/2},$$

where M and ϵ are as in Corollary 16.3.

Proof. A step in the string s will be called "up" if it is of the form $k \to k+1$ and will be called "down" if it is of the form $k \to k-1$. Let n_u be the number of up steps and n_d the number of down steps in s. Then clearly $n_u + n_d = n$ and $n_u \leq n_d + N$. Hence $n_d \geq \frac{1}{2}(n - N)$. By Corollary 16.3

$$\begin{aligned}
\sum_{W \in \mathcal{W}(s)} \mathrm{diam}(f_\mu(W)) &\leq M^{n_u} \epsilon^{n_d} \\
&\leq M^N (M\epsilon)^{n_d} \\
&\leq M^N (M\epsilon)^{\frac{1}{2}(n-N)} \\
&= (\frac{M}{\epsilon})^{N/2} (M\epsilon)^{n/2}.
\end{aligned}$$

□

We can now finish Lemma 17.1. The sum over all domains is bounded by

$$\begin{aligned}\sum_W \operatorname{diam}(f_\mu(W)) &\le \sum_n \sum_{s:|s|=n} \sum_{W\in\mathcal{W}(s)} \operatorname{diam}(f_\mu(W)) \\ &\le \sum_n \sum_{s:|s|=n} (\frac{M}{\epsilon})^{N/2}(M\epsilon)^{n/2} \\ &\le (\frac{M}{\epsilon})^{N/2}\sum_n 2^n (M\epsilon)^{n/2} \\ &\le 2(M/\epsilon)^{N/2},\end{aligned}$$

if $M\epsilon < 1/4$. This completes the proof. □

Now we can prove that $f_\mu(\mathbb{T})$ has dimension 1 if μ is chosen appropriately.

Lemma 17.4. *Suppose notation is as above. Suppose $\varphi(t) : [0,\infty) \to \infty$ is increasing, continuous and satisfies $\varphi(t) = o(t)$. There is a sequence $r_n \nearrow \infty$ so that if $\rho(\operatorname{supp}(\mu_n), \operatorname{supp}(\mu_{n+1})) \ge r_n$, then $\mathcal{H}^\varphi(f_\mu(\mathbb{T})) = 0$.*

Proof. Suppose we have already chosen $\mu_1, \dots, \mu_n$. Then by Lemma 17.1, $f_n(\mathbb{T})$ is rectifiable. Thus there is a cover of $f_n(\mathbb{T})$ by open balls $\{B(x_j, r_j)\}$ such that $\sum_j \varphi(r_j) \le 2^{-n}$. Then $U_n = \cup_j B(x_j, r_j)$ is an open neighborhood of $f_n(\mathbb{T})$ and if r_n is large enough we have $f_\mu(\mathbb{T}) \subset U_n$ as well (since $f \to f_n$ uniformly on compact sets as $r_n \to \infty$). This proves the lemma. □

Taking $\varphi(t) = t \log \frac{1}{t}$, for example, and μ satisfying the lemma shows that $f_\mu(\mathbb{T})$ can have dimension 1, as desired.

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