Conformal Mapping in Linear Time

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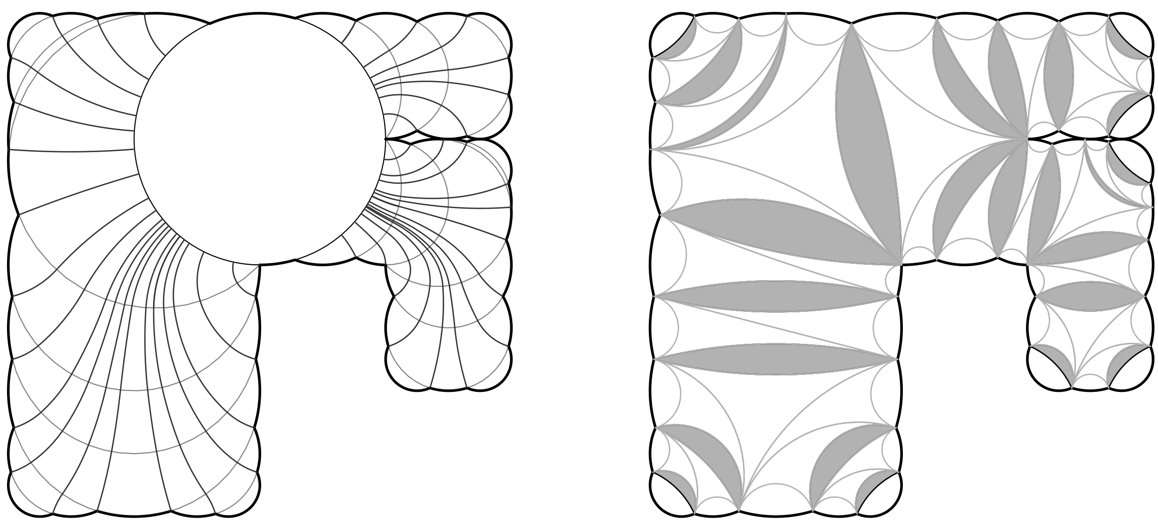
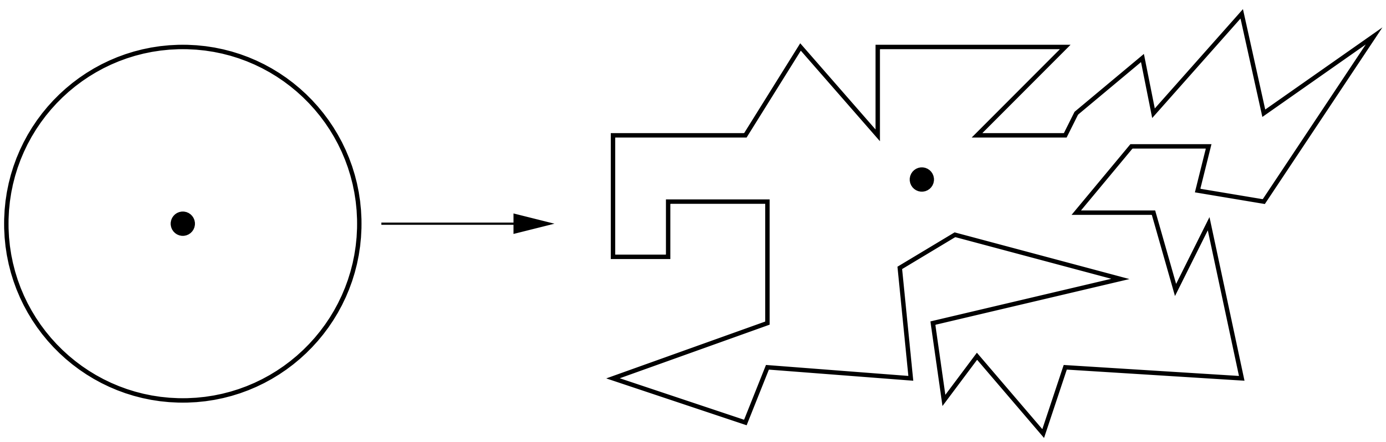


copies of lecture slides available at
`www.math.sunysb.edu/~bishop/lectures`

Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \Omega \to \mathbb{D}$.



The Schwarz-Christoffel formula:

$$f(z) = A + C \int^z \prod_{k=1}^n (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

Gives conformal map to polygon. α's are interior angles, z's map to vertices.

Many numerical methods:

- "Zipper": Don Marshall
- Circle packing: Ken Stephenson
- S-C solvers: Trefethen, Banjai, Driscoll, Vavasis
- holomorphic forms and Ricci flow: Yau and Gu
- Random waks: Braverman *et. al.*
- Methods of Wegmann, Theodorsen, Fornberg
- See surveys by Delillo (1994) and Porter (2007).

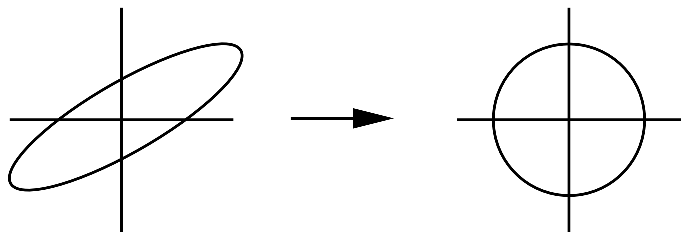
Porter divides methods into two classes:

$$\text{easy methods } \Omega \to \mathbb{D}$$

$$\text{fast methods } \mathbb{D} \to \Omega$$

Can we approximate a conformal map to an n-gon with accuracy ϵ in time $C(n, \epsilon)$?

A map is K-quasiconformal if preimages of small disks are ellipses of eccentricity $\leq K$.



$$f_z = \frac{1}{2}(f_x - i f_y), \quad f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$$

$$|\mu_f| = |\frac{f_{\bar{z}}}{f_z}| \leq \frac{K-1}{K+1} < 1$$

Fact: If $\|\mu\|_\infty < 1$ $\exists$ QC f so $\mu = \mu_f$.

- f is conformal iff $\mu_f \equiv 0$
- If $\mu_g = \mu_f$ then $f \circ g^{-1}$ is conformal.
- $\mathrm{QC_K}$ = K-QC maps. If $\mathbf{w}, \mathbf{z} \subset \mathbb{T}$ are n-tuples,

$$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists h \in \mathrm{QC_K}, \mathrm{h}(\mathbf{w}) = \mathbf{z}\}$$

Theorem: If $\partial\Omega$ is an n-gon we can compute a $(1+\epsilon)$-QC map $\mathbb{D} \to \Omega$ in time $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Maps are stored as $O(n)$ p-term power series where $p = O(\log \frac{1}{\epsilon})$. Only $O(1)$ FFT's per vertex.

Theorem: Suppose $\partial\Omega$ is an n-gon. We can construct points $\mathbf{w} = \{w_1, \dots, w_n\} \subset \mathbb{T}$ so that:

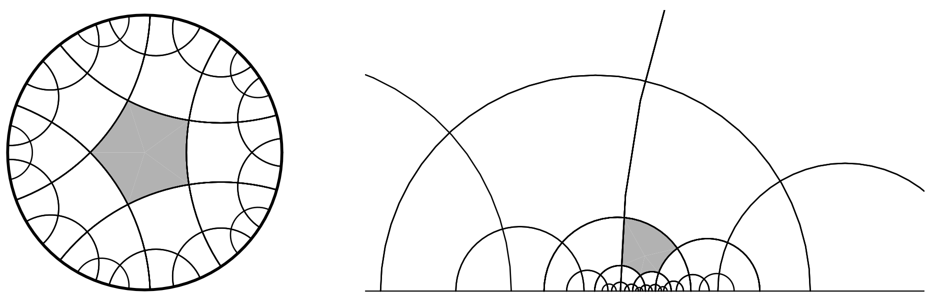
1. uses at most $C(\epsilon)n$ steps.
2. $d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$.

where $\mathbf{z} = f^{-1}(\mathbf{v})$ are conformal prevertices and $C(\epsilon) = C + C \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}$

Hyperbolic disk: Metric on $\mathbb{D}$,

$$d\rho = |dz|/1 - |z|^2 \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}$$

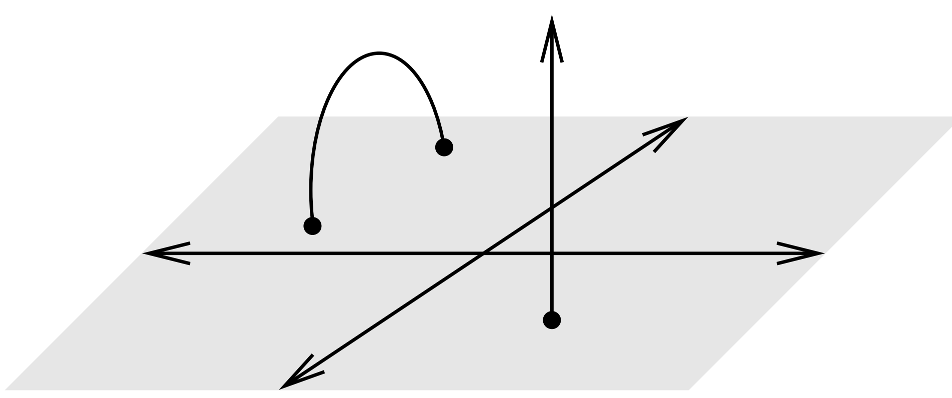
Geodesics are circles or lines orthogonal to $\partial\mathbb{D}$.



Hyperbolic space: Metric on $\mathbb{R}^3_+$,

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}^2$.



- f is a **bi-Lipschitz** if

$$\frac{1}{A}\rho(x,y) \leq \rho(f(x),f(y)) \leq A\rho(x,y).$$

- f is a **quasi-isometry** if

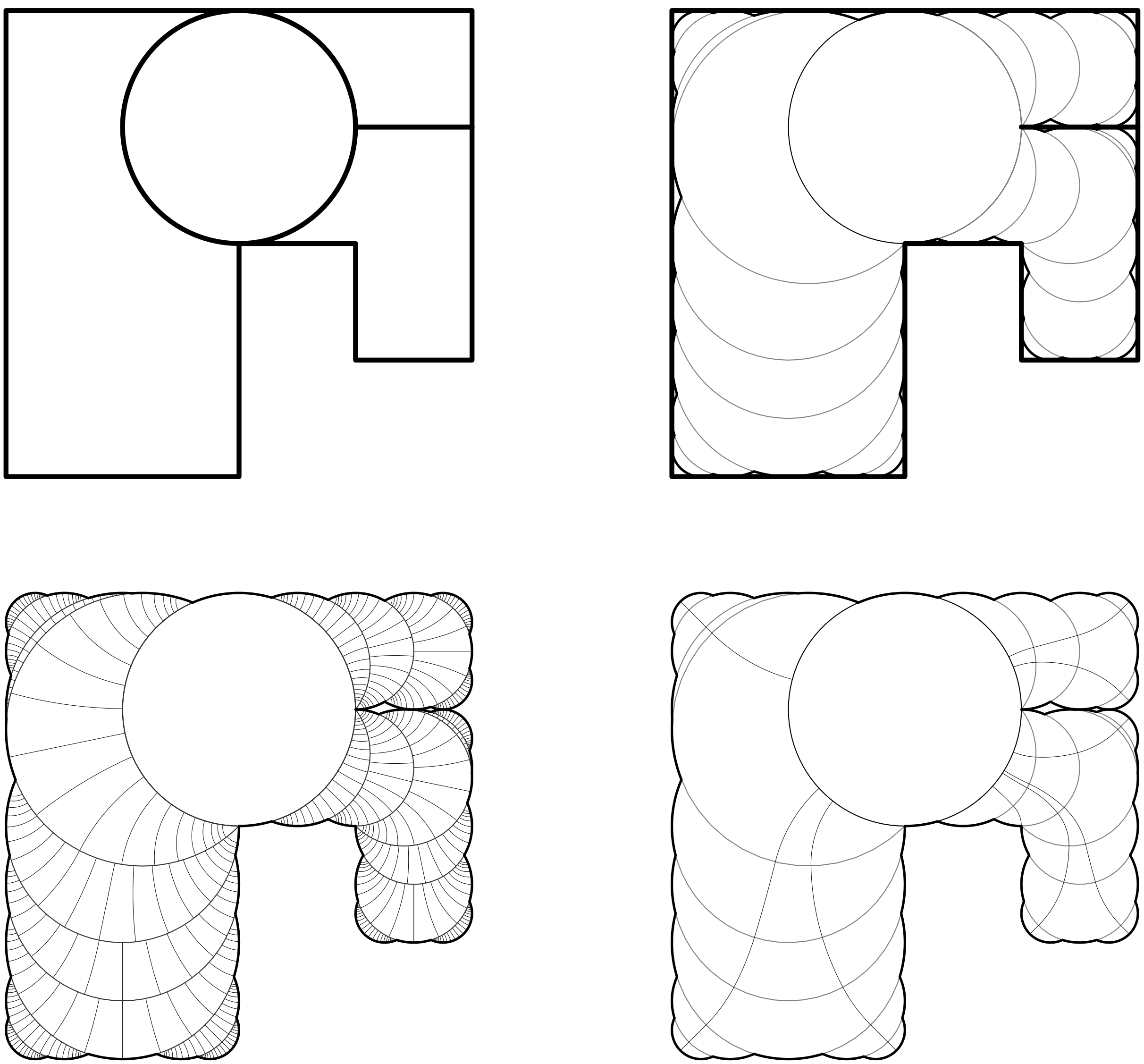
$$\frac{1}{A}\rho(x,y) - B \leq \rho(f(x),f(y)) \leq A\rho(x,y) + B.$$

- QI=BL at "large scales".

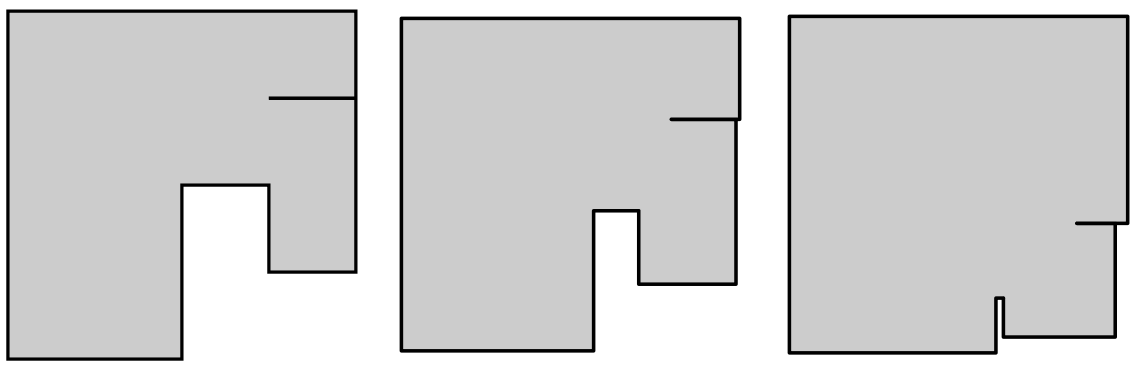
- On hyperbolic disk, BL $\Rightarrow$ QC $\Rightarrow$ QI.

Theorem: $f : \mathbb{T} \to \mathbb{T}$ has a QC-extension to interior iff it has QI-extension (hyperbolic metric) iff it has a BL-extension.

Fast Almost Riemann Mapping Theorem:
Can construct a K-QC map from n-gon Ω to disk in $O(n)$ time, and K independent of n and Ω.

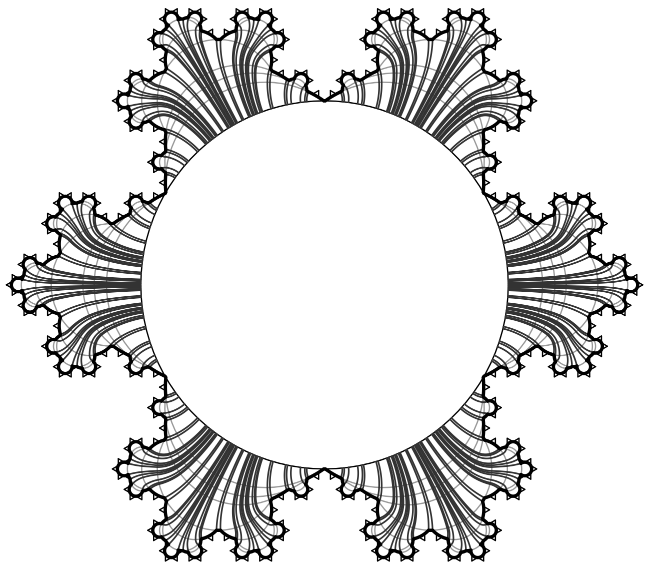


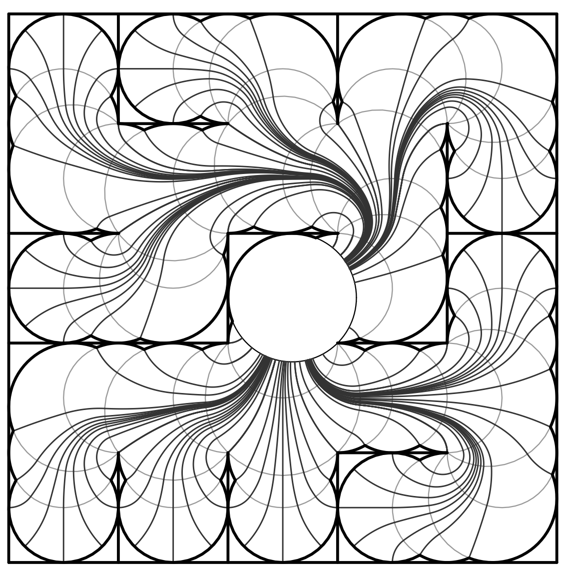
Schwarz-Christoffel maps:



If we plug in ι-images of vertices we almost get the correct polygon (center). Using uniformly spaced points is clearly worse (right).

Map for more complicated domains



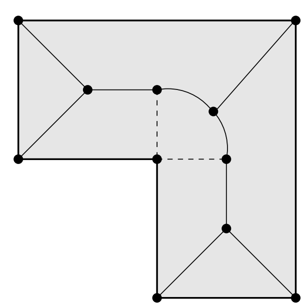


Approximate by disks such that $\#(\partial D \cap \partial \Omega) \geq 2$. Centers are the **medial axis**.

For polygons is a finite tree with 3 types of edges:
- point-point bisectors (straight)
- edge-edge bisectors (straight)
- point-edge bisector (parabolic arc)

Chin-Snoeyink-Wang (1998) gave $O(n)$ algorithm. Uses Chazelle' theorem (1991): an n-gon can be triangulated in $O(n)$ time.

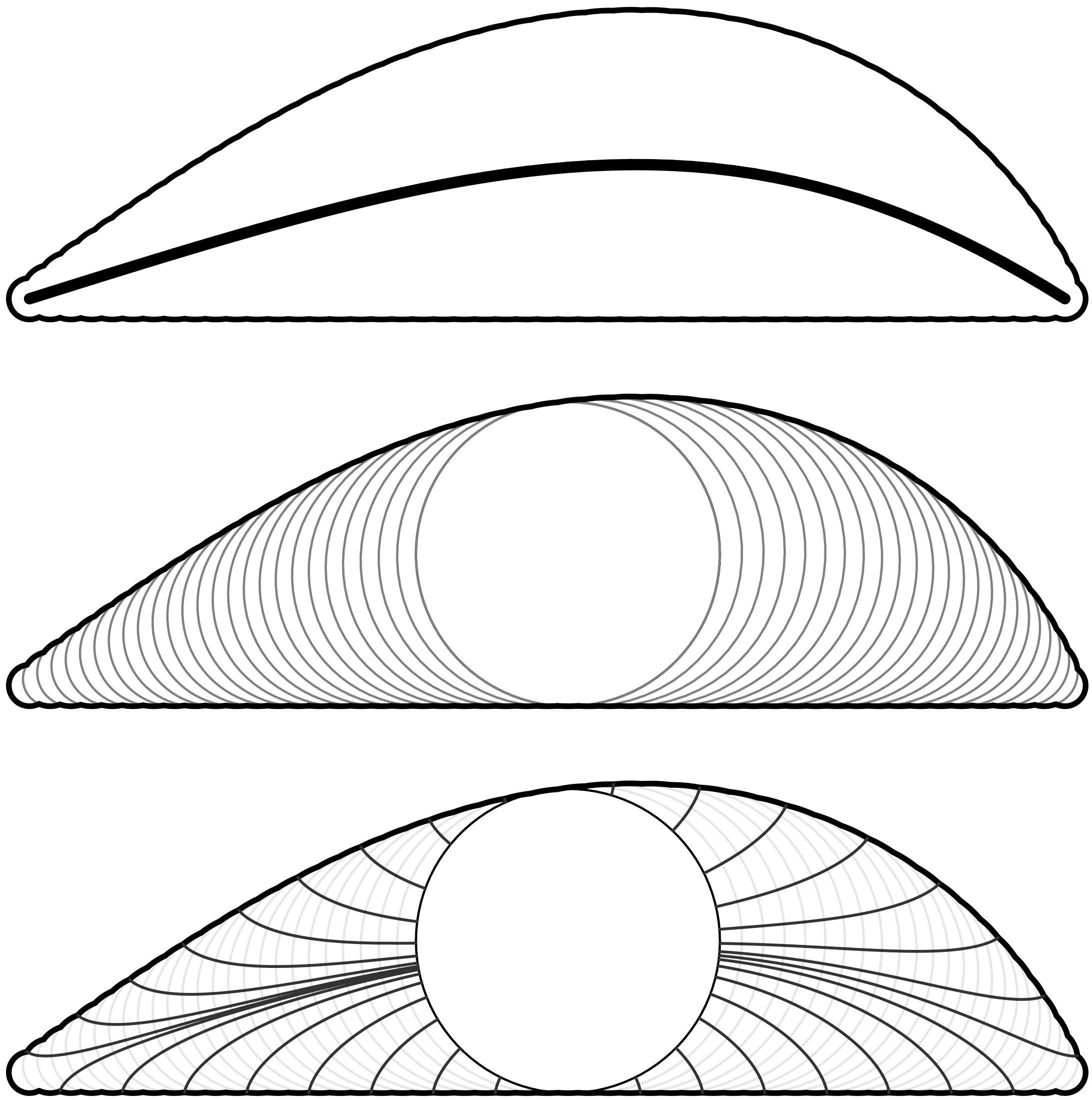
Medial axis is subset of boundary of Voronoi cells where sites are edges of polygon.



For applications see:

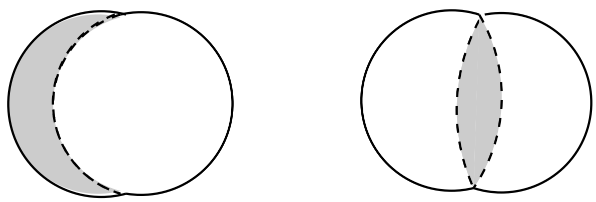
`www.ics.uci.edu/~eppstein/gina/medial.html`

Similar flow for any simply connected domain.

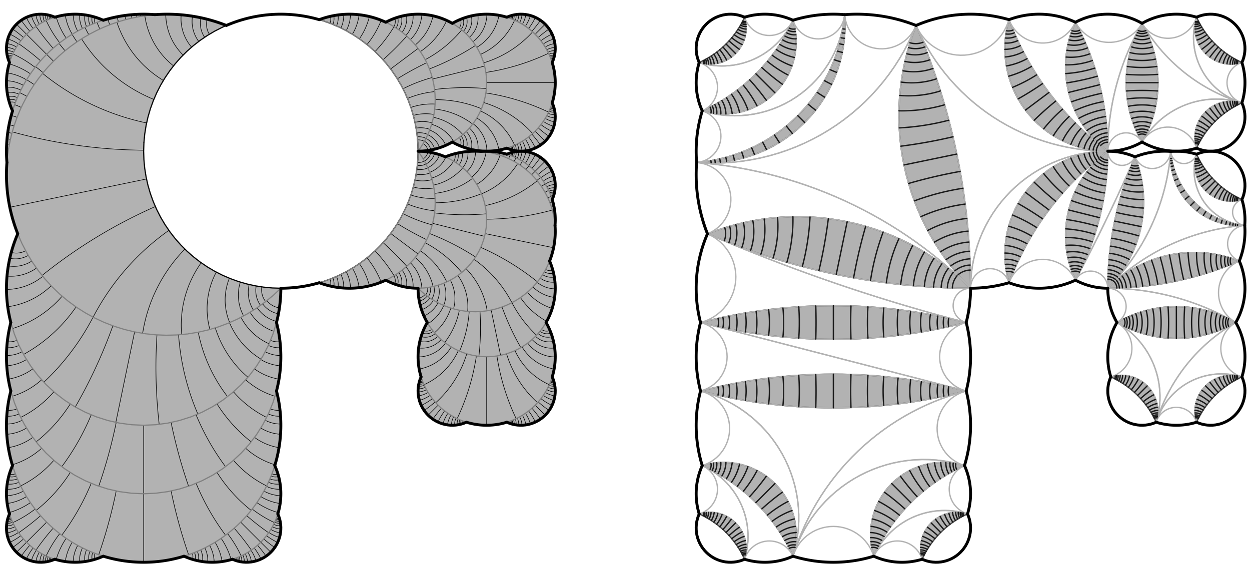


The obvious "collapse tangential crescents map" is not a quasi-isometry (it even maps some interior points to boundary!)

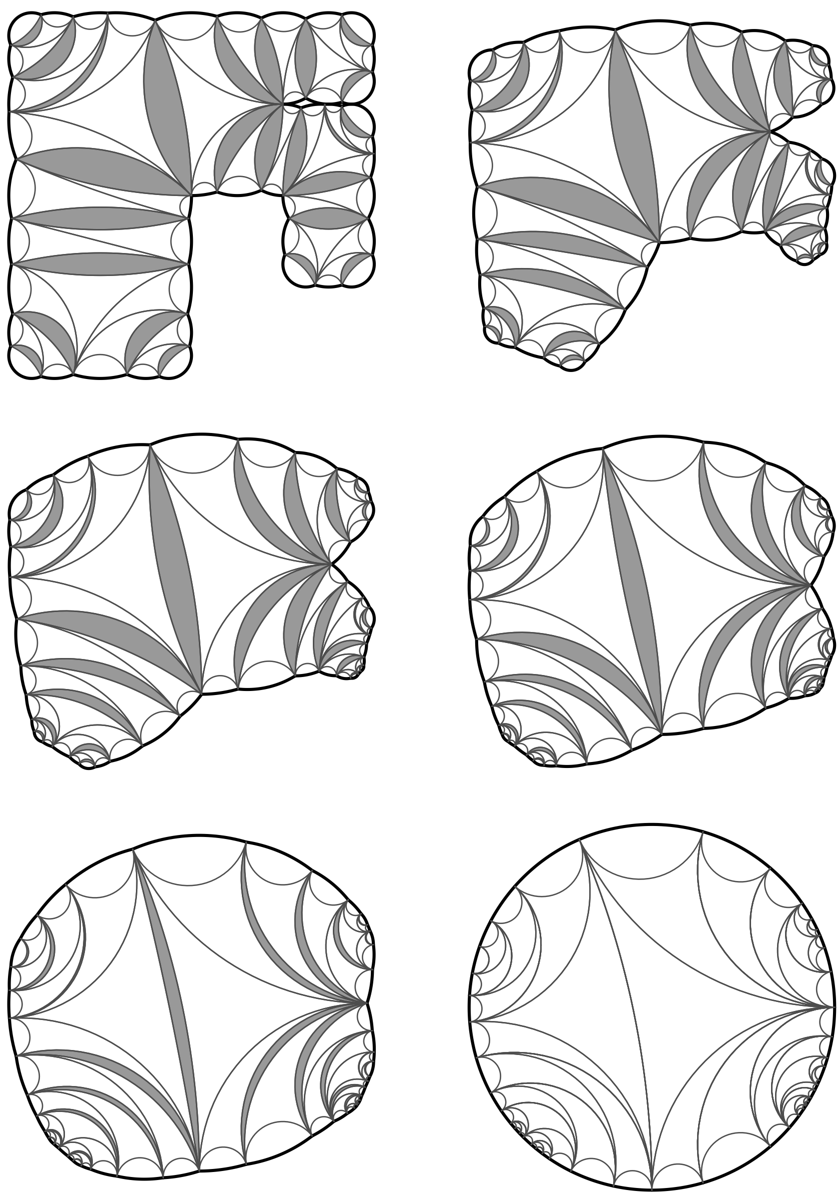
Two ways to write union of disks using crescents.



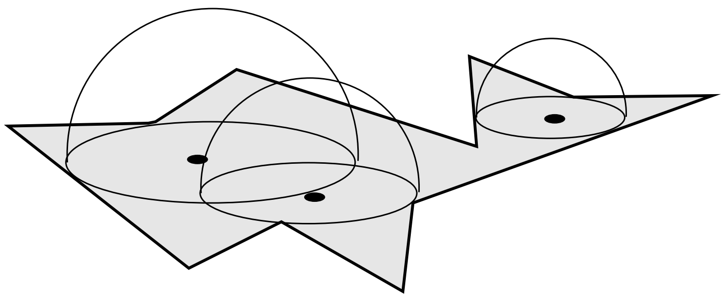
We call these **tangential** and **normal** crescents.



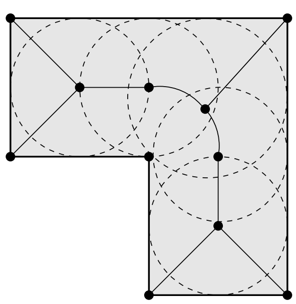
Collapsing normal crescents is fast to compute and gives quasi-isometry. (Angle scaling family)

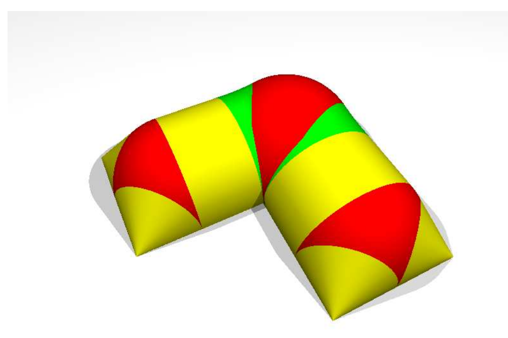


The **dome** of Ω is boundary of union of all hemispheres with bases contained in Ω.



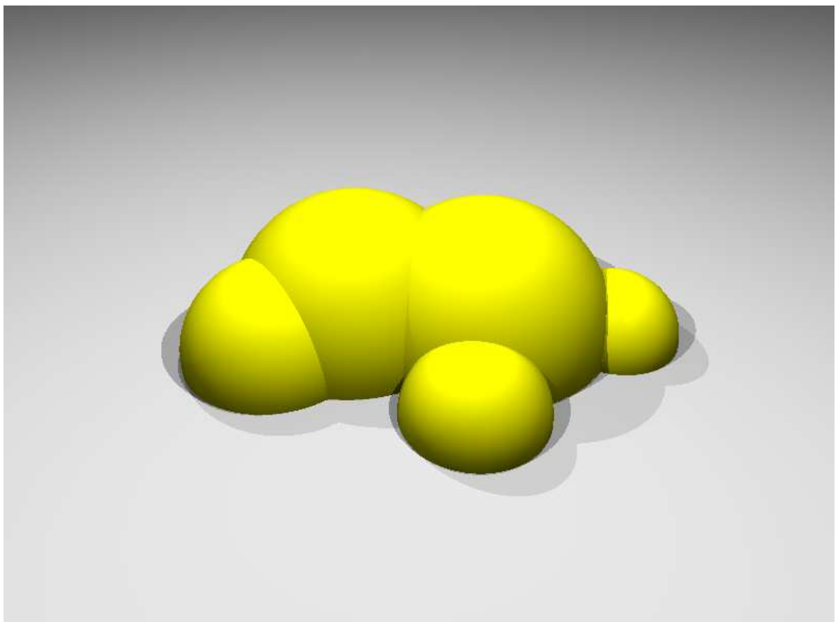
Equals boundary of hyperbolic convex hull of Ω^c. Only need consider medial axis disks.

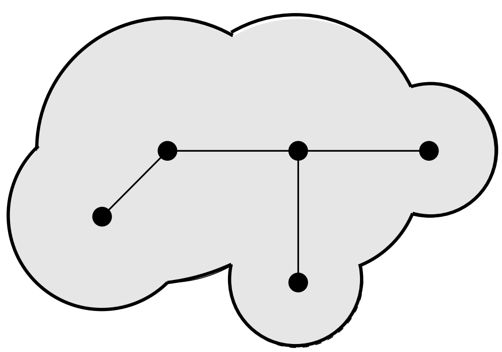


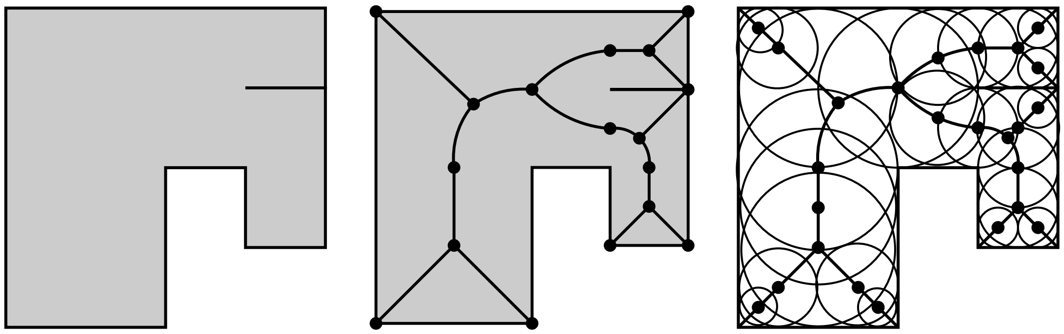


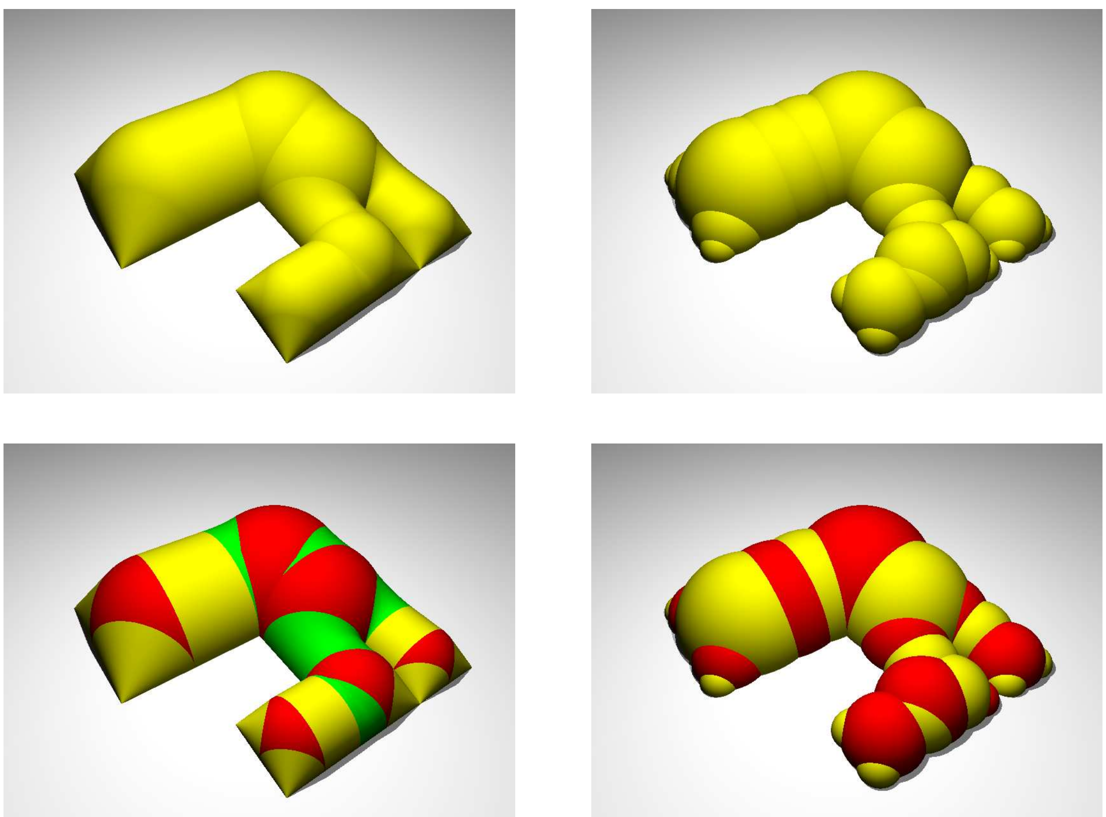
Google("medial axis")= 26,300
Google("hyperbolic convex hull")= 71

Finitely bent domain (= finite union of disks).

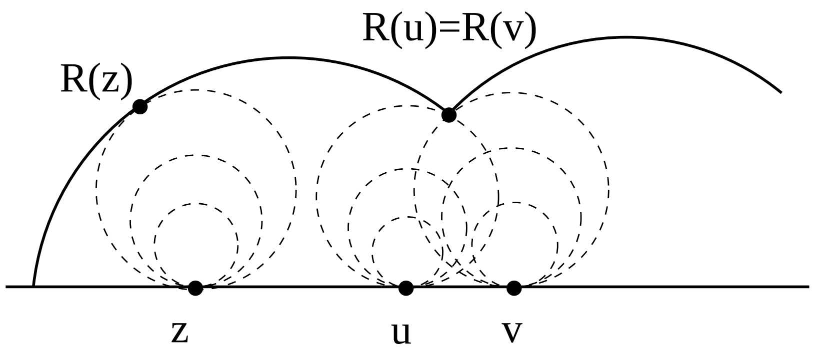








Nearest point retraction $R : \Omega \to \mathrm{Dome}(\Omega)$**:** Expand ball tangent at $z \in \Omega$ until it hits dome.



$$\text{normal crescents} = R^{-1}(\text{bending lines})$$
$$\text{gaps} = R^{-1}(\text{faces})$$

Theorem: R is a quasi-isometry.

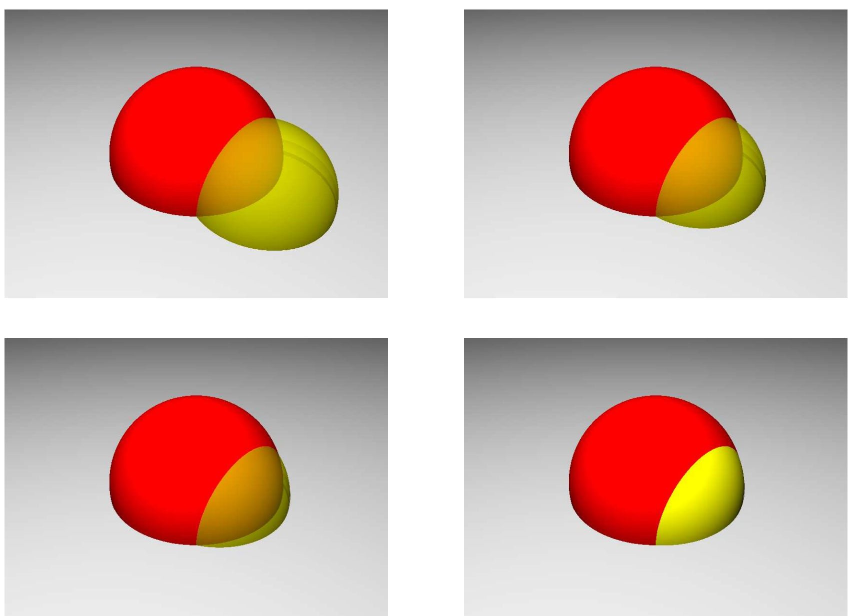
Corollary (Sullivan, Epstein-Marden): $\exists$ K-QC $\sigma : \Omega \to \mathrm{Dome}$, $\sigma = \mathrm{Id}$ on $\partial\Omega$.

Thurston conjectured $K = 2$. False by Epstein and Markovic. $K < 7.82$ is known.

Let ρ be the hyperbolic path metric on Dome.

Theorem (Thurston): There is an isometry ι from (Dome, ρ) to the hyperbolic disk.

For finitely bent domains rotate around each bending geodesic by an isometry to remove the bending (more obvious if vertices are 0 and ∞).



$\iota \circ R : \Omega \to \text{Dome} \to \mathbb{D}$ is uniformly QI map! Equals crescent collapsing map on boundary.

Theorem: If $\partial\Omega$ is an n-gon we can compute a $(1+\epsilon)$-QC map $\mathbb{D} \to \Omega$ in time $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Proof of theorem is in two steps:

Step 1: Given $\epsilon < \epsilon_0$ and $(1+\epsilon)$-QC $f_n : \Omega \to \mathbb{D}$ construct $(1+\epsilon^2)$-QC map $f_{n+1} : \Omega \to \mathbb{D}$. Takes time $O(1 + \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Step 2: Discretize angle scaling family

$$(\Omega_0, V_0), \ldots, (\Omega_N, V_N)$$

- $\Omega_0 = \mathbb{D}$,
- $\Omega_N = \Omega$, $V_N = V$,
- δ-QC maps $g_k : \Omega_k \to \Omega_{k+1}$, $V_k \to V_{k+1}$.

If $\delta < \epsilon_0/2$ then can find conformal maps by Step 1 and induction.

Amazing Fact 1: ϵ_0 independent of Ω and n.

Amazing Fact 2: N independent of Ω and n.

Consequence: Get ϵ_0 approximation in time $O(n)$ (independent of Ω).

Then just repeat Step 1 until get ϵ accuracy :

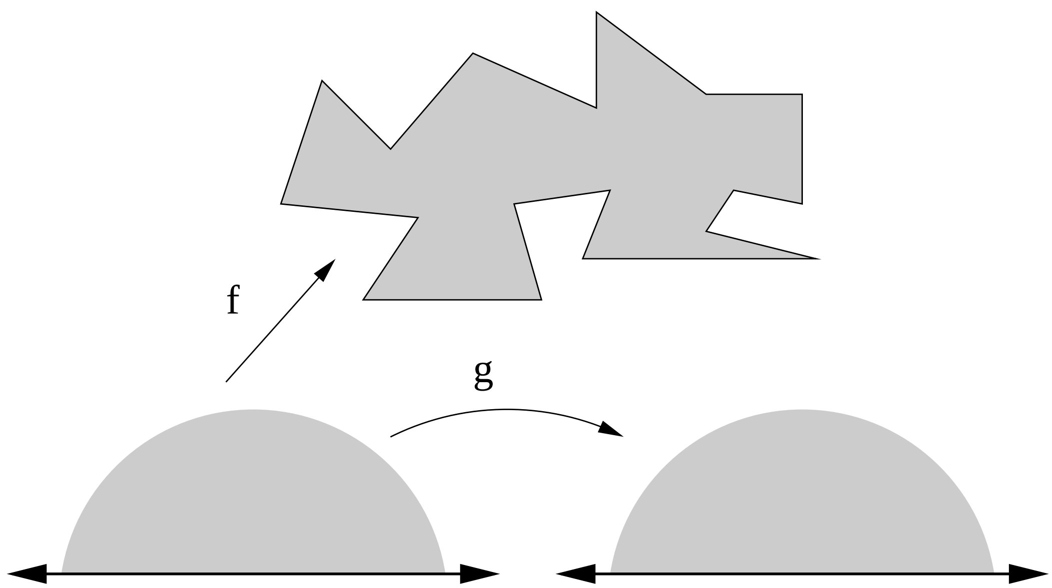
$$\epsilon_0, C\epsilon_0^2, \ldots, C^k\epsilon_0^{2^k}.$$

About $\log\log\epsilon$ iterations suffice and time for kth iteration is $O(k2^{2k})$, so final step dominates.

Idea for Step 1: Suppose

$$f : \mathbb{H} \to \Omega, \qquad g : \mathbb{H} \to \mathbb{H}, \qquad \mu_f = \mu_g.$$

Then $f \circ g^{-1} : \mathbb{H} \to \Omega$ is conformal.

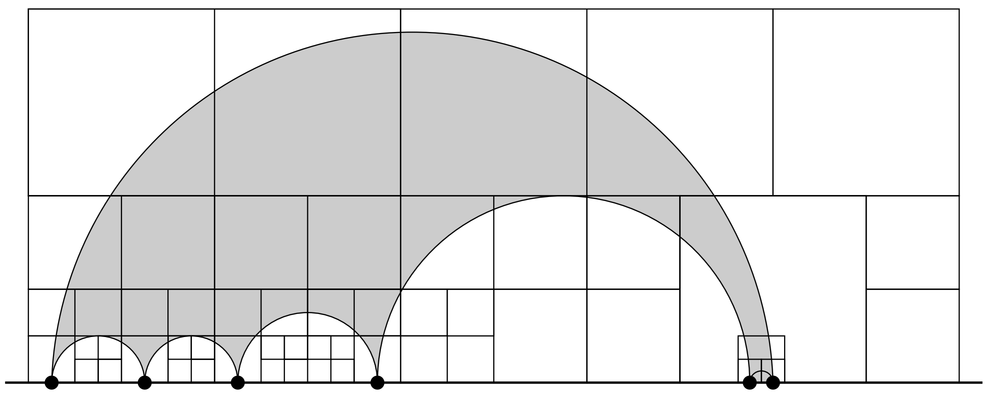


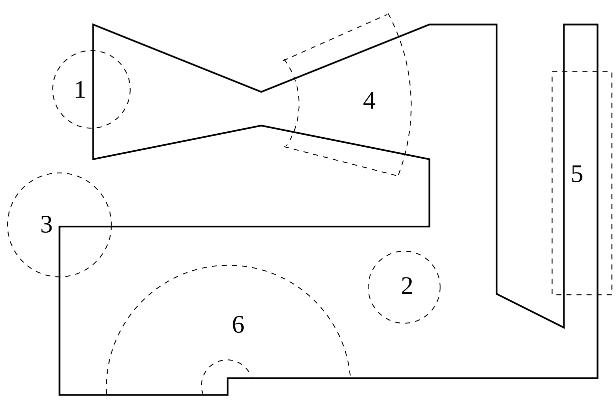
Can't solve $\mu_g = \mu_f$ (i.e., $g_{\bar{z}} = \mu g_z$) exactly in finite time, but can quickly solve

$$g_{\bar{z}} = (\mu_f + O(\|\mu_f\|^2))g_z.$$

Then $f \circ g^{-1}$ is $(1 + O(\|\mu_f\|^2))$-QC.

Cut $\mathbb{H}$ into $O(n)$ pieces on which f, f^{α} or $\log f$ has nice series representation. Need $p = O(|\log \epsilon|)$ terms on each piece to get ϵ accuracy.



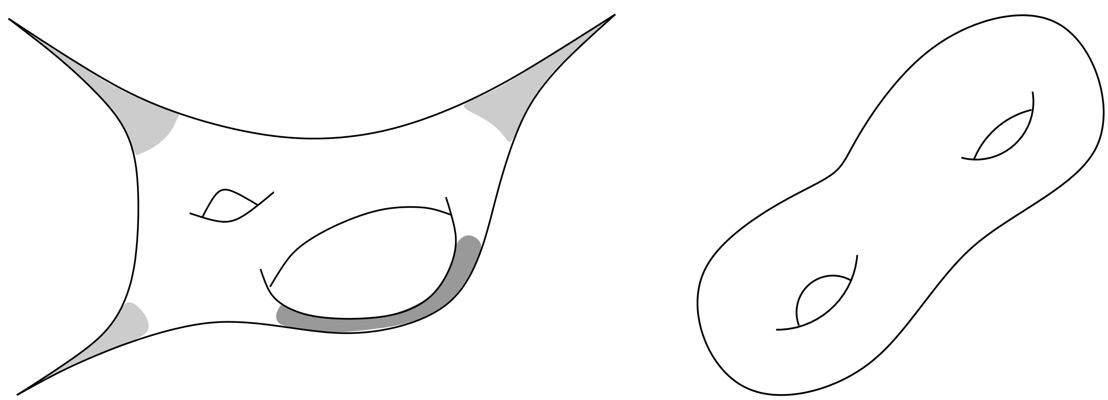


Note that 4,5,6 correspond to arches. Need arches for linear complexity bound.

Thick/Thin decompositions of polygons.
Needed to get only $O(n)$ pieces in decomposition.

A hyperbolic manifold can be partitioned into thick and thin parts based on the size of the injectivity radius at each point.

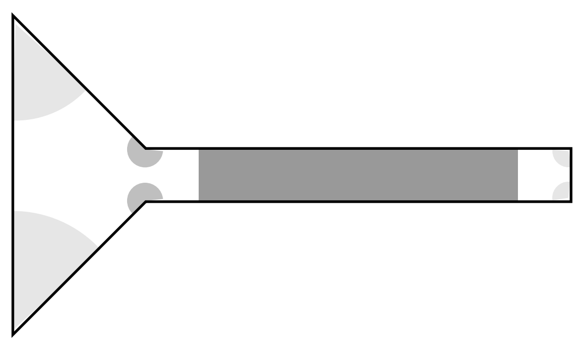
Thin parts often cause technical difficulties, but there are only a few types of thin parts and each has a well understood shape.

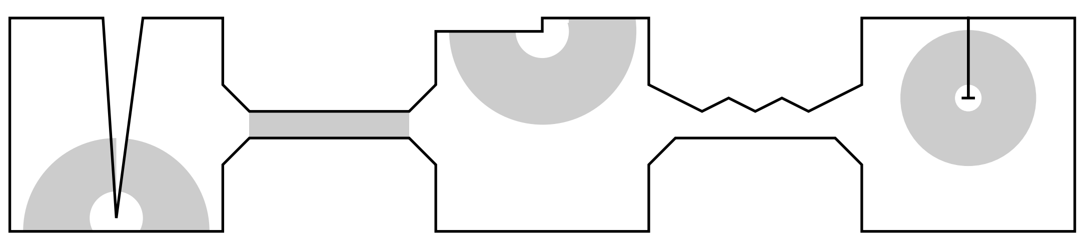


There is analogous decomposition of polygons.

An ϵ-thin part corresponds to two edges whose extremal distance in Ω is $< \epsilon$. (Roughly, distance apart is small compared to minumum diameter.)

Parabolic thin parts occur at every vertex. Hyperbolic thins parts use non-adjacent edges.





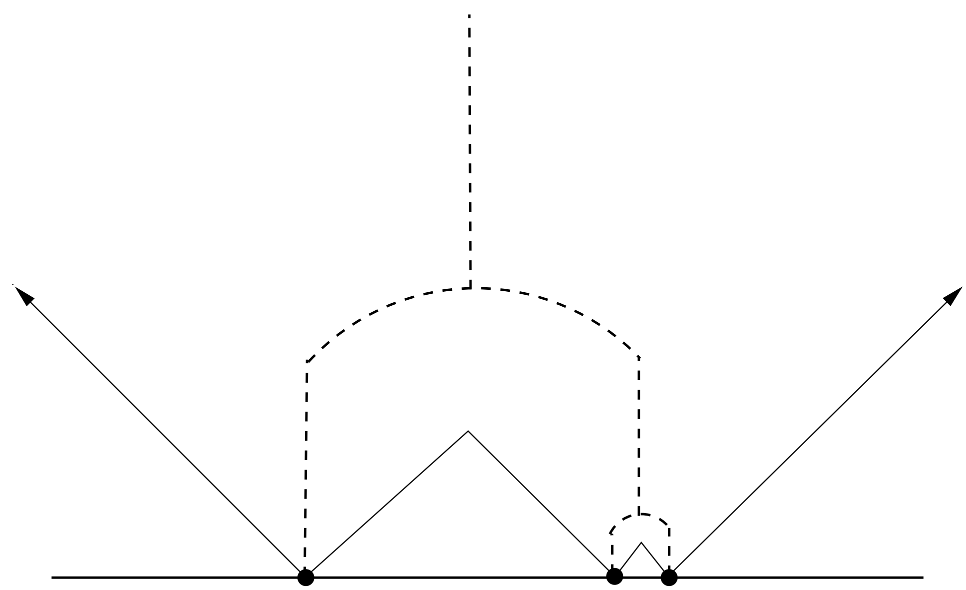
There is a version of Mumford-Bers compactness.

• At most $O(n)$ thin parts.

• Can be located in linear time using iota map.

• Conformal maps onto thin parts "explicitly known".

• Remaining thick components have good approximations by $O(n)$ disks.



Application to meshing:
Marshall Bern and David Eppstein showed any n-gon has quadrilateral mesh with all angles $\leq 120°$ which can be found in time $O(n \log n)$.

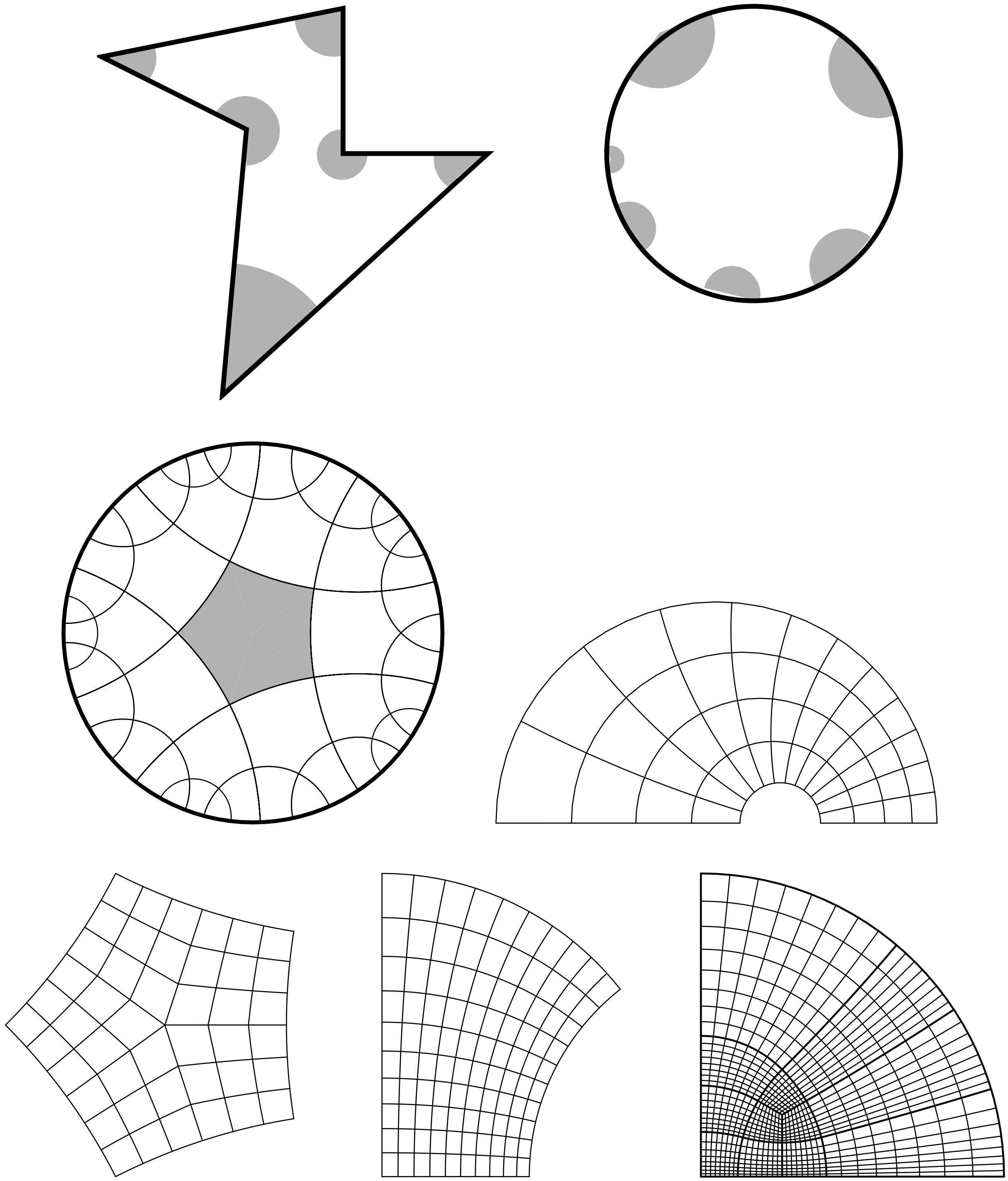
Theorem: Any n-gon has quadrilateral mesh with all new angles between $60°$ and $120°$ which can be found in time $O(n)$.

Idea of proof

- Decompose polygon into thick and thin parts.

- Find explicit meshes in thin parts (known shapes).

- Conformally map thick parts to disk and use hyperbolic geometry to mesh. Map back to domain.



- Is the estimate $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$ optimal?

What is bit complexity? Joe Mitchell pointed out to me that linear time in infinite-precision need not be linear time in finite precision.

- Find optimal QC maps with constraints?

Quasiconformal Jacobian problem: given $J > 0$, is there a QC map f so that

$$\text{area}(f(E)) \simeq \int_E J dx dy,$$

Given J how can we compute f?

Similar problem will be discussed by Saalfeld.

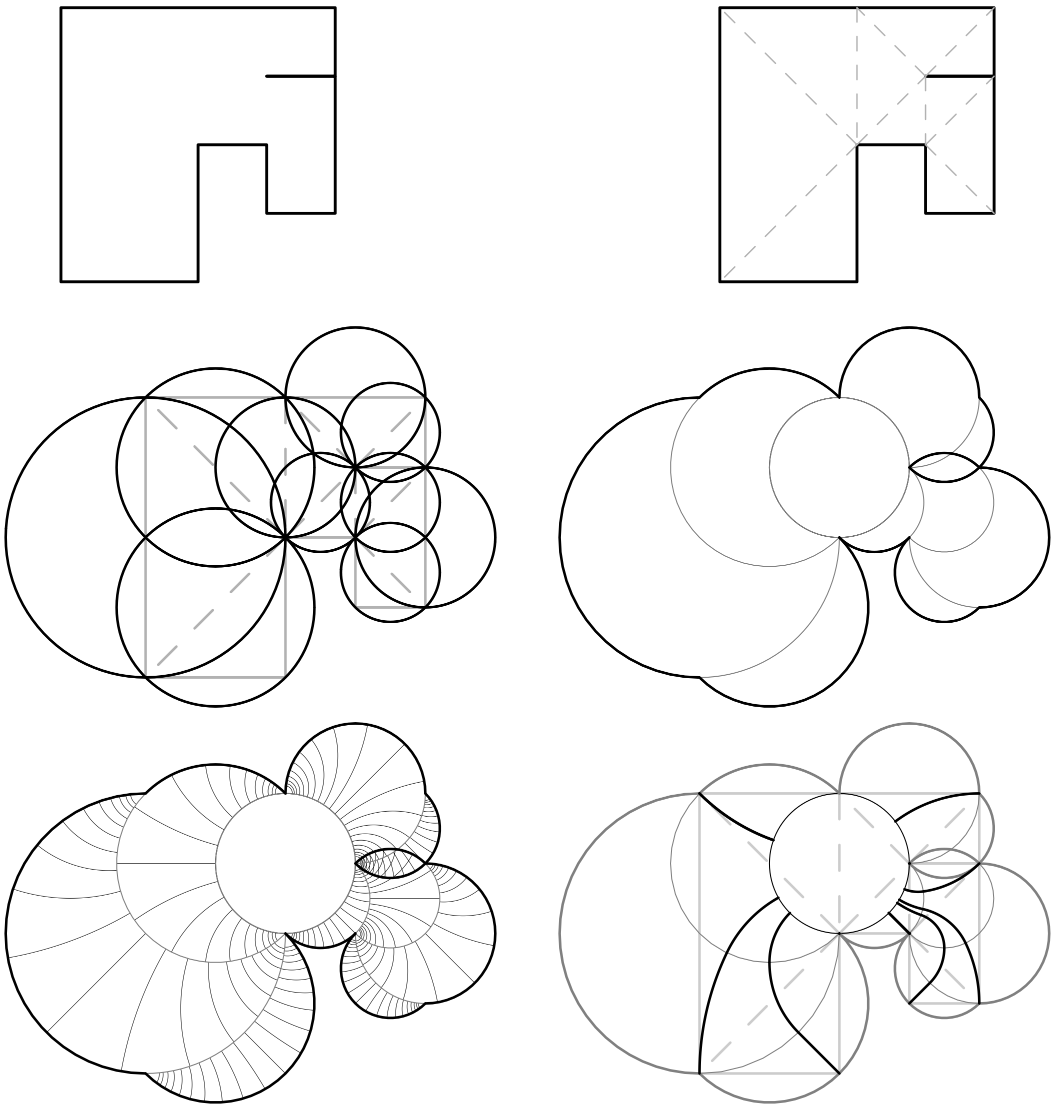
Conformal = Möbius in 3-D and not every topological ball can be QC mapped to a round ball.

Can we quickly test if a polyhedron can be QC mapped to a ball and estimate the best QC constant? Solve Dirichlet problem in linear time?

Given a polyhedron can we find a "simplier" one which is a QC image? E.g., convert $10,000$-gon to a 100-gon with almost the same "shape".

David Hamilton has shown a 3-D topological ball is a QC image of a round ball iff its boundary is a "nice" image of a 2-sphere. This could reduce 3-D problems to 2-D surface mapping problems (e.g., Stephenson, Gu).

CRDT algorithm of Driscoll and Vavasis (1998).



What other way to approximate by disks gives a better map? Can a few “rules of thumb” give a very fast map which is adequate for applications?

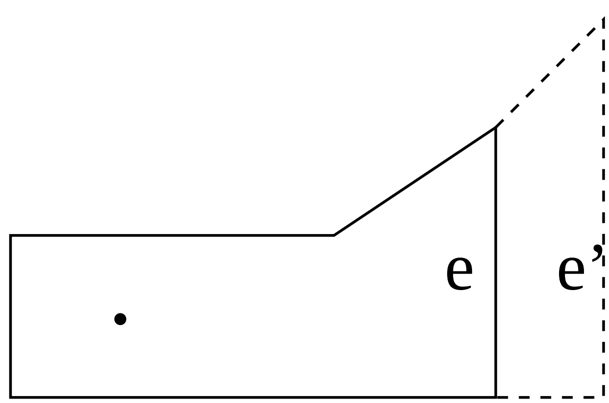
Davis's algorithm: Suppose P is a polygon with vertices $\mathbf{v} = \{v_1, \ldots, v_n\} \in \mathbb{C}^n$ and

$$\mathbf{w} = \{w_1, \ldots, w_n\} \in \mathbb{T}^n$$

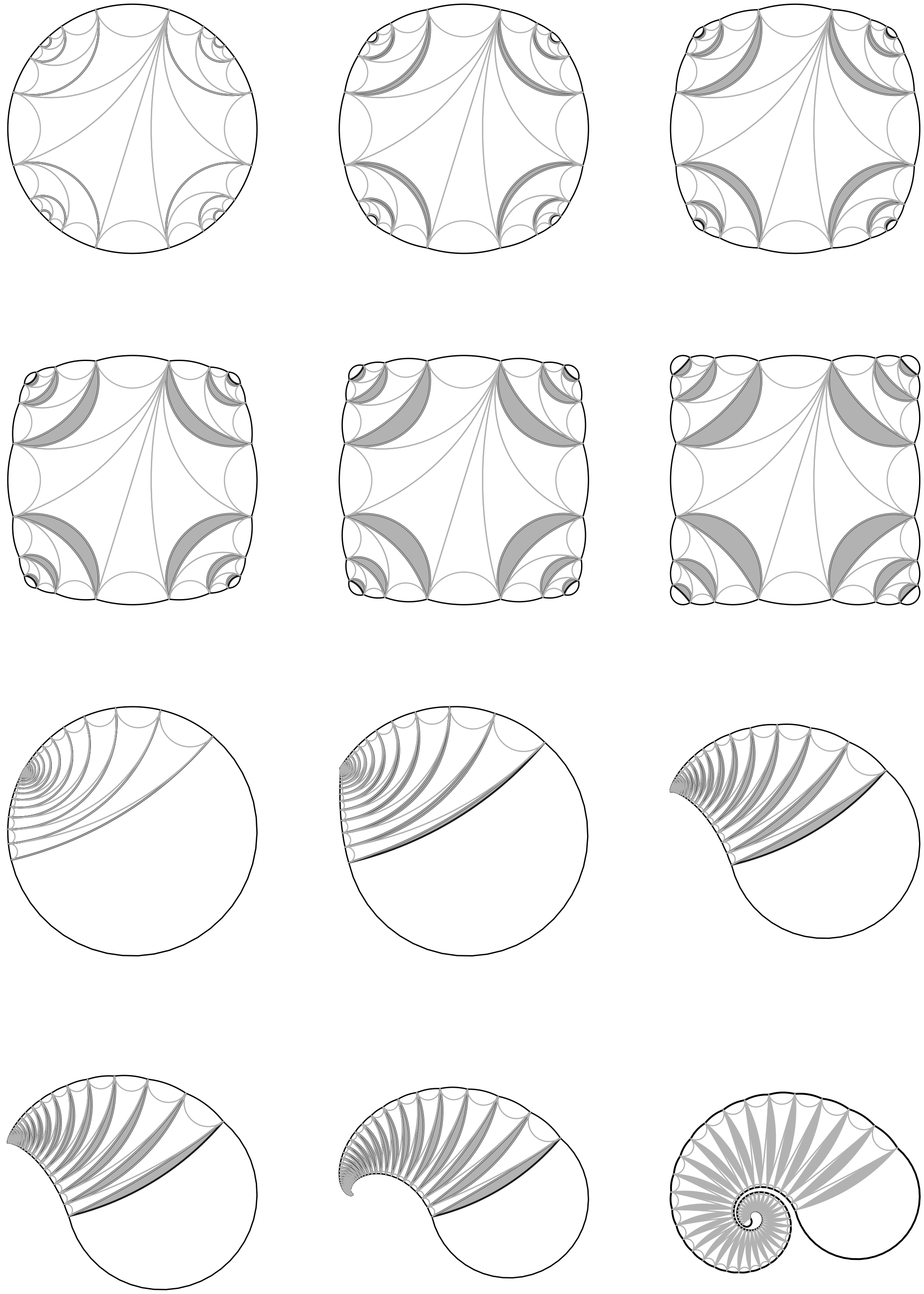
is the guess for the prevertices. Apply Schwarz-Christoffel to $\mathbf{w}$ to get $\mathbf{v}' = \{v_1', \ldots, v_n'\}$. and modify guess by

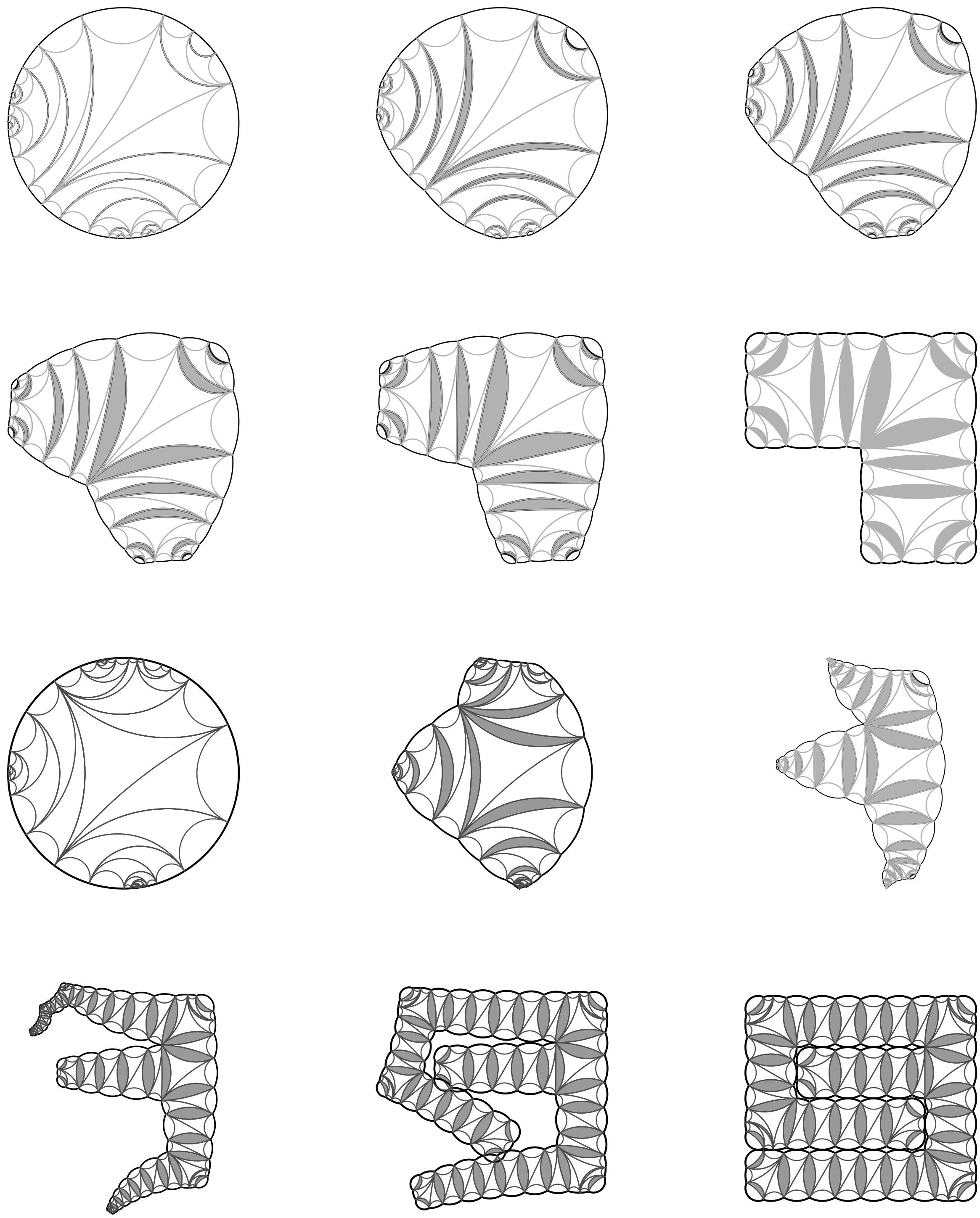
$$|w_k' - w_{k+1}'| = k|w_k - w_{k+1}| \frac{|v_k - v_{k+1}|}{|v_k' - v_{k+1}'|}$$

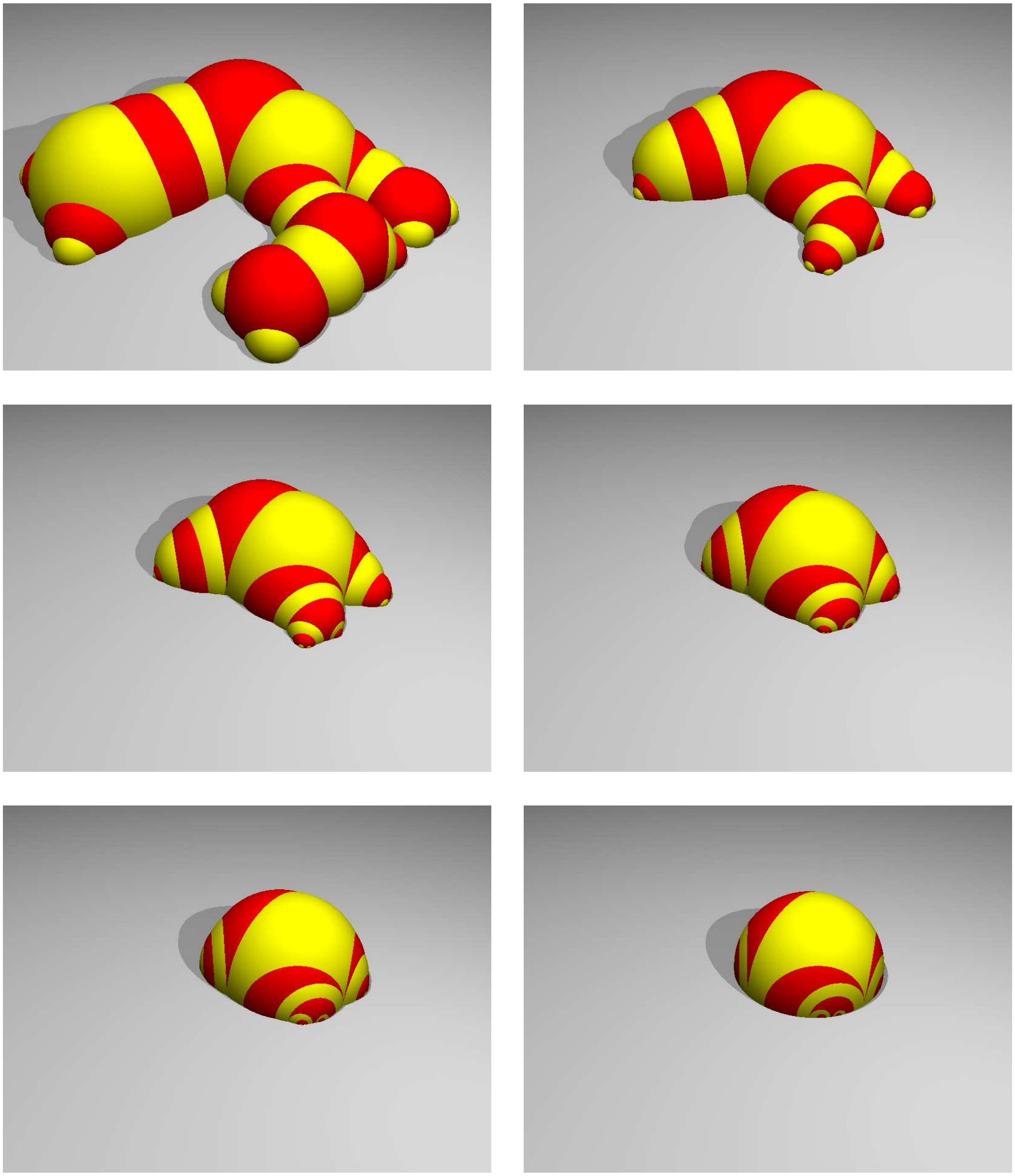
Method can fail (even locally):



What if we use medial axis edges to modify guess? Experiment with modification rules.

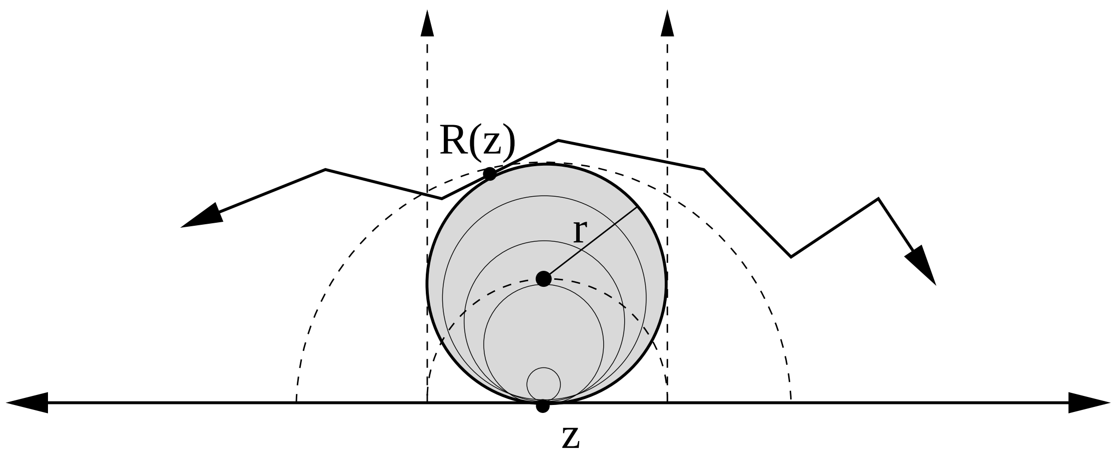






Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \operatorname{dist}(z, \partial\Omega) \simeq \operatorname{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



Fact 2: R is Lipschitz.

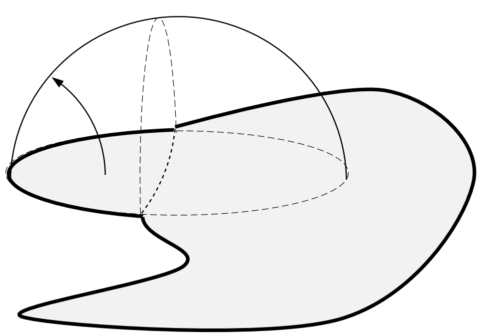
Ω simply connected $\Rightarrow$

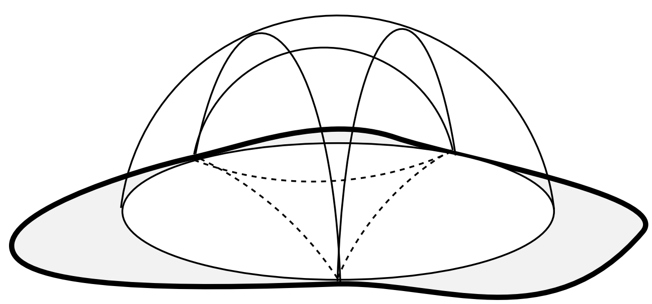
$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \mathrm{Dome(D)} \Rightarrow$

$$\mathrm{dist}(z, \partial\Omega)/\sqrt{2} \leq \mathrm{dist}(z, \partial D) \leq \mathrm{dist}(z, \partial\Omega)$$

$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\mathrm{Dome}}(R(z)).$$

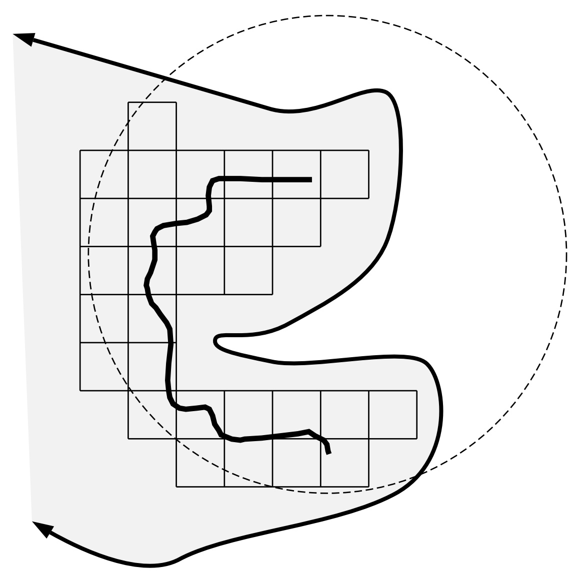




Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $\operatorname{dist}(R(z), \mathbb{R}^2) = r$ and γ is geodesic from z to w.

$$\begin{aligned}
\Rightarrow \qquad & \operatorname{dist}(\gamma, \mathbb{R}^2) \simeq r \\
\Rightarrow \qquad & \operatorname{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\
& \quad R^{-1}(\gamma) \subset D(z, Cr) \\
\Rightarrow \qquad & \rho_\Omega(z, w) \leq C
\end{aligned}$$



Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

$$|g'(z)| \simeq \frac{\operatorname{dist}(g(z), \partial\mathbb{D})}{\operatorname{dist}(z, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\operatorname{dist}(z, \partial\Omega) &\simeq \operatorname{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \operatorname{dist}(g(z), \partial D)
\end{aligned}$$