SURFACES WHICH APPROXIMATE THE THRICE PUNCTURED SPHERE

CHRISTOPHER J. BISHOP

Abstract. We prove some results that quantify the well known fact that the thrice punctured sphere has trivial deformation space. More precisely, given any $K < \infty, \delta > 0$ there is an $\epsilon > 0$ so that if $R = \mathbb{D}/G$ is a union of Y-pieces all with boundary lengths $\leq \epsilon$ then $\dim(\Lambda) \leq 1 + \delta$ for the limit set of any K-quasiconformal deformation of G. We also show that $R = \mathbb{D}/G$ looks like the thrice punctured sphere near ∞ then it has no big deformations near infinity; this condition arises from the study of Ruelle's property, i.e., the analytic dependence of dimension on the deformation. Our main tools are an estimate involving the Schwarzian derivative and an estimate which says that a quasiconformal mapping of a Y-piece with short boundaries can always be replaced by an equivalent map with dilatation concentrated near the boundary.

Date: September 14, 2000.

1991 *Mathematics Subject Classification.* Primary: 30F35 .

Key words and phrases. Hausdorff dimension, quasi-Fuchsian groups, quasiconformal deformation, critical exponent, convex core, Schwarzian derivative, Y-pieces .

The author is partially supported by NSF Grant DMS 9800924.

1. Introduction

Let $R_0 = \mathbb{C} \setminus \{0,1\}$ be the thrice punctured sphere. It is well known that the Teichmüller space of R_0 is trivial, i.e., the only Riemann surface quasiconformally equivalent to R_0 is itself. Equivalently, if $R_0 = \mathbb{D}/G_0$ then that the only deformations of G_0 to a quasi-Fuchsian group are Möbius transformations, i.e., the limit set is a circle. In this paper we will show that if a Riemann surface R is a union of pieces which approximate thrice punctured spheres, then it is hard to deform R in the sense that a large quasiconformal constant is required to increase the dimension of the limit set. Before stating the precise result, we review some definitions.

A generalized Y-piece in a Riemann surface R is a region bounded by three simple closed geodesics (or punctures) which is homeomorphic to a 2-sphere minus three disks (or points). If all three boundary components have length $\leq L$ we say the Y-piece is L-bounded (punctures count as zero length). We say that R has a L-bounded Y-piece decomposition if it can be written as a union of L-bounded Y-pieces with disjoint interiors. Let Γ be the union of all simple closed geodesics which occur as boundary arcs in the Y-piece decomposition and let $\Gamma^\epsilon \subset \Gamma$ denote all those with lengths $\geq \epsilon$. By the collar lemma there is a $C > 0$ (depending only on L) so that the hyperbolic C-neighborhoods of elements of Γ are pairwise disjoint. It is tempting to think that any surface with an upper bound on its injectivity radius has a decomposition into L-bounded Y-pieces, but this is not so. For example, it is a nice exercise to prove that $\mathbb{R}^2 \setminus (\mathbb{Z} + i\mathbb{Z})$ has uniformly bounded injectivity radius, but no such decomposition for any $L < \infty$ (the surface can be written as a disjoint union of Y-pieces, but the boundary lengths cannot be uniformly bounded).

A conformal mapping $f : \mathbb{D} \to \Omega$ is called a deformation of the Fuchsian group G if for every $g \in G$, $f \circ g \circ f^{-1}$ is Möbius transformation restricted to Ω. It is called a quasiconformal deformation if f has a quasiconformal extension to the whole plane. If μ is a G invariant Beltrami coefficient on $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$, i.e., satisfies

$$\mu(g(z)) = \mu(z) g'(z)/|g'(z)|,$$

then there is a corresponding quasiconformal deformation which we will denote f_μ (unique up to Möbius transformations).

Theorem 1.1. *Given $K < \infty$ and $\eta > 0$ there is an $\epsilon > 0$, so that if $R = \mathbb{D}/G$ is Riemann surface which has an ϵ-bounded Y-piece decomposition, then* $\dim(f(\mathbb{T})) \leq 1 + \eta$ *for every K-quasiconformal deformation of G.*

Since the critical exponent of the Poincaré series satisfies $\delta(g) \leq \dim(\Lambda)$ by a result in [8], we see the same inequality also holds for the critical exponent.

Theorem 1.1 can considered as a special case of the general question: how can the dimension of $\mathbb{T}$ (or a subset of $\mathbb{T}$) be changed by a G-invariant quasiconformal map? One of the fundamental results is that of Bowen [12] which says that for co-compact G, a deformation is either trivial or strictly increases the dimension of $\mathbb{T}$. For more recent results in this direction see [5] and its references. The even more general question of how dimension of sets can be changed under general quasiconformal maps is currently an active area of investigation (e.g. see [1], [6], [10], [11], [13]).

I was led to consider Theorem 1.1 by results in [4] on Ruelle's property. Ruelle's theorem [14] states that if $t \to G_t$ is an analytic family of deformations of a cocompact Fuchsian group G, then $t \to \dim(\Lambda(G_t))$ is a real analytic function, i.e., the Hausdorff dimension of the limit set changes analytically with the deformation. Astala and Zinsmeister showed this fails for many infinitely generated Fuchsian groups (e.g., [2], [3]). In [4] it is shown that Ruelle's property fails if G has big deformations near infinity, i.e., there is a sequence of G-invariant Beltrami coefficients $\{\mu_n\}$ so that $\limsup_n \|\mu_n\|_\infty < 1$, the supports of μ_n leave every compact set of R and $\liminf_n \delta(\mu_n) > 1$ (where $\delta(\mu_n)$ denotes the critical exponent of the deformed group $G_n = f_{\mu_n} \circ G \circ (f_{\mu_n})^{-1}$). In [4] it was also shown that a large number of infinitely generated groups have this property (e.g., all divergence type groups with a positive lower bound on injectivity radius) and it was asked it any infinitely generated groups fail to have it. In this paper we produce a class of infinitely generated groups which do not have big deformations near infinity.

Theorem 1.2. *Suppose $R = \mathbb{D}/G$ has a L-bounded Y-piece decomposition and that for every $\epsilon > 0$ all but finitely many of the Y-pieces are ϵ-bounded. Then G does not have big deformations near infinity.*

Thus for such examples, the question of whether Ruelle's property holds if still open. There clearly are such groups; for example, Figure 1 shows a divergence type surface written as a union of Y-pieces in which the lengths of the gluing geodesics tend to zero.

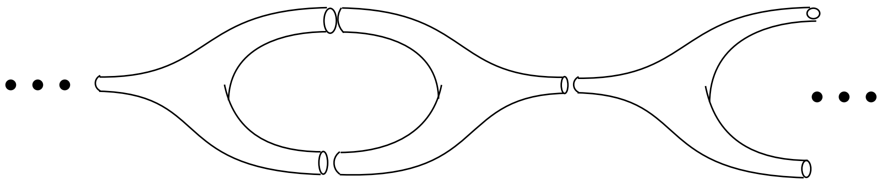

FIGURE 1. A divergence type surface

If $\{G_n\}$ is a sequence of groups acting on 3 dimensional hyperbolic space (consider the ball model), we will say that $\{G_n\} \to G$ geometrically if $G_n(0) \cup S^2$ converges to $G(0) \cup S^2$ in the Euclidean Hausdorff metric. We say that G' is a geometric boundary group of G (and write $G' \in \partial_g G$) if there is a sequence of Möbius transformations $\{g_n\}$ such that $g_n \circ G \circ g_n^{-1} \to G'$. If the injectivity radius at $g_n(0)$ tends to zero then the resulting limit (if it exists) is elementary and non-discrete. Thus we will only take limits over sequences where the injectivity radius is bounded away from zero. Given any such sequence there is always a subsequence which converges to a discrete group. We will call a Fuchsian group exceptional if it is the covering group of the sphere minus m disks and n points where $m+n \leq 3$ and denote this family by $\mathcal{E} = \cup_{m+n\leq 3} \mathcal{E}_{m,n}$. In [4] it is proven that Ruelle's property fails if either (1) $\partial_g G \not\subset \mathcal{E}$, or (2) G is divergence type and $\mathcal{E}_{0,3} \notin \partial_g G$. In this notation, Theorem 1.2 say that if $\partial_g G = \mathcal{E}_{0,3}$ then G has no big deformations near infinity. It is an open problem to characterize the groups which have big deformations near infinity; based on the comments above this will probably involve some measure of how often and how closely R looks like a thrice punctured sphere.

Rather than give separate proofs of Theorems 1.1 and 1.2, we will prove the following result which contains both results as special cases

Theorem 1.3. *Given $L, K < \infty$ and $\eta > 0$ there are $\epsilon > 0$ and $r < \infty$ so that the following holds. Suppose $R = \mathbb{D}/G$ is a Riemann surface which has a decomposition into L-bounded Y-pieces. Suppose $F : R \to S$ is a K-quasiconformal map with Beltrami coefficient μ and* $\mathrm{dist}(\mathrm{supp}(\mu), \Gamma^\epsilon) > r$. *Then the corresponding quasi-Fuchsian deformation of G has limit set of dimension $\leq 1 + \eta$.*

Theorem 1.1 follows because $\Gamma^\epsilon = \emptyset$ (and so every μ satisfies the hypothesis) and Theorem 1.2 follows because Γ^ϵ is compact and hence eventually the supports of $\{\mu_n\}$ will be far from it and because $\delta(G) \leq \dim(\Lambda)$.

The proof of Theorem 1.3 rests on two estimates. The first is from [7] for the dilatation of quasiconformal mappings between Y-pieces; in particular it says that for Y-pieces with very short boundaries, the dilatation is concentrated near the boundary. The second is from [9] estimates the Hölder continuity of f at points which are far from the support of μ. Both estimates are recalled in Section 2. In Section 3 we prove Theorem 1.3.

2. Two estimates

We will think of two Y-pieces as being close to each other if their boundary lengths are close, i.e., we define

$$d(Y_1, Y_2) = \max(|\log \frac{a_1}{a_2}|, (|\log \frac{b_1}{b_2}|, (|\log \frac{c_1}{c_2}|).$$

For generalized Y-pieces we interpret $|\log \frac{a_1}{a_2}|$ as zero if $a_1 = a_2 = 0$ and as $+\infty$ if one is zero and the other is not. Similarly for the b and c terms. The following is from [7].

Lemma 2.1. *Suppose Y_1 and Y_2 are two L-bounded generalized Y-pieces with boundary lengths (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively. Let $D = d(Y_1, Y_2)$. Then there is a quasiconformal map $f : Y_1 \to Y_2$ with constant $K = K(L, D)$ which is affine on each of the boundary components. Moreover, the dilatation K_f of f satisfies*

$$|K_f(z)| \leq 1 + C(L, D) \exp(-2\text{dist}(z, \partial Y))$$

In particular, if Y is 1-bounded and D is fixed, then the quasiconformal dilatation is $\leq 1 + \nu$ except on a A-neighborhood of the boundary, where $A < \infty$ depends only on D and ν. If Y has very short boundaries, this means the dilatation is concentrated near the boundaries and the map is almost conformal on most of Y.

A Carleson square Q in $\mathbb{D}$ is a region of the form

$$Q = Q_I = \{z = re^{i\theta} : \theta \in I, 1 - |I| < r < 1\},$$

for some arc $I \subset \mathbb{T}$. The arc $I = I_Q$ is called the base of Q. For an arc I on the unit circle and $\lambda > 0$ we let λI denote the concentric interval with length $\lambda|I|$. We let

$\ell(Q) = |I|$ and

$$T(Q) = \{z = re^{i\theta} : \theta \in I, 1 - |I| < r < 1 - \frac{1}{2}|I|\},$$

denote the top half of Q. We let z_Q denote the center of $T(Q)$. A dyadic Carleson subsquare of Q is a Carleson square whose base is a dyadic subinterval of I_Q. A particular kind of domain which we will use are the so called "stopping time" regions. These are of the form $W = Q \setminus \cup_j Q_j$ where $\{Q_j\}$ is a disjoint (except for boundaries) collection of dyadic Carleson subsquares of Q.

Now suppose f is a conformal map of the disk which has quasiconformal extension to the plane. Suppose the Beltrami coefficient of this extension is μ (which is obviously supported in $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$). Let ρ denote the hyperbolic metric on either $\mathbb{D}$ or $\mathbb{D}^*$. We wish to know that f is Hölder with exponent close to 1 at points far from $\operatorname{supp}(\mu)$ with respect to the hyperbolic metric on $\mathbb{D}^*$. The following is from [9].

Lemma 2.2. *Given $K < \infty$ and $\eta > 0$ there are $C < \infty$ and $r < \infty$ so that the following holds. Suppose f is a conformal map of the disk with a K-quasiconformal extension to the plane with Beltrami coefficient μ. Also suppose that* $\operatorname{diam}(f(\mathbb{D})) \leq K|f'(0)|$. *Suppose $W = Q \setminus \cup_j Q_j$ is a stopping time region and $\rho(W^*, \operatorname{supp}(\mu)) \geq r$. Then*

$$\sum_j \operatorname{diam}(f(Q_j))^{1+\eta} \leq C \operatorname{diam}(f(Q))^{1+\eta}.$$

The proof involves two steps. First one shows that the Schwarzian derivative of Φ decays (exponentially fast) away from the support of μ. Second, one proves that if the Schwarzian is small enough on W then Φ satisfies a Hölder condition with exponent close to 1. See [9] for details.

3. Proof of Theorem 1.3

Suppose $F : R \to S$ is K-quasiconformal and let $f : \mathbb{D} \to \mathbb{D}$ be a lift to the unit disk. Suppose $Y_1 \subset R$ is a Y-piece. Let Ω_1 be a lift of (the interior of) Y_1 to the unit disk. Suppose γ is a boundary geodesic of Y_1 and that γ lifts to $(-1, 1) \subset \partial\Omega_1$. Normalize f so that it fixes -1 and 1.

Let $E_1 = \partial\Omega_1 \cap \mathbb{T}$ and let $E_2 = f(E_1)$. Let Ω_2 be the hyperbolic convex hull of E_2. Then Ω_2 projects to a Y-piece Y_2 in S and by Lemma 2.1 there is a quasiconformal map $g : \Omega_1 \to \Omega_2$ which agrees with f on E_1 and projects to a map from Y_1 to Y_2.

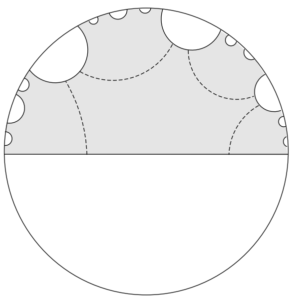

FIGURE 2. Ω, a lift of a Y-piece

The map g is affine on the components of $\partial\Omega_1 \cap \mathbb{D}$ and has quasiconformal constant depending only on $\mathrm{dist}(Y_1, Y_2)$. Let Γ be as in the introduction, and given $\gamma \in \Gamma$, let $\lambda_\gamma = \ell(g(\gamma))/\ell(\gamma)$.

By piecing together maps corresponding to all lifts of all Y-pieces in the decomposition of R, we obtain a map of the disk to the disk which agrees with f on the boundary, but which may be discontinuous across the lift of any boundary geodesic γ. The maps on either side of γ are both affine, so the discontinuity consists of a translation along γ, say of length t_γ (we think of γ as oriented and τ as a signed distance). Thus the map can be made continuous across γ by applying a skew map of the following form. Assume γ is the positive imaginary axis in the upper half plane and Ω lies in the right quadrant. Define

$$\tau(z) = \begin{cases} e^{a(z)} z, & \pi/2 - \theta_0 \arg(z) \leq \pi/2 \\ z, & \arg(z) < \pi/2 - \theta_0, \end{cases}$$

where θ_0 is chosen so small that $\{z : \pi/2 - \theta_0 \arg(z) \leq \pi/2\}$ lies inside the C-neighborhood of γ which is disjoint from the C-neighborhoods of all other boundary geodesics and $a(z)$ is the linear function of $\arg(z)$ which is $-t_\gamma$ when $\arg(z) = \pi/2$ and is 0 when $\arg(z) = \pi/2 - \theta_0$. The map τ is clearly quasiconformal with constant t_γ/θ_0 except for a discontinuity along of the imaginary axis (which is a hyperbolic translation of size $-t_\gamma$) and is the identity outside a C-neighborhood of γ.

Lemma 3.1. *Suppose $R = \mathbb{D}/G$ is a Riemann surface with a L-bounded Y-piece decomposition and suppose γ is a boundary arc for one of the Y-pieces. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal conjugation of G to another Fuchsian group and the Beltrami coefficient of f is μ. Then $\frac{1}{1+C} \leq \lambda_\gamma \leq 1 + C$ and $-C \leq t_\gamma \leq C$*

for some C which depends only on K and L. Given any $\epsilon > 0$ there is an $r < \infty$ (depending only on ϵ, K and L) so that $C \leq \epsilon$ if $\rho(\gamma, \mathrm{supp}(\mu)) \geq r$.

Proof. First we would like to identify λ_γ and t_γ in terms of f's action on the circle. Assume γ lifts to $(-1,1) \subset \mathbb{D}$. Suppose γ on the boundary of two Y-pieces Y_1 and Y_2 (not necessarily distinct) and choose a second boundary component $\gamma_1 \in \partial Y_1$ and $\gamma_2 \in \partial Y_2$ (these may be punctures) and let σ_1 and σ_2 be the shortest curves connecting γ to γ_1 and γ_2 respectively. Let $\tilde{\gamma}_j$, $j = 1, 2$ be a lift of γ_j which is connected to $(-1,1)$ by a lift $\tilde{\sigma}_j$. If γ_j is a puncture then $\tilde{\gamma}_j$ will be a point on the circle. Since γ has length $\leq L$, we may also choose these lifts so that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ both hit $(-1,1)$ within hyperbolic distance $L/2$ of the origin. Let $g \in G$ be a element with translation axis $(-1,1)$. By replacing g by a power of itself we may assume that $1 \leq \rho(0, g(0)) \leq 1 + L$.

Given a ordered set of six points on the unit circle $E = (x_1, x_2, y_1, y_2, z_1, z_2)$, let γ_x, γ_y and γ_z denote the hyperbolic geodesics in $\mathbb{D}$ with endpoints (x_1, x_2), (y_1, y_2) and (z_1, z_2) respectively. We allow $y_1 = y_2$ or $z_1 = z_2$; in either case, the geodesics γ_y and γ_z are just interpreted as the points on the circle. However, all other pairs must be distinct. Let p_y and p_z be the points on γ_x which are closest to γ_y and γ_z. Let $d(E) = \rho(p_y, p_z)$. See Figure 3.

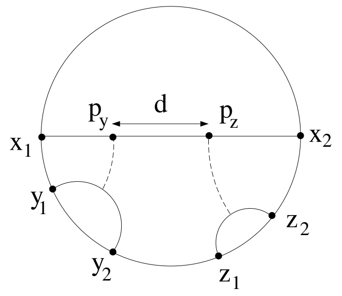

FIGURE 3. Definitions of $d(E)$.

Let $x_1 = -1, x_2 = 1$, let y_1 and y_2 be the endpoints of $\tilde{\gamma}_1$ and let z_1 and z_2 be the endpoints of $g(\tilde{\gamma}_1)$. Then clearly, $\lambda_\gamma = d(f(E))/d(E)$. Since f is quasisymmetric on the circle it is easy to see that this will be bounded and bounded away from zero depending only on the quasiconformal constant of f. If $\rho(\gamma, \mathrm{supp}(\mu)) \geq r$, then with our normalization, μ is supported in the annulus $\{z : 1 - Ce^{-r} \leq |z| \leq 1 + Ce^{-r}\}$ for some absolute $C < \infty$. As $r \to \infty$ and K remains fixed, the compactness of the family of K-quasiconformal mappings implies f must tend to the identity (we have normalized f so it fixes -1, 1 and ∞). Thus for r large enough, every point on the unit circle is moved less than ϵ by f and hence it is easy to see that $d(f(E))$ differs from $d(E)$ by at most $C\epsilon$. Since $\rho(p_y, p_z) \geq 1$ and both points are at most hyperbolic distance $L/2$ from the orgin, it is easy to deduce that $1 - C\epsilon \leq d(f(E))/d(E) \leq 1 + C\epsilon$.

We can compute t_γ in almost exactly the same way, except that now we let z_1 and z_2 be the endpoints of $\tilde{\gamma}_2$ and then $t_\gamma = d(f(E)) - d(E)$. As above, this is clearly bounded in terms of K and if γ is far from the support of μ, the same proof as above shows $|t_\gamma| \leq C\epsilon$. This proves the lemma. □

Applying these estimates and Lemma 2.1 to the map g constructed above we obtain a quasiconformal map h of the disk to itself which agrees with f on the circle and which satisfies the following properties:

1. h maps Y-pieces to Y-pieces
2. The quasiconformal constant of h is bounded depending only on K, the quasiconformal constant of f.
3. Given K there is an $\epsilon > 0$ such that on any ϵ-bounded Y-piece the quasiconformal dilatation $K_h(z)$ of h is bounded by $1 + C_1(K)\exp(-2\text{dist}(z, \partial Y))$.
4. Given any $K < \infty$ and $\nu > 0$ there is an $r < \infty$ such that on any Y-piece whose finite boundary is at least distance r from the support of μ, $K_h(z)$ is bounded by $1 + \nu$.

By assumption in Theorem 1.3, every Y-piece in the decomposition of R is either ϵ-bounded or has boundary geodesics at least distance r from the support of μ. Thus the quasiconformal map h can be factored as $h = h_2 \circ h_1$ where h_1 is also $C(K)$-quasiconformal and has dilatation supported in an C neighborhood of the ϵ-geodesics and h_2 is is $1 + \nu$ quasiconformal. Hence h_2 is Hölder continuous with exponent $(1 + \eta)^{-1}$ (if ν is small enough) and hence can only raise the Hausdorff dimension on any set by at most a factor of $1 + \eta$. Thus it suffices to show $h_2(\mathbb{T})$ has dimension at most $1 + \eta$. All we need to prove is the following (which does not involve any group).

Lemma 3.2. *Suppose we are given $K < \infty$, $\eta > 0$ and $s < \infty$. Then there is a $r < \infty$ so that the following holds. Suppose $\{\gamma_n\}$ is a collection of geodesics in the hyperbolic disk such that any two of them are at least hyperbolic distance r apart.*

Suppose f is a quasiconformal map of the plane which is conformal inside the disk and whose dilatation outside the disk satisfies $K_f(z) \le K$ if $\rho(z^*, \cup_m \gamma_n) \le s$ and is $= 1$ otherwise. Then the Hausdorff dimension of $f(\mathbb{T})$ is less than $1+\eta$.

Proof. Divide disk into two types of regions; those which are near a geodesic and those which are not. More precisely, let $t \ge s$ (to be chosen below) and let $\{A_n\}$ be the t-neighborhoods of these geodesics. By taking r large enough (e.g., $\ge 2t+2s+10$) we may assume that if $n \ne m$ then $\rho(A_n, A_m) \ge 2t+10$. We will say a dyadic Carleson square Q is type 1 if $T(Q)$ hits $A = \cup_n A_n$ and Q is type two if $T(Q)$ is disjoint from A. For each type 1 square form a stopping time region $W \subset Q$ by removing all maximal type 2 squares. If Q is type 2, then define $W \subset Q$ by removing all maximal type 1 subsquares. This process divides the disk into type 1 and type 2 regions, as illustrated in Figure 4

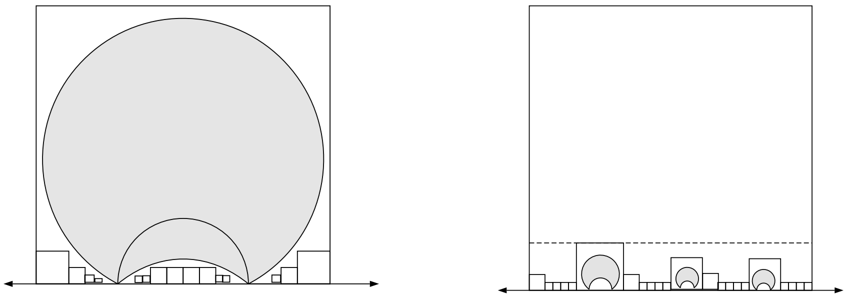

FIGURE 4. Type 1 and type 2 regions

Note that if $W = Q \setminus \cup_j Q_j$ is a type 2 region, then $T(Q)$ is within distance $t+1$ of one geodesic and $T(Q_j)$ is within distance $t+1$ of a different one. Thus the top halves of these squares are more than $r - 2t - 4$ apart. Thus

$$\text{(1)} \qquad \operatorname{diam}(Q_j) \le C\exp(-r+2t)\operatorname{diam}(Q) \le C\exp(-r/2)\operatorname{diam}(Q),$$

if we assume $r \ge 4t$.

Suppose $\eta > 0$. If $W = Q \setminus \cup_j Q_j$ is a type 1 region then

$$\sum_j \operatorname{diam}(f(Q_j))^{1+\eta} \le M\operatorname{diam}(f(Q))^{1+\eta},$$

where M depends only on K, η and s, since f is Hölder of some positive order (since it is K-quasiconformal) and the diameters of the Q_j's decay geometrically. If it is a type 2 region, we claim that

$$\sum_j \mathrm{diam}(f(Q_j))^{1+\eta} \leq m\mathrm{diam}(f(Q))^{1+\eta},$$

where m depends on K, η, s and r and for a fixed K may be taken as close to zero as we wish by making r large enough. To prove the claim, choose r_0 so that Lemma 2.2 holds for K and $\eta/2$. Then for $r \geq 4r_0$, Lemma 2.2 and (1) imply

$$\begin{aligned}
\sum_j \left(\frac{\mathrm{diam}(f(Q_j))}{\mathrm{diam}(f(Q))}\right)^{1+\eta} &\leq C(K,\eta) \sum_j \frac{\mathrm{diam}(Q_j)}{\mathrm{diam}Q})^{1+\eta/2} \\
&\leq C(K,\eta) \sup_j \left(\frac{\mathrm{diam}(Q_j)}{\mathrm{diam}(Q)}\right)^{\eta/2} \sum_j \frac{\mathrm{diam}(Q_j)}{\mathrm{diam}Q} \\
&\leq C(K,\eta) C e^{-r\eta/4}.
\end{aligned}$$

This proves the claim if r is large enough (depending only on η and K).

If we take m so small that $mM < 1$ then dropping down n generations we get a covering of $f(\mathbb{T})$ by balls B_j which satisfies $\sum_j \mathrm{diam}(B_j)^{1+\eta} < (mM)^n \to 0$. Thus $\delta(\mu) \leq \dim(f_\mu(\mathbb{T})) \leq 1+\eta$. This completes the proof of Theorem 1.3. ◻

References

[1] K. Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173(1):37–60, 1994.

[2] K. Astala and M. Zinsmeister. Holomorphic families of quasi-Fuchsian groups. *Ergod. Th. & Dynam. Sys.*, 14:207–212, 1994.

[3] K. Astala and M. Zinsmeister. Abelian coverings, Poincaré exponent of convergence and holomorphic deformations. *Ann. Acad. Sci. Fenn.*, 20:81–86, 1995.

[4] C.J. Bishop. A criterion for the failure of Ruelle's property. 1999. Preprint.

[5] C.J. Bishop. Divergence groups have the Bowen property. 1999. Preprint.

[6] C.J. Bishop. Quasiconformal mappings which increase dimension. *Ann. Acad. Sci. Fenn. Math.*, 24(2):397–407, 1999.

[7] C.J. Bishop. Non-rectifiable limit sets of dimension one. 2000. Preprint.

[8] C.J. Bishop and P.W. Jones. Hausdorff dimension and Kleinian groups. *Acta. Math*, 179:1–39, 1997. also appeared as Stony Brook IMS preprint 1994/5.

[9] C.J. Bishop and P.W. Jones. Compact deformations of Fuchsian groups. 1998. preprint.

[10] C.J. Bishop and J. Tyson. The conformal dimension of the antenna set. to appear in Proc. Amer. Math. Soc.

[11] C.J. Bishop and J. Tyson. Locally minimal sets for quasiconformal maps. to appear in Ann. Acad. Sci. Fenn.

[12] R. Bowen. Hausdorff dimension of quasicircles. *Publ. I. H. E. S.*, 50:11–25, 1979.

[13] F. W. Gehring and J. Väisälä. Hausdorff dimension and quasiconformal mappings. *J. London Math. Soc. (2)*, 6:504–512, 1973.

[14] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dym. Sys.*, 2:99–107, 1982.

C.J. Bishop, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651
E-mail address: bishop@math.sunysb.edu