# MAT 627, Spring 2022, Stony Brook University

# TOPICS IN COMPLEX ANALYSIS

# CONFORMAL FRACTALS, PART I: DLA

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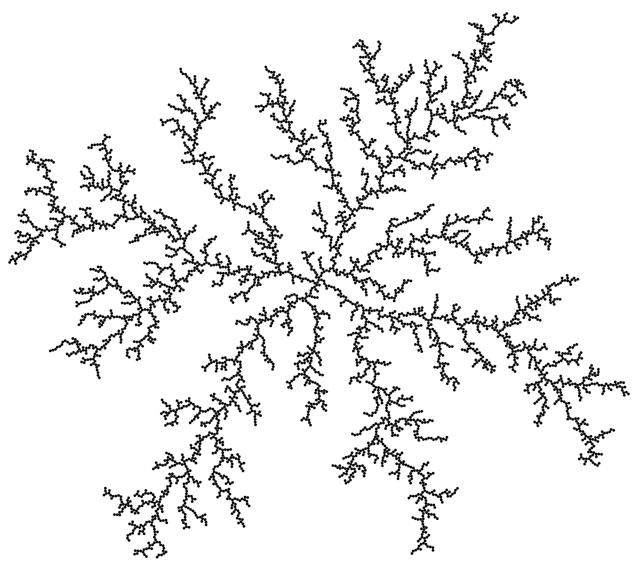

Extremal length

Symmetry and Koebe's theorem

Hyperbolic metric

Boundary continuity

Logarithmic capacity

Pfluger's theorem

Harmonic measure

Kesten's theorem on growth of DLA

This course is about fractals with some sort of invariance under conformal maps.

DLA, Brownian motion, Harmonic Measure

A fundamental tool for understanding such sets are conformal invariants, i.e., numerical values that can be associated to a certain geometric configurations and that remain unchanged (or at least change in predictable ways) under the application of conformal or holomorphic maps.

There are three conformal invariants that will be particularly important through the book: extremal length, harmonic measure and hyperbolic distance.

Of these, extremal length is the most important because it can be defined in many situations and estimated by direct geometric arguments.

The other two are defined on the disk and then transferred to other domains by a conformal map.

Our goal is to estimate harmonic measure on simply connected domains using extremal length.

Ahlfors distortion theorem and Beurling projection theorem are examples of this. These are used in Kesten's theorem on DLA.

Much of material we will cover can be found in:

"Harmonic Measure", Garnett and Marshall

"Conformal Invariants", Ahlfors

# 1. Extremal length

Suppose $\Omega$ is a domain (open, connected set).

Consider a positive function $\rho$ on a domain $\Omega$. We think of $\rho$ as analogous to $|f'|$ where $f$ is a conformal map on $\Omega$.

Just as the image area of a set $E$ can be computed by integrating $\int_E |f'|^2 dxdy$, we can use $\rho$ to define areas by $\int_E \rho^2 dxdy$. Similarly, just as we can define $\ell(f(\gamma)) = \int_\gamma |f'(z)|ds$, we can define the $\rho$-length of a curve $\gamma$ by $\int_\gamma \rho ds$.

For this to make sense, we need $\gamma$ to be rectifiable (so the arclength measure $ds$ is defined) and it is convenient to assume that $\rho$ is Borel (so that its restriction is also Borel and hence measurable for length measure on $\gamma$).

Suppose $\mathcal{F}$ is a family of locally rectifiable paths in a planar domain $\Omega$ and $\rho$ is a non-negative Borel function on $\Omega$.

We say $\rho$ is admissible for $\mathcal{F}$ if

$$\ell(\mathcal{F}) = \ell_\rho(\mathcal{F}) = \inf_{\gamma \in \mathcal{F}} \int_\gamma \rho ds \geq 1.$$

"paths" need not always be connected. Often are in examples.

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$$\ell(\mathcal{F}) = \ell_\rho(\mathcal{F}) = \inf_{\gamma \in \mathcal{F}} \int_\gamma \rho ds \geq 1.$$

Define the modulus of the path family $\mathcal{F}$ as

$$\mathrm{Mod}(\mathcal{F}) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible $\rho$ for $\mathcal{F}$.

The extremal length of $\mathcal{F}$ is defined as $\lambda(\mathcal{F}) = 1/M(\mathcal{F})$.

**Lemma 1.1** (Conformal invariance)**.** *If $\mathcal{F}$ is a family of curves in $\Omega$ and $f$ is conformal from $\Omega$ to $\Omega'$ then $M(\mathcal{F}) = M(f(\mathcal{F}))$.*

conformal = 1-1, holomorphic = angle and orientation preserving.

*Proof.* This is just the change of variables formulas

$$\int_\gamma \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dx dy = \int_{f(\Omega)} \rho dx dy.$$

These imply that if $\rho \in \mathcal{A}(f(\mathcal{F}))$ then $|f'| \cdot \rho \circ f^{-1} \in \mathcal{A}(f(\mathcal{F}))$, and thus $M(f(\mathcal{F})) \leq M(\mathcal{F})$. We get the other direction by considering $f^{-1}$. $\square$

**Lemma 1.2** (Monotonicity). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are path families so that every $\gamma \in \mathcal{F}_1$ contains some curve in $\mathcal{F}_2$, then*

$$M(\mathcal{F}_1) \leq M(\mathcal{F}_2) \quad \textit{and} \quad \lambda(\mathcal{F}_1) \geq \lambda(\mathcal{F}_2).$$

The proof is immediate since $\mathcal{A}(\mathcal{F}_1) \supset \mathcal{A}(\mathcal{F}_2)$.

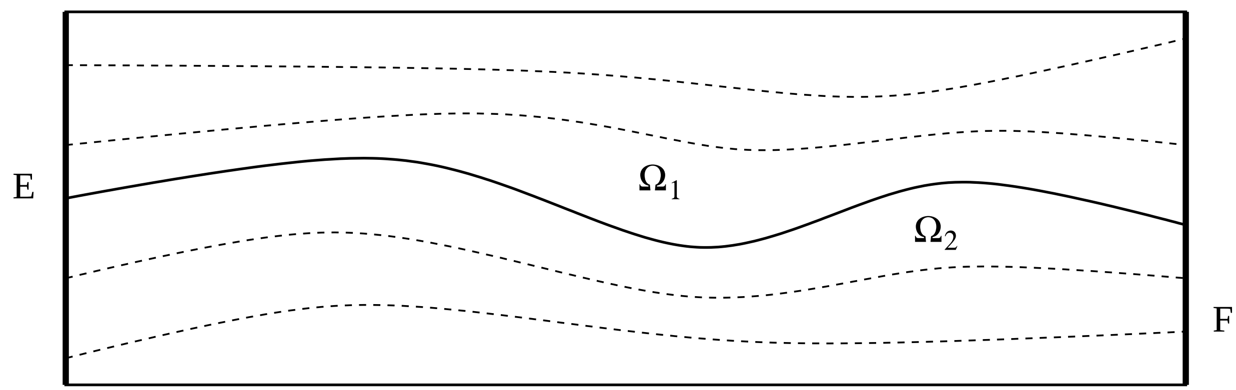
**Lemma 1.3** (Grötsch Principle). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains then $M(\mathcal{F}_1 \cup \mathcal{F}_2) = M(\mathcal{F}_1) + M(\mathcal{F}_2)$.*

*Proof.* Suppose $\rho_1$ and $\rho_2$ are admissible for $\mathcal{F}_1$ and $\mathcal{F}_2$. Take $\rho = \rho_1$ and $\rho = \rho_2$ in their respective domains. Then it is easy to check that $\rho$ is admissible for $\mathcal{F}_1 \cup \mathcal{F}_2$ and, since the domains are disjoint, $\int \rho^2 = \rho_1^2 + \int \rho_2^2$.

Thus $M(\mathcal{F}_1 \cup \mathcal{F}_2) \leq M(\mathcal{F}_1) + M(\mathcal{F}_2)$. By restricting an admissible metric $\rho$ to each domain, a similar argument proves the other direction. $\square$

The Grötsch principle and monotonicity combine to give

**Corollary 1.4** (Parallel Rule). *Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are path families in disjoint domains $\Omega_1, \Omega_2$ that connect sets $E, F$ and that $\Omega_1 \cup \Omega_2 \subset \Omega$. If $\mathcal{F}$ is the path family connecting $E$ and $F$ in $\Omega$, then $M(\mathcal{F}) \geq M(\mathcal{F}) + M(\mathcal{F})$.*




A rectangle $\Omega$ is split into two "parallel" subregions. The path families $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ connect the left ($E$) and right ($F$) sides of the rectangle and satisfy $M(\mathcal{F}) \geq M(\mathcal{F}) + M(\mathcal{F})$.

**Lemma 1.5** (Series Rule). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains and every curve of $\mathcal{F}$ contains both a curve from both $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2)$.*

**Lemma 1.5.** (Series Rule) *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are families of curves in disjoint domains and every curve of $\mathcal{F}$ contains both a curve from both $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2)$.*

*Proof.* If $\rho_1 \in \mathcal{A}(\mathcal{F}_i)$ for $i = 1, 2$, then $\rho = t\rho_1 + (1-t)\rho_2$ is admissible for $\mathcal{F}$.

Since the domains are disjoint we may assume $\rho_1\rho_2 = 0$ everywhere so taking For $0 \leq t \leq 1$, take

$$= t^2\rho_1 + (1-t^2)\rho_2.$$

it is easy so see that this is admissible and since the domains are disjoint we may assume $\rho_1\rho_2 = 0$.

Integrating $\rho^2$ then shows $M(\mathcal{F}) \leq t^2 M(\mathcal{F}_1) + (1-t^2)M(\mathcal{F}_2)$ for each $t$.

To find the optimal $t$ set $a = M(\mathcal{F}_1)$, $b = M(\mathcal{F}_2)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

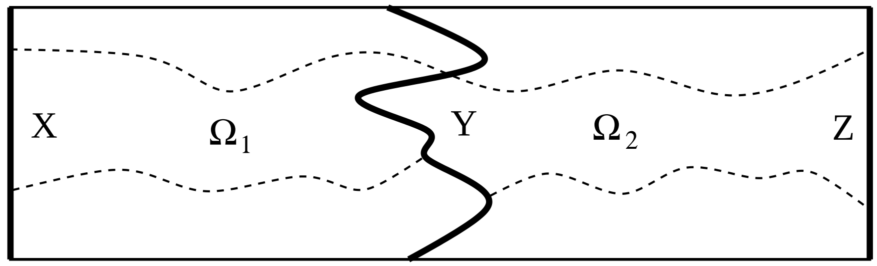
$$\begin{aligned} M(\mathcal{F}) \;&\leq\; t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2} \\ &=\; \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}} \end{aligned}$$

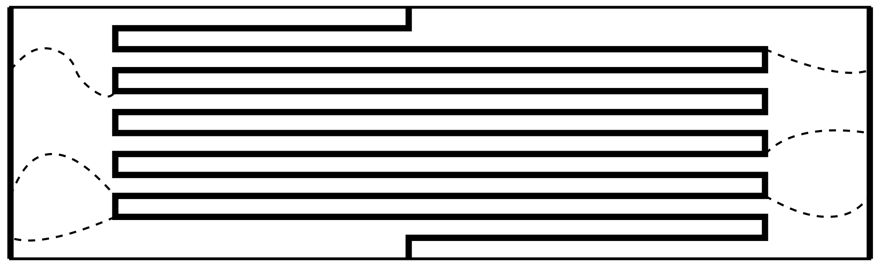
or

$$\frac{1}{M(\mathcal{F})} \geq \frac{1}{M(\mathcal{F}_1)} + \frac{1}{M(\mathcal{F}_2)},$$

which, by definition, is the same as

$$\lambda(\mathcal{F}) \geq \lambda(\mathcal{F}_1) + \lambda(\mathcal{F}_2). \quad \square$$



The series rule says that the extremal distance from $X$ to $Z$ in the rectangle is greater than the sum the extremal distance from $X$ to $Y$ in $\Omega_1$ plus the extremal distance from $Y$ to $Z$ in $\Omega_2$.

The bottom figure show a more extreme case where the extremal distance between opposite sides of the rectangle is much larger than either of the other two terms.

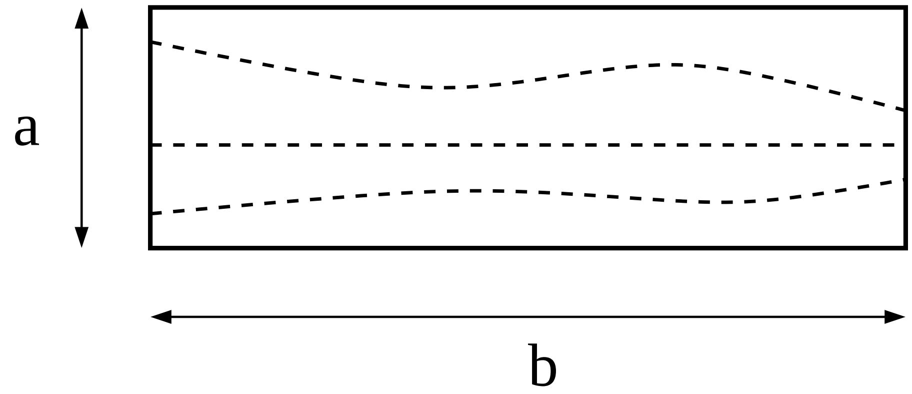
Given a Jordan domain $\Omega$ and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between $E$ and $F$ (in $\Omega$) is the extremal length of the path family in $\Omega$ connecting $E$ to $F$ (paths in $\Omega$ that have one endpoint in $E$ and one endpoint in $F$).

The series rule is a sort of "reverse triangle inequality" for extremal distance.

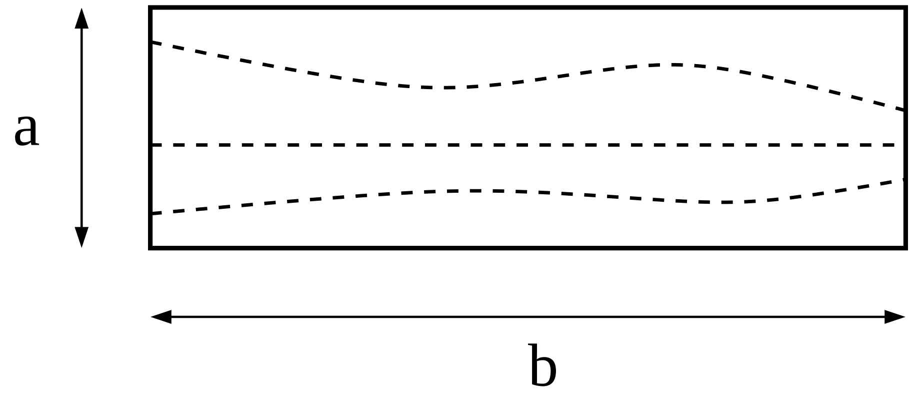
Extremal distance can be particularly useful when both $E$ and $F$ are connected. In this case, their complement in $\partial\Omega$ also consists of two arcs, and the extremal distance between these is the modulus of the arcs separating $E$ and $F$.

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound. Giving a lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases.

However, in the special case of extremal distance between arcs $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family. Thus estimates of both upper and lower bounds can be given by producing metrics. This is often the easiest thing to do.




The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates.

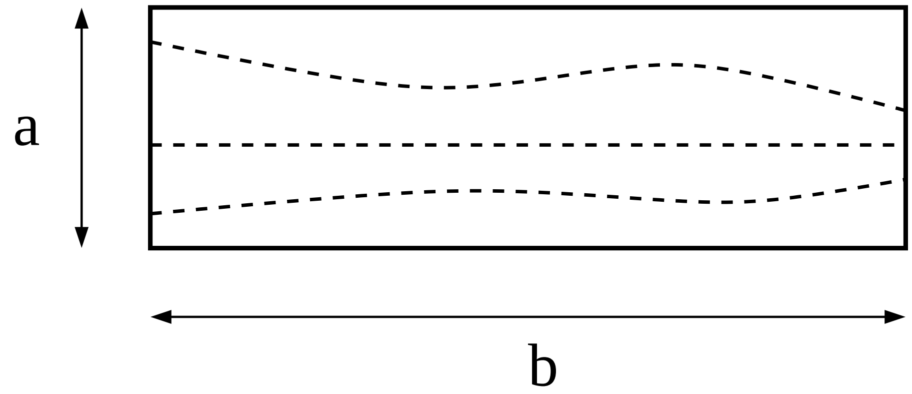



So suppose $R = [0, b] \times [0, a]$ is a $b$ wide and $a$ high rectangle and $\Gamma$ consists of all rectifiable curves in $R$ with one endpoint on each of the sides of length $a$. Then each such curve has length at least $b$, so if we let $\rho$ be the constant $1/b$ function on $R$ we have

$$\int_\gamma \rho ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dx dy = \frac{1}{b^2} ab = \frac{a}{b}.$$

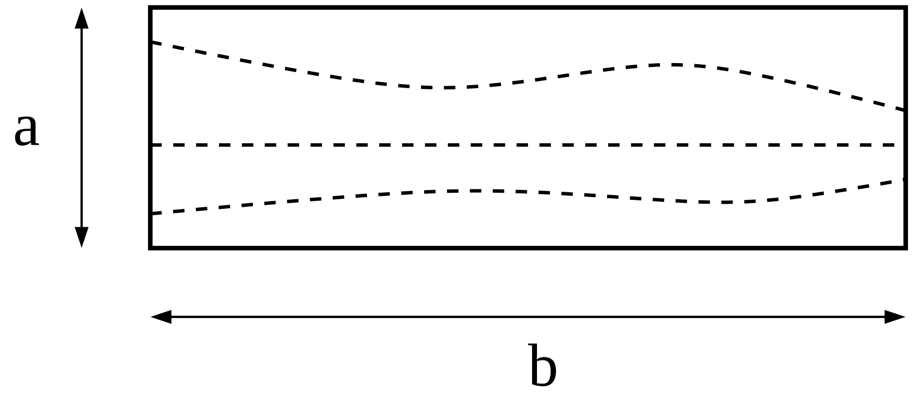



To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fg dx)^2 \leq (\int f^2 dx)(\int g^2 dx).$$

To apply this, suppose $\rho$ is an admissible metric on $R$ for $\gamma$. Every horizontal segment in $R$ connecting the two sides of length $a$ is in $\Gamma$, so since $\gamma$ is admissible,

$$\int_0^b \rho(x,y) dx \geq 1,$$




So by Cauchy-Schwarz

$$1 \leq \int_0^b (1 \cdot \rho(x,y))dx \leq \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to $y$ to get

$$\int_0^a 1dy \leq b \int_0^a \int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \leq \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \geq \frac{b}{a}$. Thus we must have equality.

**Lemma 1.6.** *If* $A = \{z : r < |z| < R\}$ *then the modulus of the path family connecting the two boundary components is* $\frac{1}{2\pi} \log \frac{R}{r}$*. More generally, if* $\mathcal{F}$ *is the family of paths connecting* $r\mathbb{T}$ *to a set* $E \subset R\mathbb{T}$*, then* $M(\mathcal{F}) \geq |E| \log \frac{R}{r}$*.*

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*Proof.* By conformal invariance, we can rescale and assume $r = 1$. Suppose $\rho$ is admissible for $\mathcal{F}$. Then for each $z \in E \subset \mathbb{T}$,

$$1 \leq (\int_1^R \rho dr)^2 \leq (\int_1^R \frac{dr}{r})(\int_1^R \rho^2 r dr) = \log R \int_1^R \rho^2 r dr$$

and hence we get

$$\int_0^{2\pi} \int_1^R \rho^2 r dr d\theta \geq \int_E \int_r^R \rho^2 r dr d\theta$$

$$\geq |E| \int_1^R \rho^2 r dr \geq |E| \log R \quad \square$$

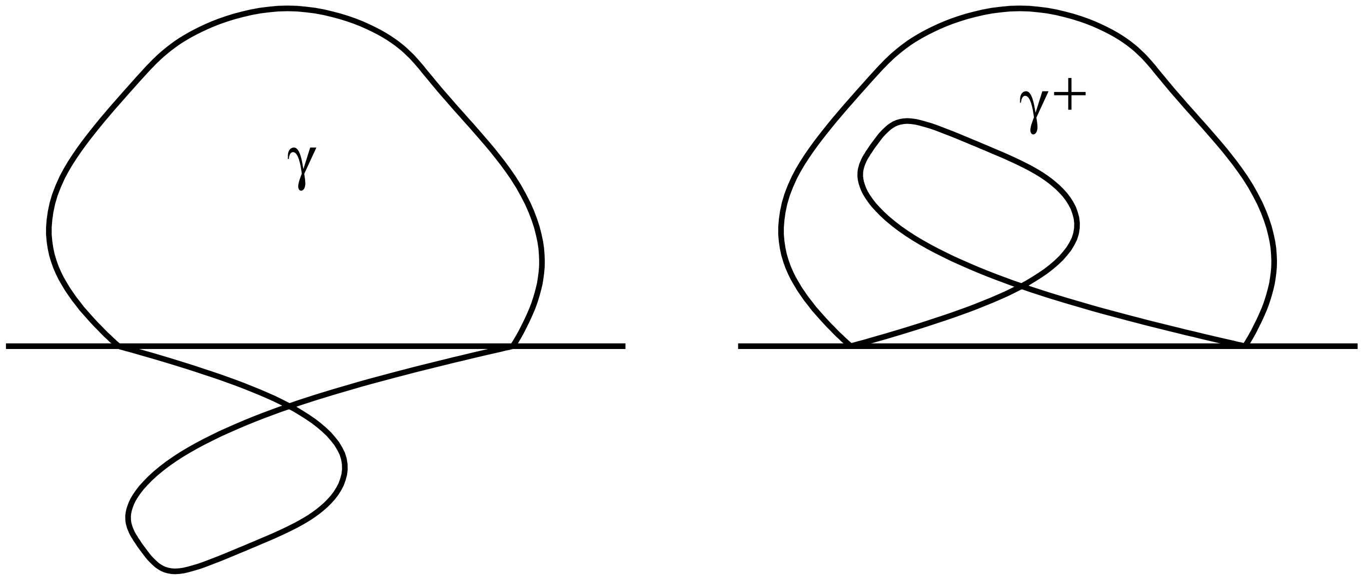
# 2. Symmetry of extremal length and Koebe's $\frac{1}{4}$-theorem

The standard proof of Koebe's theorem uses Green's theorem to estimate the power series coefficients of conformal map (proving the Bieberbach conjecture for the second coefficient). However here we will take the less traveled path and present a proof, due to Mateljevic, that uses a symmetry property of extremal length.

For the "usual" proof, see Chapter I of "Harmonic Measure" by Garnett and Marshall.

If $\gamma$ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let $\gamma^+ = (\gamma \cap \mathbb{H}) \cup \overline{\gamma \cap \mathbb{H}_l}$, where $\mathbb{H}, \mathbb{H}_l$ denote the upper and lower half-planes.

For a path family $\Gamma$, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma^+ = \{\gamma^+ : \gamma \in \Gamma\}$.



**Lemma 2.1** (Symmetry Rule)**.** *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma^+)$*.*

*Proof.* We start by proving $M(\Gamma) \leq 2M(\Gamma^+)$.

Given a metric $\rho$, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\int_{\gamma^+} \sigma ds \geq \int_{\gamma^+} \rho ds \geq \inf_{\gamma \in \Gamma} \int_{\gamma} \rho ds.$$

Thus if $\rho$ admissible for $\Gamma^+$, then $\sigma$ is admissible for $\Gamma$. Hence $M(\Gamma) \leq \int \sigma^2$.

Therefore, since $\max(a, b)^2 \leq a^2 + b^2$,

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z)dxdy + \int \rho^2(\bar{z})dxdy \leq 2\int \rho^2(z)dxdy.$$

Taking the infimum over admissible $\rho$'s for $\Gamma^+$ makes the right hand side equal to $2M(\Gamma^+)$, proving the claim.

Conversely, given $\rho$ define

$$\sigma(z) = \rho(z) + \rho(\bar{z}) \text{ for } z \in \mathbb{H}$$

and

$$\sigma = 0 \text{ for } z \in \mathbb{H}_l.$$

Then

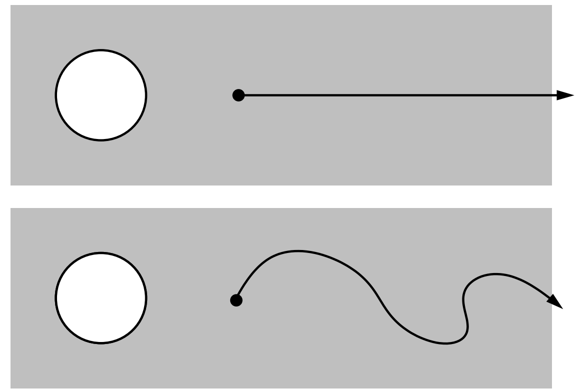
$$\begin{aligned}
\int_{\gamma^+} \sigma ds &= \int_{\gamma^+} \rho(z) + \rho(\bar{z}) ds \\
&= \int_{\gamma \cap \mathbb{H}} \rho(z) ds + \int_{\gamma \cap \mathbb{H}} \rho(\bar{z}) ds + \int_{\gamma \cap \mathbb{H}_l} \rho(z) + \int_{\gamma \cap \mathbb{H}_l} \rho(\bar{z}) ds \\
&= \int_{\gamma} \rho(z) ds + \int_{\bar{\gamma}} \rho(z) ds \\
&\geq 2 \inf_{\rho} \int_{\gamma} \rho ds.
\end{aligned}$$

Thus if $\rho$ is admissible for $\Gamma$, $\frac{1}{2}\sigma$ is admissible for $\Gamma^+$.

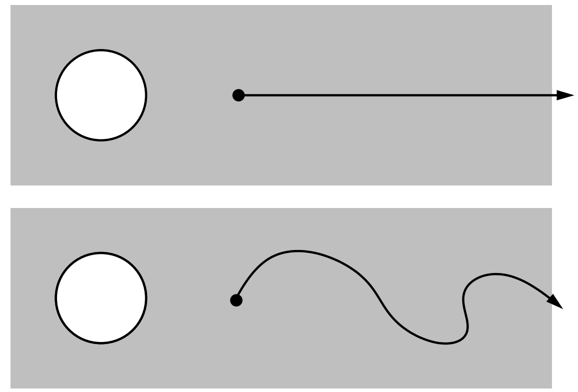
Since $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned}
M(\Gamma^+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\
&= \frac{1}{4}\int_{\mathbb{H}} (\rho(z)+\rho(\bar{z}))^2 dxdy \\
&\leq \frac{1}{2}\int_{\mathbb{H}} \rho^2(z)dxdy + \int_{\mathbb{H}} \rho^2(\bar{z})dxdy \\
&= \frac{1}{2}\int \rho^2 dxdy.
\end{aligned}$$

Taking the infimum over all admissible $\rho$'s for $\Gamma$ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. $\square$

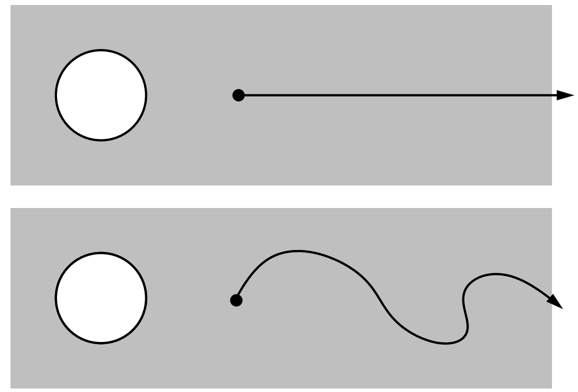


**Lemma 2.2.** *Let* $\mathbb{D}^* = \{z : |z| > 1\}$ *and* $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ *for some* $R > 1$. *Let* $\Omega = \mathbb{D}^* \setminus K$, *where* $K$ *is a closed, unbounded, connected set in* $\mathbb{D}^*$ *which contains the point* $\{R\}$. *Let* $\Gamma_0, \Gamma$ *denote the path families in these domains with separate the two boundary components. Then* $M(\Gamma_0) \leq M(\Gamma)$.



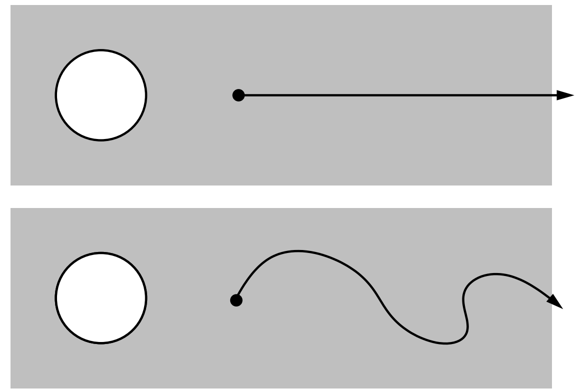
*Proof.* We use the symmetry principle we just proved.

The family $\Gamma_0$ is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma^+) = \frac{1}{2}M(\Gamma_0)$. The family $\Gamma$ may not be symmetric, but we can replace it by a larger family that is.



Let $\Gamma_R$ be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0. Clearly $\Gamma \subset \Gamma_R$ and $\Gamma_R$ is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$.

Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.



Suppose $\gamma \in \Gamma_R$. Since $\gamma$ has non-zero winding around 0 it must cross both the negative and positive real axes. If it never crossed $(0, R)$ then the winding around 0 and $R$ would be the same, which false, so $\gamma$ must cross$(0, R)$ as well.

Choose points $z_- \in \gamma \cap (-\infty, 0)$ and $z_+ \in \gamma \cap (0, R)$. These points divide $\gamma$ into two subarcs $\gamma_1$ and $\gamma_2$.Then $\gamma^+ = \gamma_1^+ \cup \gamma_2^+$. But if we reflect $\gamma_2^+$ into the lower half-plane and join it to $\gamma_1^+$ it forms a closed curve $\gamma_0$ that is in $\Gamma_0$ and $\gamma_0^+ = \gamma^+$. Thus $\gamma^+ \in \Gamma_0^+$, as desired. $\square$

Let $\Omega_{\epsilon,R} = \{z : |z| > \epsilon\} \setminus [R, \infty)$, (as illustrated on earlier slide). Thus $\Omega_{1,R}$ is the domain considered in the previous lemma.

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We can estimate the moduli of these domains using the Koebe map
$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$
which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4}, \infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$.

Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\epsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius $\frac{\epsilon}{4R}(1 \pm O(\frac{\epsilon}{R}))$. Thus the modulus of $\Omega_{\epsilon,R}$ is

$$2\pi \log \frac{4R}{\epsilon} + O(\frac{\epsilon}{R}). \tag{2.1}$$

**Theorem 2.3** (The Koebe $\frac{1}{4}$ Theorem)**.** *Suppose $f$ is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof follows "Quasiconformal and quasiregular harmonic analogues of Koebe's theorem and applications", by Miodrag Mateljević, Ann. Acad. Sci. Fenn. Math., 32, 2007-301–315.

*Proof.* Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components).

Let $R = \mathrm{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\epsilon,r} = \{z : \epsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\epsilon} = M(A_{\epsilon,1}) = M(f(A_{\epsilon,1})).$$

Let $\delta = \min_{|z|=\epsilon} |f(z)|$. Since $f'(0) = 1$, we have $\delta = \epsilon + O(\epsilon^2)$.

Note that $f(A_{\epsilon,1}) \subset f(\mathbb{D}) \setminus D(0,\delta)$, so

$$M(f(A_{\epsilon,1})) \leq M(f(\mathbb{D}) \setminus D(0,\delta)).$$

By Lemma 2.2 and (2.1),

$$M(f(\mathbb{D}) \setminus D(0,\delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\epsilon}.$$

Taking $\epsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. $\square$

# 3. The hyperbolic metric

The **hyperbolic metric** on $\mathbb{D}$ is given by $d\rho(z) = |dz|/(1-|z|^2)$. This means that the hyperbolic length of a rectifiable curve $\gamma$ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2}, \tag{3.1}$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of $z$ and $w$).

$$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2},$$

In many sources, there is a "2" in the numerator of (3.1), but we follow Garnett-Marshall's book, where the definition is as given in (3.1). For most things, this makes no difference, but the reader is warned that some of our formulas may differ by a factor of 2 from the analogous formulas in some papers and books.

One version gives hyperbolic disk curvature -1 and the other -4.

We define the **hyperbolic gradient** of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ as

$$D_H^H f(z) = |f'(z)| \frac{1 - |z|^2}{1 - |f(z)|^2}.$$

More generally, given a map $f$ between metric spaces $(X, d)$ and $(Y, \rho)$ we define the gradient at a point $z$ as

$$D_d^\rho f(z) = \limsup_{x \to z} \frac{\rho(f(z), f(x))}{d(x, z)}.$$

The use of the word "gradient" is not quite correct; a gradient is usually a vector indicating both the direction and magnitude of the greatest change in a function. We use the term in a sense more like the term "upper gradient" that occurs in metric measure theory to denote a function $\rho \geq 0$ that satisfies

$$|f(b) - f(a)| \leq \int_\gamma \rho ds,$$

for any curve $\gamma$ connecting $a$ and $b$.

In these notes, the most common metrics we will use are the

usual Euclidean metric on $\mathbb{C}$,

the spherical metric on the Riemann Sphere, $S^2$

$$\frac{ds}{1+|z|^2},$$

the hyperbolic metric on the disk or on some other hyperbolic planar domain.

To simplify notation, we use E, S and H to denote whether we are taking a gradient with respect to Euclidean, Spherical or Hyperbolic metrics. For example if $f : U \to V$, the symbol $D_H^H f$ means that we are taking a gradient from the hyperbolic metric on $U$ to the hyperbolic metric on $V$ (assuming the domains are clear from context; otherwise we write $D_U^V$ or $D_{\rho_U}^{\rho_v}$ it we need to be very precise.

In this notation, the spherical derivative of a function, usually denoted

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2},$$

is written $D_E^S f(z)$ since it is a limit of quotients where the numerator is measured in the spherical metric and the denominator is measured in the Euclidean metric.

Similarly $D_H^S$ denotes a gradient measuring expansion from a hyperbolic to the spherical metric. This particular gradient is important in the theory of normal families.

Another variation we will use is $D^E_{\mathbb{D}} f$. If this is bounded on the disk, then $f$ is a Lipschitz function from the hyperbolic metric on the disk to the Euclidean metric on the plane. Holomorphic function with this property are called Bloch functions, and are fundamental to Makarov's theorem on the dimension of harmonic measure.

A **linear fractional transformation** is a map of the form

$$z \to a + bxc + dz,$$

where $a, b, c, d \in \mathbb{C}$. These exactly the 1-to-1, holomorphic maps of the Riemann sphere to itself. Such maps are also called **Möbius transformation**

**Lemma 3.1.** *Möbius transformations of $\mathbb{D}$ to itself are isometries of the hyperbolic metric.*

*Proof.* When $f$ is a Möbius transformation of the disk we have

$$f(z) = \frac{z-a}{1-\bar{a}z}, \qquad f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus

$$\begin{aligned}
D_H^H f(z) &= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|f(z)|^2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2-|z-a|^2} = \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z)(1-a\bar{z})-(z-a)(\bar{z}-\bar{a})} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z-a\bar{z}+|az|^2)-(|z|^2-a\bar{z}-z\bar{a}+|a|^2)} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1+|az|^2-|z|^2-|a|^2)} = 1.
\end{aligned}$$

Note that

$$\ell_\rho(f(\gamma)) \leq \int_\gamma D_H^H f(z) \frac{|dz|}{1-|z|^2}.$$

Thus Möbius transformations multiply hyperbolic length by at most one. Since the inverse also has this property, we see that Möbius transformation preserve hyperbolic length. $\square$

The segment $(-1, 1)$ is clearly a geodesic for the hyperbolic metric and since isometries take geodesics to geodesics, we see that geodesics for the hyperbolic metric are circles orthogonal to the boundary.

On the disk it is convenient to define the quasi-hyperbolic metric

$$T(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

The hyperbolic metric between two points can then be expressed as

$$\rho(w,z) = \frac{1}{2}\log\frac{1+T(w,z)}{1-T(w,z)}. \tag{3.2}$$

On the upper half-plane the corresponding function is

$$T(z,w) = \left|\frac{z-w}{w-\bar{z}}\right|,$$

and $\rho$ is related as before.

**Lemma 3.2** (Schwarz's Lemma)**.** *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then $|f'(0)| \leq 1$ with equality iff $f$ is a rotation. Moreover, $|f(z)| \leq |z|$ for all $|z| < 1$, with equality for some $z \neq 0$ iff $f$ is a rotation.*

*Proof.* Define $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$. This is a holomorphic function since if $f(z) = \sum a_n z^n$ then $a_0 = 0$ and so $g(z) = \sum a_n z^{n-1}$ has a convergent power series expansion. Note that

$$\max_{|z|=r} |g(z)| \leq \frac{1}{r} \max_{|z|=r} |f| \leq \frac{1}{r}.$$

So by the maximum principle $|g| \leq \frac{1}{r}$ on $\{|z| < r\}$. Taking $r \nearrow 1$ shows $|g| \leq 1$ on $\mathbb{D}$. Thus $|f(z)| \leq |z|$.

Equality $|f(z)| = |z|$ anywhere implies $|g(z)| = 1$, which implies $g \equiv 1$ is constant. Thus $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$ and equality implies $f$ is a rotation. $\square$

In terms of the hyperbolic metric this says that

$$\rho(f(0), f(z)) = \rho(0, f(z)) \leq \mathbb{H}_r(0, z),$$

which shows the hyperbolic distance from 0 to any point is non-increasing. For an arbitrary holomorphic self-map of the disk $f$ and any point $w \in \mathbb{D}$ we can always choose Möbius transformations $\tau, \sigma$ so that $\tau(0) = w$ and $\sigma(f(w)) = 0$, so that $\sigma \circ f \circ \tau(0) = 0$. Since Möbius transformations are hyperbolic isometries, this shows

**Corollary 3.3.** *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic then $\rho(f(w), f(z)) \leq \rho(w, z)$.*

A family $\mathcal{F}$ of meromorphic functions on a planar domain $\Omega$ is a **normal family** if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on every compact set or converges uniformly to $\infty$ on every compact set. The following can be found in several texts, e.g., Folland's text *Real Analysis*.

**Theorem 3.4** (Arzela-Ascoli)**.** *A family $\mathcal{F}$ of continuous functions from a planar domain $\Omega$ to a metric space $(X, d)$ is normal if and only if*

(1) *$\mathcal{F}$ is equicontinuous on every compact $E \subset \Omega$.*

(2) *For any $z \in \Omega$, $\{f(z) : f \in \mathcal{F}\}$ is pre-compact (lies in a compact subset).*

By the Cauchy estimates, a holomorphic map $f$ from a planar domain $\Omega$ to the unit disk satisfies

$$|f'(z)| \leq C/\mathrm{dist}(z, \partial\Omega).$$

By the Arzela-Ascoli Theorem, the family of such functions is normal; we call this the "first version" of Montel's theorem.

**Hurwitz's Theorem:** Let $\Omega$ be a connected, open set and $\{f_n\}$ be a sequence of holomorphic functions which converge uniformly on compact subsets of $\Omega$ to a holomorphic function $f$. If each $f_n$ is nonzero everywhere in $\Omega$, then $f$ is either identically zero or also is nowhere zero.

*Proof.* If not, then $f$ has a zero, which must be isolated. Take a small curve around the zero on which $f$ has non-zero winding number around zero. Then for $n$ large enough $f_n$ also has non-zero winding number. Hence $f_n$ has a zero inside the curve, by the argument principle This contradicts assumption that $f_n$ is everywhere non-zero. $\square$

**Lemma 3.5.** *If $\{f_n\}$ are holomorphic functions on a domain $\Omega$ that converge uniformly on compact sets to $f$ and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.*

*Proof.* We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around $z$. Let $E = \{z_n\}_1^\infty \cup \{z\}$. This is a compact set so it has a positive distance $d$ from $\partial\Omega$. The points within distance $d/2$ of $E$ form a compact set $F$ on which the functions $\{f_n\}$ are uniformly bounded on $E$, say by $M$.

By the Cauchy estimate the derivatives are bounded by a constant $M'$ on $E$. Thus

$|f(z) - f_n(z_n) \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \leq |f(z) - f_n(z)| + M'|z - z_n|$,

and both terms on the right tend to zero by hypothesis. $\square$

A planar domain $\Omega$ is called **hyperbolic** if $\mathbb{C} \setminus \Omega$ has at least two points.

**Uniformization theorem for hyperbolic planar domains:**

**Theorem 3.6.** *Every hyperbolic plane domain $\Omega$ is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to $\Omega$).*

We will prove this in three steps: bounded domains, simply connected domains and finally the general case.

*Uniformization for bounded domains.* If $\Omega$ is bounded, then by a translation and rescaling, we may assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$. We will define a sequence of domains $\{\Omega_n\}$ with $\Omega_0 = \Omega$ and covering maps $p_n : \Omega_n \to \Omega_{n-1}$ such that $p(0) = 0$.

We will show that $\Omega_n$ contains hyperbolic disks centered at 0 of arbitrarily large radius and that the covering map $q_n = p_1 \circ \cdots \circ p_n : \Omega_n \to \Omega_0 = \Omega$ converges uniformly on compacta to a covering map $q : \mathbb{D} \to \Omega$.

If $\Omega_0 = \mathbb{D}$ we are done, since the identity map will work.

In general assume that we have $q_n : \Omega_n \to \Omega_0$ and that there is a point $w \in \mathbb{D} \setminus \Omega_n$. Let $\tau$ and $\sigma$ be Möbius transformations of the disk to itself so that $\tau(w) = 0$, choose a square root $\alpha$ of $\tau(0)$ and choose $\sigma$ so $\sigma(\alpha) = 0$.

Then $p_{n+1}(z) = \sigma(\sqrt{\tau(z)})$ and let $\Omega_{n+1}$ be the component of $U = p_{n+1}^{-1}(\Omega_n)$ that contains the origin (the set $U$ will have one or two components; two if $w$ is in a connected component of $\mathbb{D} \setminus \Omega_n$ that is compact in $\mathbb{D}$, and one otherwise). Since $\sigma$ and $\tau$ are hyperbolic isometries and $\sqrt{z}$ expands the hyperbolic metric, we see that $\Omega_{n+1}$ contains a larger hyperbolic ball around 0 than $\Omega_n$ did.

More precisely, suppose $d_n = \text{dist}(\partial\Omega_n, 0) < r < 1$ for all $n$. Since $f(z) = z^2$ maps the disk to itself, it strictly contracts the hyperbolic metric; a more explicit computation shows

$$D_H^H f(z) = |2z|\frac{1-|z|^2}{1-|z|^4} = \frac{2|z|}{1+|z|^2} < 1.$$

Thus $g(z) = \sqrt{z}$ is locally an expansion of the hyperbolic metric, at least on a subdomain $W \subset \mathbb{D}$ where it has a well defined branch. For $z \neq 0$,

$$D_H^H g(z) = |\frac{1}{2\sqrt{z}}|\frac{1-|z|^2}{1-|z|} \geq \frac{1+|z|}{2\sqrt{z}}. \tag{3.3}$$

Then (3.3) says that

$$D_H^H p_n(0) = D_H^H \sqrt{z}(\tau(0)) > \frac{1+r}{2\sqrt{r}} > 1,$$

since $|\tau(0)| = |w| < r$. Hence $D_H^H q_n(0)$ increases by this much at every step. But $D_H^H q_n(0) \leq 1$, which is a contradiction. Thus $d_n \to 1$.

Thus $\{q_n\}$ is a sequence of uniformly bounded holomorphic functions on the disk. By Montel's theorem, there a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $q : \mathbb{D} \to \Omega$.

It is non-constant since it has non-zero gradient at the origin; moreover, by Hurwitz's theorem $q'$ never vanishes on $\mathbb{D}$ since it is the locally uniform limit of the sequence $\{q_n'\}$, and these functions never vanish since they are all derivatives of locally univalent covering maps. Next we show that $q$ is a covering map $\mathbb{D} \to \Omega$.

Fix $a \in \Omega$ and let $d = \mathrm{dist}(a, \partial\Omega)$. Since $\Omega$ is bounded, this is finite. Let $D = D(a, d) \subset \Omega$. Since $q_n$ is a covering map, every branch of $q_n^{-1}$ is 1-to-1 holomorphic map of $D$ into $\mathbb{D}$ and hence each $q_n$ is a contraction from the hyperbolic metric on $D$ to the hyperbolic metric on $\mathbb{D}$. Thus every preimage of $\frac{1}{2}D$ has uniformly bounded hyperbolic diameter.

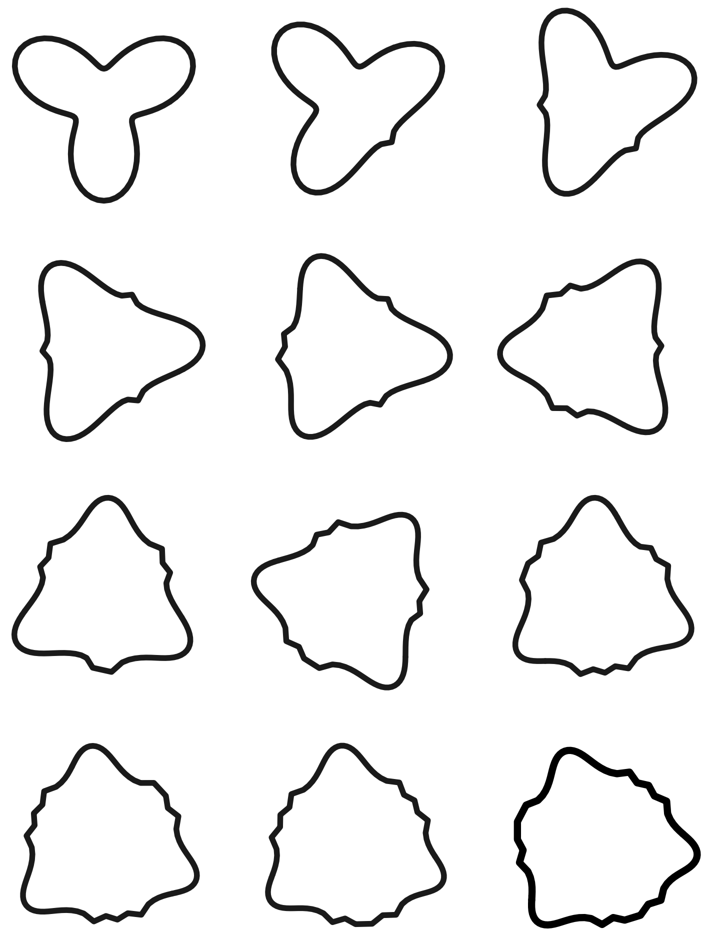
Now fix a point $b \in q^{-1}(a)$. Since $q_n(b) \to q(b) = a$, $q_n(b) \in \frac{1}{2}D$ for $n$ large enough, so there is branch of $q_n^{-1}$ that contains $b$. Since these branches are uniformly bounded holomorphic functions, by Montel's theorem we can pass to a subsequence so that they converge to a holomorphic function $g$ from $\frac{1}{2}D$ into $\mathbb{D}$. Moreover,

$$q(g(z)) = \lim_n q_n(q_n^{-1}(z)) = z,$$

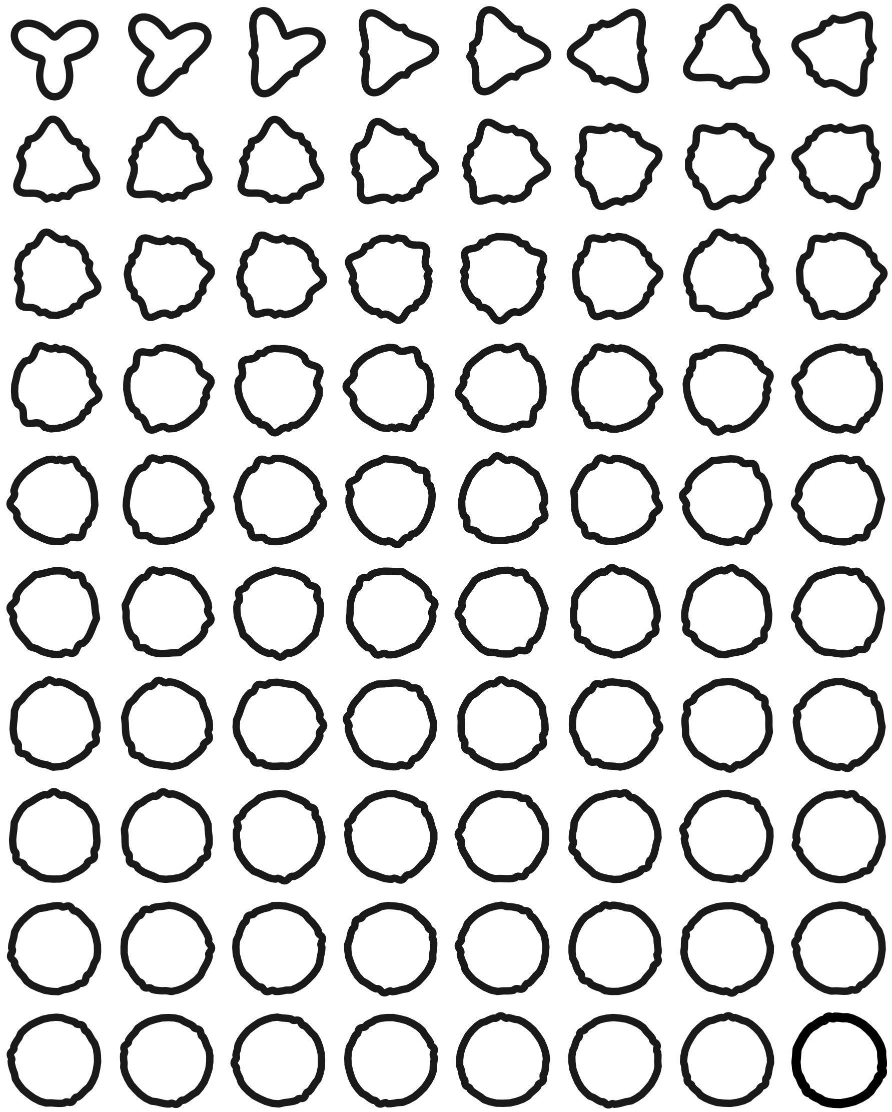
by Lemma 3.5. $\square$

*Proof of Riemann mapping theorem.* It suffices to show any simply connected planar domain, except for the plane itself, can be conformally mapped to a bounded domain. If the domain $\Omega$ is bounded, there is nothing to do.

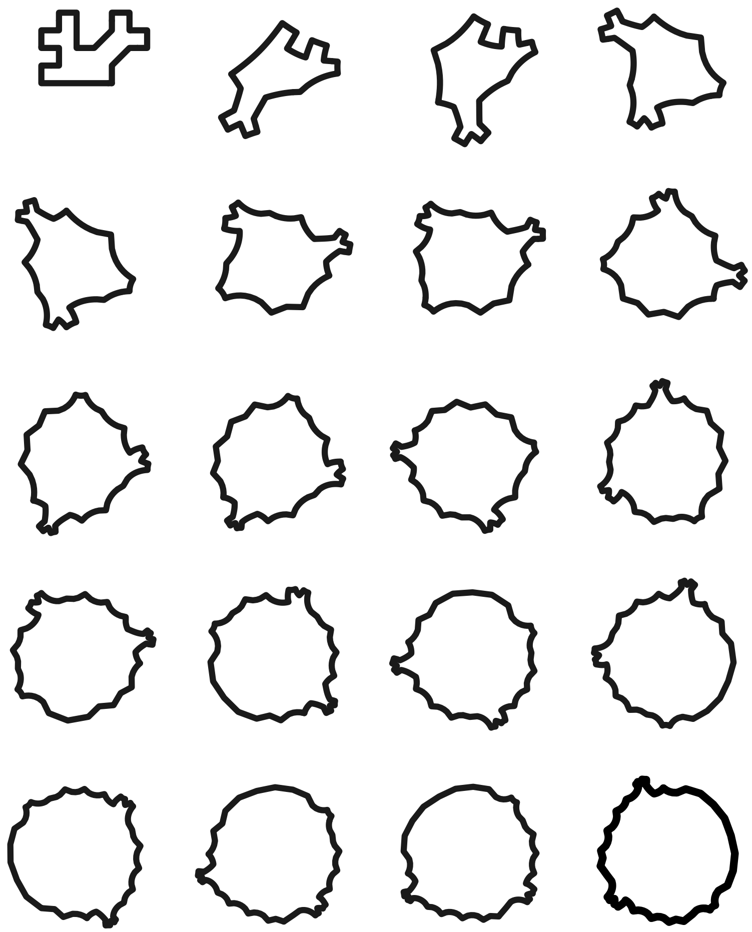
If $\Omega$ omits a disk $D(x,r)$ then the map $z \to 1/(z-x)$ conformal maps $\Omega$ to a bounded domain. Otherwise, translate the domain so that 0 is on the boundary and consider a continuous branch of $\sqrt{z}$. The image is a 1-1, holomorphic image of $\Omega$, but does not contain both a point and its negative. Since the image contains some open ball, it also omits an open ball and hence can be mapped to a bounded domain by the previous case. $\square$

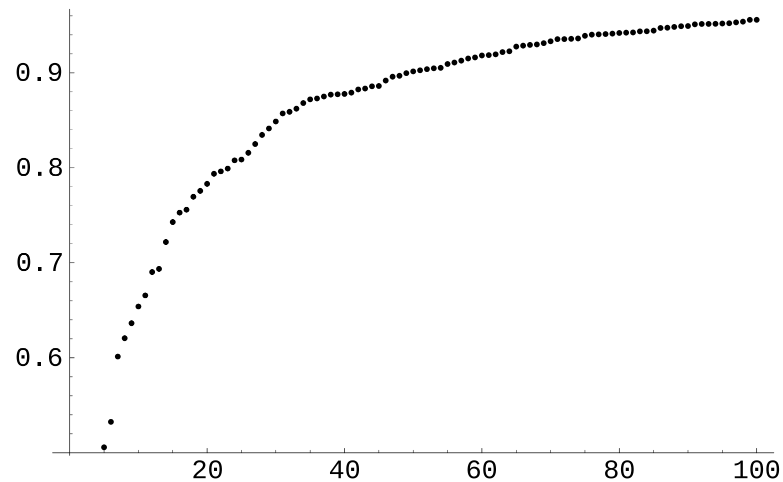


On the top left is a subdomain of the disk whose boundary is parameterized by $\gamma(t) = e^{it}\frac{1}{3}(3+\sin(t)))$. This is a polygon with 100 vertices defined by the points $t = k/100$, $k = 1, \ldots, 100$. The next 11 figures show the first 11 iterations of Koebe's method. The next figure show more iterations.



This shows the first 80 iterations of Koebe's method for the same domain.






Koebe's method applied to a polygon. We have added 19 new, equally spaced vertices to the interior of each edge. On the bottom we have graphed the absolute value of the vertex closest to the origin at each iteration, up to 100 iterations.

The final step is to deduce the uniformization theorem for all hyperbolic plane domains (we have only proved it for bounded domains so far).

It suffices to show that any hyperbolic plane domain has a covering map from some bounded domain $W$, for then we can compose the covering maps $\mathbb{D} \to W$ and $W \to \Omega$.

We can reduce to the following special case of $\mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$

**Theorem 3.7.** *There is a holomorphic covering map from $\mathbb{D}$ to $\mathbb{C}^{**}$.*

*Proof.* Let

$$\Omega = \{z = x + iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of $\Omega$. Every such reflection of $\Omega$ stays in $\mathbb{H}$ and maps $\mathbb{H}$ to either the lower or upper half-planes.

Continuing this forever gives a covering map from a simply connected subdomain $U$ of $\mathbb{H}$ to $W$. Since $U$ is simply connected and not the whole plane (it is a subset of $\mathbb{H}$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. (Actually $U = \mathbb{H}$, but we do not need this stronger result.) $\square$

*Proof of Uniformization of general planar domains.* Let $q : \mathbb{D} \to \mathbb{C}^{**} = \mathbb{C} \setminus \{0, 1\}$. be a covering map of the twice punctured plane. If $\{a, b\} \in \mathbb{C} \setminus \Omega$ then $h(z) = bq(z) + a$ is a covering map from $U = h^{-1}(\Omega) \subset \mathbb{D}$ to $\Omega$. Any connected component of $U$ shows that $\Omega$ has a covering from a bounded plane domain, finishing the proof. $\square$

We the covering maps to define a hyperbolic metric $\rho_\Omega(z)ds$ on any hyperbolic domain $\Omega$.

The function $\rho$ should be defined so that the covering map $p$ is locally an isometry, i.e.,

$$\begin{aligned} 1 &= D_{\mathbb{D}}^{\Omega} p(w) \\ &= D_{\mathbb{D}}^{E} \mathrm{Id}(w) \cdot D_{E}^{E} p(w) \cdot D_{E}^{\rho_\Omega} \mathrm{Id}(p(w)) \\ &= \frac{1}{\rho_{\mathbb{D}}(w)} \cdot |p'(w)| \cdot \rho_\Omega(z) \end{aligned}$$

and so we take

$$\rho_\Omega(z) = \frac{|p'(w)|}{1-|w|^2} = |p'(w)|\rho_{\mathbb{D}}(w) \text{ where } p(w) = z.$$

Different choices of $p$ and $w$ give the same value for $\rho_\Omega(z)$ since they differ by an isometry of $\mathbb{D}$. Thus every hyperbolic planar domain has a hyperbolic metric.

The quasi-hyperbolic metric of a planar domain is defined as

$$\widetilde{\rho}_\Omega(z)ds = \frac{ds}{\mathrm{dist}(z, \partial\Omega)}.$$

For simply connected domains, we will prove below that $\rho$ and $\widetilde{\rho}$ are boundedly equivalent; for more general domains this can fail (e.g., punctured disk), but some useful estimates are still available.

The first observation is that if $f : U \to V$ is conformal and $\rho_U(z)ds$ and $\rho_V(z)ds$ are the densities of the hyperbolic metrics on $U$ and $V$ then

$$\rho_V(f(z)) = \rho_U(z)/|f'(z)|.$$

Applying this to the map $\tau(z) = (z+1)/(z-1)$ that maps the right half-plane $\mathbb{H}_r = \{x + iy : x > 0\}$ to the unit disk $\mathbb{D}$, we see that the hyperbolic density for the half-plane is

$$\rho_{\mathbb{H}_r}(z) = |\tau'(z)|\rho_{\mathbb{D}}(\tau(z)) = \frac{2}{|z-1|^2}\frac{1}{1-|\tau(z)|^2} = \frac{1}{2x} = \frac{1}{2\text{dist}(z, \partial\mathbb{H}_r)}.$$

Thus the hyperbolic density on a half-plane is approximately the same as the quasi-hyperbolic metric. Using Koebe's theorem we can deduce that that this is true for any simply connected domain.

**Lemma 3.8.** *For simply connected domains, the hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent, i.e.,*

$$d\rho_\Omega \leq d\widetilde{\rho}_\Omega \leq 4 d\rho_\Omega. \tag{3.4}$$

*Proof.* Using Koebe's theorem,

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \leq \rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z), \partial\Omega} = \frac{1}{\mathrm{dist}(f(z), \partial\Omega} = \widetilde{\rho}(f(z)),$$

which is one half of the result.

The other half is similar:

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \geq \frac{1}{4}\rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z), \partial\Omega)} = \frac{1}{4}\widetilde{\rho}(f(z)).$$

$\square$

**Corollary 3.9.** *If* $f : \Omega \to \Omega'$ *is conformal, then*

$$\frac{\operatorname{dist}(f(z), \partial\Omega')}{4 \operatorname{dist}(z, \partial\Omega)} \le |f'(z)| \le \frac{4 \operatorname{dist}(f(z), \partial\Omega')}{\operatorname{dist}(z, \partial\Omega)}.$$

*Proof.* Write $f = g \circ h^{-1}$ where $g : \mathbb{D} \to \Omega'$ and $h : \mathbb{D} \to \Omega$ and use the chain rule and Koebe's theorem. $\square$

**Corollary 3.10.** *If $U \subset V$ are both hyperbolic, then $\rho_U \geq \rho_V$.*

*Proof.* If $\Pi_U : \mathbb{D} \to U$ and $\Pi_V : \mathbb{D} \to V$ are the covering maps then the inclusion map $U \to V$ can be lifted to conformal map $\mathbb{D} \to \Pi_V^{-1}(U) \subset \mathbb{D}$. Applying Schwarz's lemma to this map (and using the fact that the projections are local isometries) gives the result. $\square$

**Corollary 3.11.** *If $f : \mathbb{D} \to \Omega$ is conformal then $\varphi(z) = \log|f'(z)|$ is Lipschitz from the hyperbolic metric to the Euclidean metric, with bound that is independent of $f$.*

*Proof.* We want to bound $|\varphi'|$ uniformly on the disk, but by applying Möbius transformations, it suffices to bound $|\varphi'(0)|$. By the Cauchy estimate for derivatives, it suffices to show $|\varphi(z) - \varphi(0)|$ is uniformly bounded on a uniform neighborhood of the origin, or equivalently, that $|f(z)/f(0)|$ is uniformly bounded.

We may assume that $|z| = 1/2$, $f(0) = 0$ and $|f'(0) = 1$. Suppose $d = \operatorname{dist}(f(z), \partial\Omega) \gg 1$ and $\gamma$ is a hyperbolic geodesic from 0 to $f(z)$. Then

$$\rho(0, f(z)) \geq \frac{1}{4}\int_\gamma \frac{ds}{\operatorname{dist}(z, \partial\Omega)} \geq \frac{1}{4}\int_0^d \frac{ds}{1+s} = \frac{1}{4}\log(d+1).$$

Since $\rho(0, f(z)) = \rho(0, z) = \rho(0, \frac{1}{2})$, we see $d$ is uniformly bounded. $\square$

This result says that $f$ conformal implies $\log f'$ is in the Bloch space (with a norm bounded independent of $f$; sharp value is 6).

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|)1 - |z|^2).$$

The sharp upper bound is 6.

It is also true that any Bloch function of small norm ( $< 2$) is $\log f'$ for some conformal map.

# 4. Boundary continuity

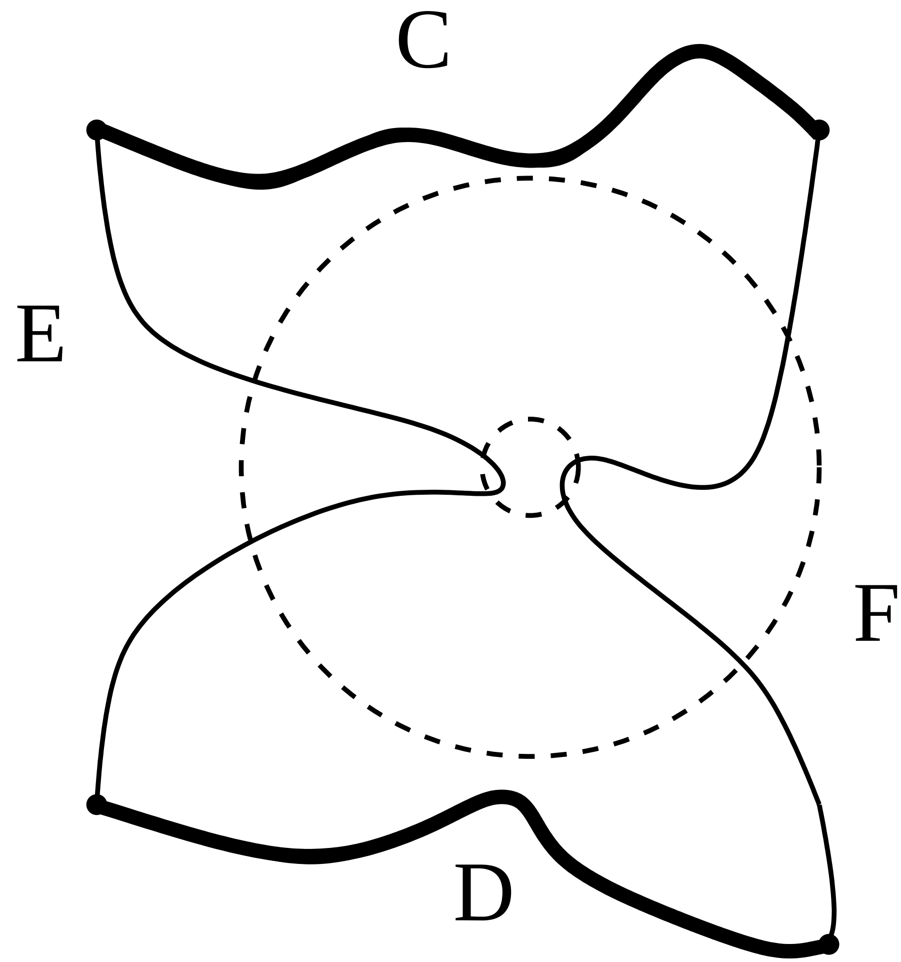
**Lemma 4.1.** *Suppose $Q$ is a quadrilateral with opposite pairs of sides $E, F$ and $C, D$. Assume*

(1) *$E$ and $F$ can be connected in $Q$ by a curve $\sigma$ of diameter $\leq \epsilon$,*

(2) *any curve connecting $C$ and $D$ in $Q$ has diameter at least 1.*

*Then the modulus of the path family connecting $E$ and $F$ in $Q$ is larger than $M(\epsilon)$ where $M(\epsilon) \to \infty$ as $\epsilon \to 0$.*

*Proof.* Define a metric on $Q$ by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\epsilon)$ for $\epsilon < |z-a| < 1/2$. Any curve $\gamma$ connecting $C$ and $D$ must cross $\sigma$ and since $\gamma$ has diameter $\geq 1$ it must leave the annulus where $\rho$ is non-zero. This shows that the modulus of the path family in $Q$ separating $E$ and $F$ is small, hence the modulus of the family connecting them is large. $\square$


C
E
F
D

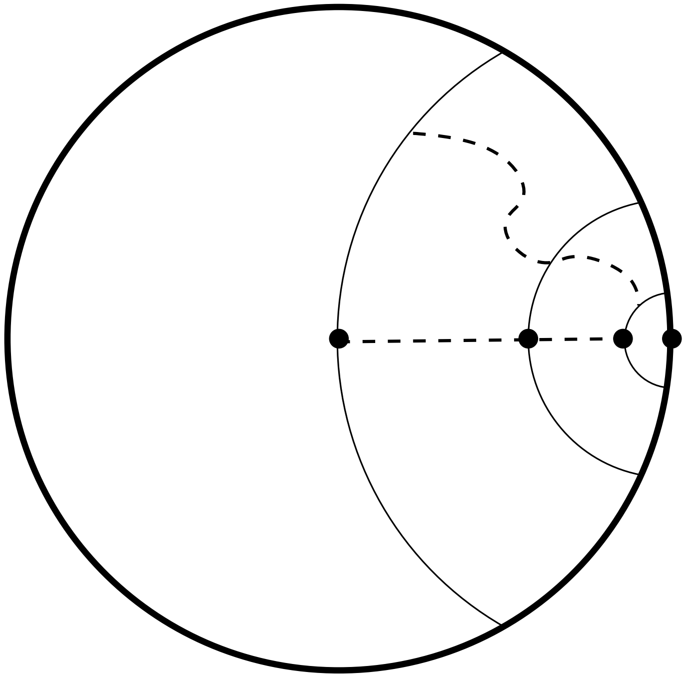
**Theorem 4.2** (Gehring-Hayman inequality)**.** *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in $\Omega$, let $\gamma$ be the hyperbolic geodesic connecting these two points and let $\sigma$ be any other curve in $\Omega$ connecting them. Then $\ell(\gamma) \leq C\ell(\sigma)$.*

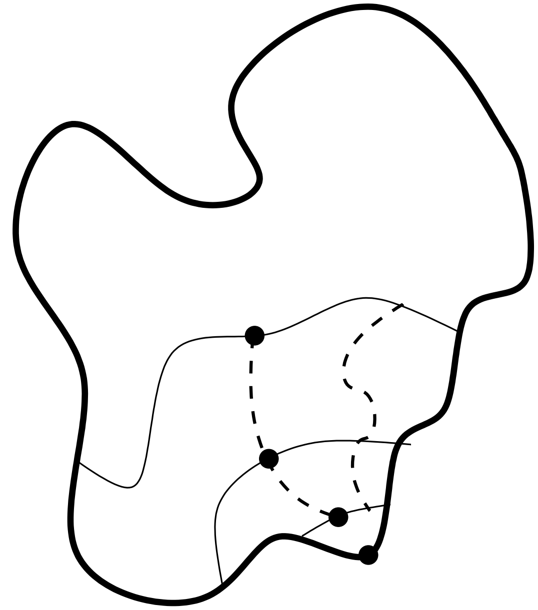
*Proof.* Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that $\gamma$ is the image of $I = [0, r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = r_N = 1 - 2^{-N}$ for some $N$. Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z - 1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z - 1| = 2^{-n}\},$$
$$z_n = \gamma_n \cap [0, 1),$$
$$d_n = \mathrm{dist}(f(z_n), \partial\Omega).$$





Let $Q_n' \subset Q_n$ be the sub-quadrilateral of points with $|\arg(1-z)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart. In particular, this means that any curve in $f(Q_n)$ separating $f(\gamma_n)$ and $f(\gamma_{n+1})$ must cross $f(Q_n')$ and hence has diameter $\gtrsim d_n$.

Since $Q_n$ has bounded modulus, so does $f(Q_n)$ and so Lemma 4.1 says that the shortest curve in $f(Q_n)$ connecting $\gamma_n$ and $\gamma_{n+1}$ has length $\ell_n \simeq d_n$. Thus any curve $\gamma$ in $Q$ connecting $\gamma_n$ and $\gamma_{n+1}$ has length at least $\ell_n$, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\sigma))$$

where $\sigma$ is any curve connecting $f(0)$ and $f(r)$. $\square$

If $f : \mathbb{D} \to \Omega$ is conformal define

$$a(r) = \text{area}(\Omega \setminus f(r \cdot \mathbb{D})).$$

If $\Omega$ has finite area (e.g., if it is bounded), then clearly $a(r) \searrow 0$ as $r \nearrow 1$.

**Lemma 4.3.** *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let*

$$E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta \text{ for some } r < s < 1\}.$$

*Then the extremal length of the path family $\mathcal{P}$ connecting $D(0, r)$ to $E$ is bounded below by $\delta^2/Ca(r)$.*

*Proof.* Let $z = f(sx)$ and suppose $w \in f(D(0, r))$. By the Gehring-Hayman estimate, the length of any curve from $w$ to $z$ is at least $1/C$ times the length of the hyperbolic geodesic $\gamma$ between them. But this geodesic has a segment $\gamma_0$ lying within a uniformly bounded distance of the geodesic $\gamma_1$ from $f(rx)$ to $z$.

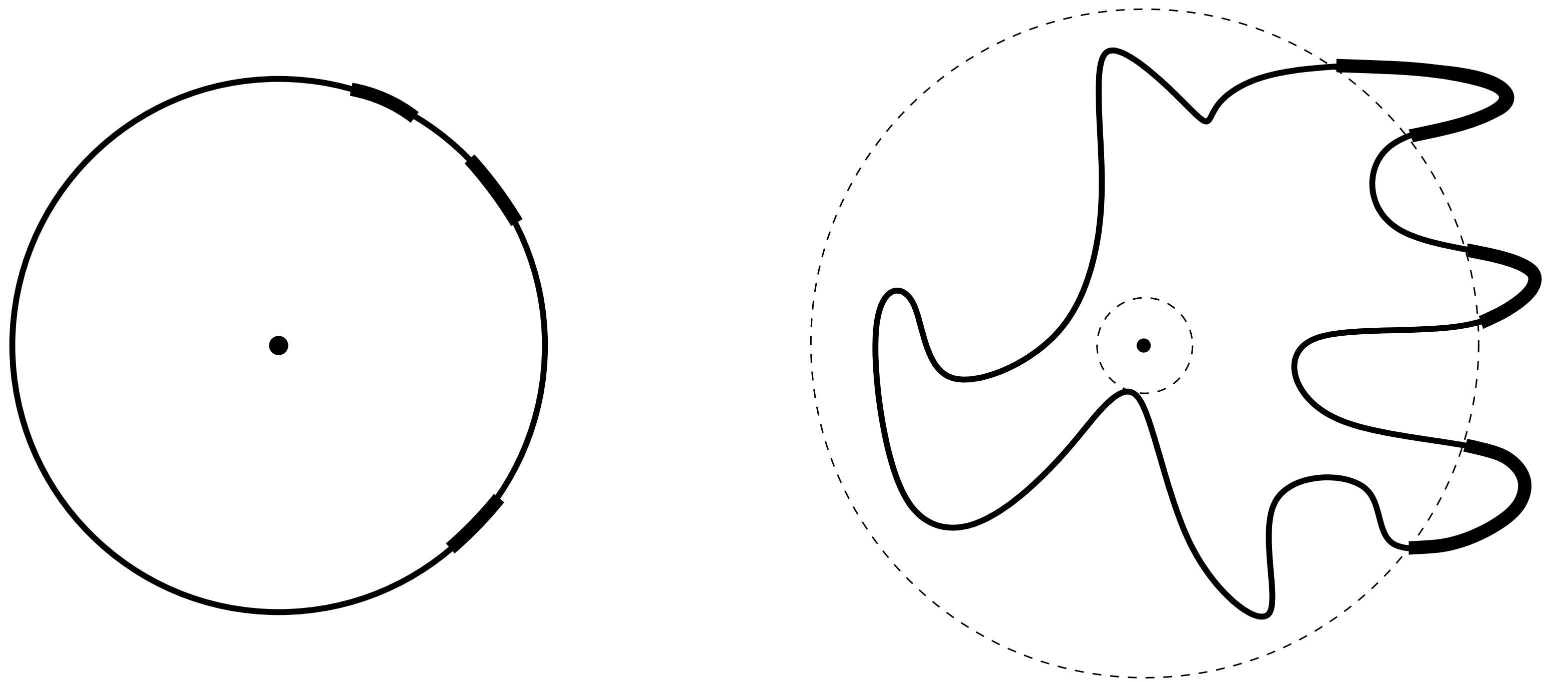
By the Koebe distortion theorem $\gamma_0$ and $\gamma_1$ have comparable Euclidean lengths, and clearly the length of $\gamma_1$ is at least $\delta$. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least $\delta/C$. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then $\rho$ is admissible for $f(\mathcal{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathcal{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

**Lemma 4.4.** *Suppose $f : \mathbb{D} \to \Omega$ is conformal, For $R \geq 1$, let*

$$E_R = \{x \in \mathbb{T} : \limsup_{r \nearrow 1} |f(rx) - f(0)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

*Then the extremal length of the path family connecting $E_R$ to $D(0, \frac{1}{2})$ in $\mathbb{D}$ is $\geq (\log R)/2\pi$.*

*Proof.* Assume $f(0) = 0$ and $\operatorname{dist}(0, \partial\Omega) = 1$ and let $\rho(z) = |z|^{-1}/\log R$ for $z \in \Omega \cap \{1 < |z| < R\}$. Then $\rho$ is admissible for the path family $\Gamma$ connecting $D(0, 1/2)$ to $\partial\Omega \setminus D(0, R)$ and $\iint \rho^2 dxdy \leq 2\pi/\log R$. By definition $M(\Gamma) \leq 2\pi/\log R$ and $\lambda(\Gamma) \geq (\log R)/2\pi$. $\square$



We say a set $E \subset \mathbb{T}$ has zero capacity if the extremal length of the path family connecting $E$ to $D(0, \frac{1}{2})$ in $\mathbb{D}$ is infinite.

Zero capacity sets are very small: zero length, even zero Hausdorff dimension.

A few facts we shall prove later:

**Fact:** If $E \subset \mathbb{T}$ has positive length, then it has positive capacity.

**Fact:** a countable union of zero capacity sets has zero capacity.

**Corollary 4.5.** *If $f : \mathbb{D} \to \Omega$ is conformal, then $f$ has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

*Proof.* Let $E_{r,\delta} \subset \mathbb{T}$ be the set of $x \in \mathbb{T}$ so that $\operatorname{diam}(f(rx, x)) > \delta$, and let $E_\delta = \cap_{0<r<1} E_{r,\delta}$. If $f$ does not have a radial limit at $x \in \mathbb{T}$, then $x \in E_\delta$ for some $\delta > 0$, and this has zero capacity by Lemma 4.3.

Taking the union over a sequence of $\delta$'s tending to zero proves the result. The set where $f$ has a radial limit $\infty$ has zero capacity by Lemma 4.4, so we deduce $f$ has finite radial limits except on zero capacity. $\square$

Combining the last two results proves

**Corollary 4.6.** *Given $\epsilon > 0$ there is a $C < \infty$ so that the following holds. If $f : \mathbb{D} \to \Omega$ is conformal, $z \in \mathbb{D}$ and $I \subset \mathbb{T}$ is an arc that satisfies $|I| \geq \epsilon(1 - |z|)$ and $\mathrm{dist}(z, I) \leq \frac{1}{\epsilon}(1 - |z|)$, then $I$ contains a point $w$ where $f$ has a radial limit and $|f(w) - f(z)| \leq C\mathrm{dist}(f(z), \partial\Omega)$.*

**Theorem 4.7** (Carathéodory)**.** *Suppose that* $f : \mathbb{D} \to \Omega$ *is conformal, and that* $\partial\Omega$ *is compact and locally path connected (for every* $\epsilon > 0$ *there is a* $\delta > 0$ *so that any two points of* $\partial\Omega$ *that are within distance* $\delta$ *of each other can be connected by a path in* $\partial\Omega$ *of diameter at most* $\epsilon$*). Then* $f$ *extends continuously to the boundary of* $\mathbb{D}$*.*

This theorem is actually due to a student of Carathéodory, Marie Torhorst, in her 1918 dissertation at Bonn University. See Lasse Rempe's article:

On prime ends and local connectivity

It is enough to show $f$ is uniformly continuous on the open disk. Since $f$ is continuous on $\mathbb{D}$, it is automatically uniformly continuous on any compact subdisk $\{|z| \leq 1 - r\}$, so we only need show that sets of small diameter near the boundary map to sets of small diameter.

*Proof.* Suppose $\eta > 0$ is small. Since $\partial\Omega$ is compact $\Omega \setminus f(\{|z| < 1 - \frac{1}{n}\})$ has finite area that tends to zero as $n \nearrow \infty$. Thus if $n$ is sufficiently large, this region contains no disk of radius $\eta$.

Choose $\{z_j\}$ to be $n$ equally spaced points on the unit circle and using Lemma 4.6 choose interlaced points $\{w_j\}$ so that $f$ has a radial limit $f(w_j)$ at $w_j$ and this limit satisfies $|f(w_j) - f(rw_j)| \leq C\eta$ where $r = 1 - 1/n$.

Given $\delta > 0$ we can choose $\eta$ and $n$ so that

$$\begin{aligned}|f(w_j) - f(w_{j+1})| \;\leq\; & |f(w_j) - f(rw_j)| \\ & +|f(rw_j) - f(rw_{j+1})| \\ & +|f(rw_{j+1}) - f(w_{j+1})| \\ \leq\; & C\delta,\end{aligned}$$

where the center term is bounded by Koebe's theorem and the other two by definition.

Fix $\epsilon > 0$ and choose $\delta > 0$ as in the definition of locally connected. Thus if $\eta$ is so small that $C\eta < \delta$, then the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ can be connected in $\partial\Omega$ by a curve of diameter at most $\epsilon$. Thus the image under $f$ of the Carleson square with base $I_j$ (the arc between $w_j$ and $w_{j+1}$) has diameter at most $C\eta + \epsilon$.

This implies $f$ is uniformly continuous on $\mathbb{D}$, and so it has a continuous extension to the boundary. $\square$

# 5. Logarithmic capacity

Logarithmic capacity associates a non-negative number to each Borel subset of the unit circle. Applying a Möbius transformation can change this value, so it is not a conformal invariant, but it will act as an intermediate between extremal and harmonic measure (a conformal invariant that will be defined later).

Suppose $\mu$ is a positive, finite Borel measure on $\mathbb{R}^2$ and define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w).$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w) = \int U_\mu(z) d\mu(z).$$

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

**Lemma 5.1.** *$U_\mu$ is lower semi-continuous, i.e.,*

$$\liminf_{z\to w} U_\mu(z) \geq U_\mu(w).$$

*Proof.* Fatou's lemma, $\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$. $\square$

**Lemma 5.2.** *If $\mu_n \to \mu$ weak*, then* $\liminf U_{\mu_n}(z) \geq U_\mu(z)$.

*Proof.* If we replace $\varphi = \log \frac{1}{|z-w|}$ by $\varphi_r = \min(r, \varphi)$ in the definition of $U$ to get $U^r$, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) = U^r_\mu(z).$$

So for any $\epsilon > 0$ we can choose $N$ so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \epsilon.$$

As $r \to \infty$ $U^r \to U$ (by the monotone convergence theorem), so for $r$ large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_\mu(z) - 2\epsilon.$$

which proves the result. $\square$

**Lemma 5.3.** *If* $\mu_n \to \mu$ *weak*, then* $\liminf I(\mu_n) \geq I(\mu)$.

*Proof.* The proof is almost the same as for the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak*, then so does the product measure $\mu_n \times \mu_n$. However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x)g(y)d\mu_n(x)d\mu_n(y).$$

and Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space. $\square$

Suppose $E$ is Borel and $\mu$ has its closed support inside $E$. We say $\mu$ is admissible for $E$ if $U_\mu \leq 1$ on $E$ and we define the **logarithmic capacity** of $E$ as

$$\operatorname{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathcal{A}(E)$.

We define the **outer capacity** (or exterior capacity) as

$$\operatorname{cap}^*(E) = \inf\{\operatorname{cap}(V) : E \subset V, V \text{ open}\}.$$

We say that a set $E$ is **capacitable** if $\operatorname{cap}(E) = \operatorname{cap}^*(E)$.

**Lemma 5.4.** *Sets of positive length have positive capacity.*

*Proof.* Let $\mu$ be Lebesgue measure restricted to $E$. Then

$$\begin{aligned} U\mu(z) &= -\int_E 2\log|z-w|d\mu(w) \\ &\leq -4\int_0^{|E|/2} \log s ds \\ &\leq 4|E|(\log 1/|E| - 1). \end{aligned}$$

Since $E$ supports a measure with bounded potential, it has positive capacity. $\square$

**Lemma 5.5.** *If $E$ supports a measure $\mu$ so that*

$$\mu(D(x,r)) \leq Cr^{\alpha}$$

*for some $C < \infty$ and $\alpha > 0$, then $E$ has positive capacity.*

Proof is left to reader (same as previous proof).

We shall see later that a set $E$ has positive Hausdorff dimension iff it supports such a measure. Thus sets of zero capacity have zero Hausdorff dimension.

In a rough sense, they are only "slightly bigger" than countable sets.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one. To be precise, we should denote logarithmic capacity as $\mathrm{cap}_{\log}$ or logcap, but to simplify notation we simply use "cap" and will often refer to logarithmic capacity as just "capacity". Since we do not use any other capacities in these notes, this abuse should not cause confusion.

Great reference is Carleson's *Selected problems in exceptional sets*.

Short book, very dense.

**WARNING:** The logarithmic capacity that we have defined is **NOT** the same as is used in other texts such as Garnett and Marshall's book, but is related to what they call the Robin's constant of $E$, denoted $\gamma(E)$. The exact relationship is $\gamma(E) = \frac{1}{\text{cap}(E)} - \log 2$. They define the logarithmic capacity of $E$ as $\exp(-\gamma(E))$.

The reason for doing this is that the logarithmic kernel $\log \frac{1}{|z-w|}$ takes both positive and negative values in the plane, so the potential functions for general measures and the Robin's constant for general sets need not be non-negative. Exponentiating takes care of this. Since we are only interested in computing the capacity of subsets of the circle, taking the extra "2" in the logarithm gave us a non-negative kernel on the unit circle, and we defined a corresponding capacity in the usual way. Since the kernel is the logarithm, we feel justified in calling the corresponding capacity the logarithmic capacity, despite the divergence with usual usage.

**Lemma 5.6.** *Compact sets are capacitable.*

*Proof.* Since $\operatorname{cap}(E) \leq \operatorname{cap}^*(E)$ is obvious, we only have to prove the opposite direction. Set $U_n = \{z : \operatorname{dist}(z, E) < 1/n\}$ and choose a measure $\mu_n$ supported in $U_n$ with $\|\mu_n\| \geq \operatorname{cap}(U_n) - 1/n$. Let $\mu$ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so $\mu$ is admissible in the definition of $\operatorname{cap}(E)$. Thus

$$\operatorname{cap}(E) \geq \limsup \|\mu_n\| = \lim \operatorname{cap}(U_n) = \lim \operatorname{cap}(U_n) = \operatorname{cap}^*(E).$$

$\square$

It is true that all Borel (indeed all analytic) sets are capacitable.

See Appendix B of: Fractals in Probability and Analysis

Analytic sets are continuous image of Borel sets. There are analytic sets that are not Borel.

Some basic facts are given in my preprint Conformal Removability is Hard.

A more comprehensive but accessible treatment of analytic sets of analytic sets is given in Chap 11 of the text by Bruckner, Bruckner and Thomson: Real Analysis.

It is clear from the definitions that logarithmic capacity is monotone

$$E \subset F \quad \Rightarrow \quad \operatorname{cap}(E) \leq \operatorname{cap}(F). \tag{5.1}$$

and satisfies the regularity condition

$$\operatorname{cap}(E) = \sup\{\operatorname{cap}(K) : K \subset E, K \text{compact}\}. \tag{5.2}$$

**Lemma 5.7** (Sub-additive)**.** *For any sets* $\{E_n\}$,

$$\operatorname{cap}(\cup E_n) \le \sum \operatorname{cap}(E_n). \tag{5.3}$$

*Proof.* We can write $\mu = \sum \mu_n$ as a sum of singular measures so that $\mu_n$ gives full mass to $E_n$. We can then restrict each $\mu_n$ to a compact subset $k_n$ of $E_n$ so that $\mu_n(K_n) \ge (1-\epsilon)\mu(E_n)$. These restrictions are admissible for each $E_n$ and hence

$$\sum \operatorname{cap}(E_n) \ge \sum \mu_n(K_n) \ge (1-\epsilon) \sum \mu_n(E_n) = (1-\epsilon)\|\mu\|.$$

Taking $\epsilon \to 0$ proves the result. $\square$

**Corollary 5.8.** *A countable union of zero capacity sets has zero capacity.*

**Corollary 5.9.** *Outer capacity is also sub-additive.*

*Proof.* Given sets $\{E_n\}$ chose open sets $V_n \supset E_n$ so that $\text{cap}(V_n) \leq \text{cap}^*(E_n) + \epsilon 2^{-n}$. By the sub-additivity of capacity

$$\text{cap}^*(\cup E_n) \leq \text{cap}(\cup V_n) \leq \sum \text{cap}(V_n) \leq \epsilon + \sum \text{cap}^*(E_n).$$

Taking $\epsilon \to$ proves the result. $\square$

Although capacity informally "measures" the size of a set, it is not additive, and hence not a measure.

**Lemma 5.10.** *If $E$ is compact and has positive capacity, there exists an admissible $\mu$ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on $E$, except possible a set of capacity zero.*

*Proof.* Let $\mu_n$ be a sequence of measures on $E$ so that $\|\mu_n\| \to \mathrm{cap}(E)$ and $U_n = U_{\mu_n}$ is bounded above by 1 on $E$ (such a sequence exists by the definition of logarithmic capacity). By Lemma 5.2 $U_\mu$ is also bounded above by 1.

Also, by a standard property of weak* convergence of positive measures $\|\mu\| = \liminf \|\mu_n\| = \mathrm{cap}(E)$, and by Lemma 5.3,

$$I(\mu) \le \liminf I(\mu_n) \le \liminf \|\mu_n\| = \mathrm{cap}(E),$$

so we must have $I(\mu) \le \mathrm{cap}(E)$.

Note that since $\mu > 0$ we must have

$$I(\mu) = \int\int \frac{2}{|z-w|} d\mu(w) d\mu(z) > 0$$

Otherwise, $\mu$ would have to be concentrated on two antipodal points of the circle, a set that clearly has capacity zero.

First we claim that $U_\mu \geq 1$ except possibly on a set of zero capacity. Otherwise let $T \subset E$ be a set of positive capacity on which $U_\mu < 1 - \epsilon$ and let $\sigma$ be a non-zero, positive measure on $T$ which potential bounded by 1. Define

$$\mu_t = (1 - t)\mu + t\sigma.$$

By replacing $\sigma$ by a small positive multiple of itself, we may assume $\|\sigma\| < I(\mu)/2$.

This is a measure on $E$ so that

$$\begin{aligned}
I(\mu_t) &\leq \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\
&\leq (1-t)^2 I(\mu) + 2t(1-t)\int U_\mu d\sigma + t^2 I(\sigma) \\
&\leq I(\mu) - 2tI(\mu) + 2t(1-t)\int U_\mu d\sigma + O(t^2) \\
&\leq I(\mu) - 2tI(\mu) + 2t(1-t)(1-\epsilon)\|\sigma\| + O(t^2) \\
&\leq I(\mu) - 2t(I(\mu) - (1-t)(1-\epsilon)\|\sigma\|) + O(t^2) \\
&\leq I(\mu) - tI(\mu) + O(t^2) \\
&< I(\mu),
\end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of $\mu$.

Thus $U_\mu \geq 1$ everywhere and we have equality except on capacity zero, hence except on a set of $\mu$-measure zero. $\square$

Let $\mu$ be the equilibrium probability measure for $E$.

This means $\mu$ attains supremum in definition of $\mathrm{cap}(E)$ and is rescaled to be a probability measure.

Let $\gamma = 1/\mathrm{cap}(E)$ be Robin's constant of $E$.

**Lemma 5.11.** *$U_\mu = \gamma$ $\mu$-almost everywhere on $E$.*

*Proof.* We proved this is true except on a set of zero log capacity. But a set $E$ of zero log capacity also has zero $\mu$-measure: otherwise restricting $\mu$ to $E$ and multiplying by a small positive constant, if necessary, gives an admissible measure on $E$, proving it has positive capacity. $\square$

A great deal is known about the subsets of $E$ where $U_\mu(z) = \gamma$ and $U_\mu(z) < \gamma$. The first is called the set of "regular points".

These were characterized by Wiener in the 1920's by a infinite series that measures much capacity $E$ has in annuli of the form

$$\{2^{-n} \le |z - w| \le 2^{-n+1}\}.$$

If the series diverges (there is a lot of $E$ near $z$) then $z$ is a regular point and otherwise it is not.

This is the same as being regular for the Dirichlet problem: every continuous function $f$ on $\partial\Omega$ has a harmonic extension $u$ to $\Omega$ and $u$ has continuous extension to the boundary except at the set of irregular points of $\partial\Omega$, always a set of zero capacity.

In the plane, every point of a continuum $K$ is regular.

# 6. Pfluger's theorem

**Theorem 6.1** (Pfluger's theorem)**.** *If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of $K$. Then there are constants $C_1, C_2$ so that following holds. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals,*

$$\frac{1}{\mathrm{cap}(E)} + C_1 \leq \pi\lambda(\mathcal{F}_E)) \leq \frac{1}{\mathrm{cap}(E)} + C_2,$$

*where $\mathcal{F}_E$ is the path family connecting $K$ to $E$. The constants $C_1, C_2$ can be chosen to depend only on $0 < r < R < 1$ if $\partial K \subset \{r \leq |z| \leq R\}$.*

*Proof.* We may assume $E \neq \mathbb{T}$, since otherwise the result is easy.

Let $K^*$ be the reflection of $K$ across $\mathbb{T}$ and let $\Omega$ be the connected component of $\mathbb{C} \setminus (E \cup K \cup K^*)$ that has $E$ on its boundary. Let $h(z)$ be the harmonic function in $\Omega$ with boundary values 0 on $K$ and $K^*$ and boundary value 1 on $E$.

By the usual theory of the Dirichlet problem, all boundary points are regular (since all boundary components are non-degenerate continua) and hence $h$ extends continuously to the boundary with the correct boundary values. Moreover, $h$ is symmetric with respect to $\mathbb{T}$, and this implies its normal derivative on $\mathbb{T}\setminus E$ is 0. Clearly $|\nabla h|$ is an admissible metric for $\mathcal{F}$, so

$$M(\mathcal{F}) \le D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

We wish to show equality holds.

By Green's theorem and the fact that $h = 1$ on $E$,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_{E} \frac{\partial h}{\partial n} ds = -\int_{E} h \frac{\partial h}{\partial n} ds.$$

and thus

$$\begin{aligned} \int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2} \int_{E} \frac{\partial (h^2)}{\partial n} ds \\ &= \frac{1}{2} \int_{\mathbb{T}\setminus E} \frac{\partial (H^2)}{\partial n} ds + \frac{1}{2} \int_{\partial K} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2} \int_{\mathbb{D}\setminus K} \Delta(h^2) dx dy. \end{aligned}$$

The first term is zero because $h$ has normal derivative zero on$\mathbb{T} \setminus E$, and hence the same is true for $h^2$.

The second term is zero because $h$ is zero on $K$ and so $\frac{\partial (h^2)}{\partial n} = 2h \frac{\partial h}{\partial n} = 0$.

To evaluate the third term, we use the identity

$$\begin{aligned}
\Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\
&= 2h\Delta h + 2\nabla h \cdot \nabla h \\
&= 2h \cdot 0 + 2|\nabla h|^2 \\
&= 2|\nabla h|^2,
\end{aligned}$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\backslash K} \Delta(h^2) dxdy = \int_{\mathbb{D}\backslash K} |\nabla h|^2 dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\backslash K} |\nabla h|^2 dxdy = D(h).$$

Thus the tangential derivative of $h$'s harmonic conjugate has integral $D(h)$ around $\partial K$ and therefore $2\pi h/D(h)$ is the real part of a holomorphic function $g$ on $\mathbb{D} \setminus K$.

Then $f = \exp(g)$ maps $\mathbb{D} \setminus K$ into the annulus

$$A = \{z : 1 < |z| < \exp(2\pi/D(h))$$

with the components of $E$ mapping to arcs of the outer circle and the components of $\mathbb{T} \setminus E$ mapping to radial slits.

The path family $\mathcal{F}$ maps to the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $1/D(h)$.

Now we have to relate $D(h)$ to the logarithmic capacity of $E$.

Let $\mu$ be the equilibrium probability measure for $E$.

This means $\mu$ attains supremum in definition of $\mathrm{cap}(E)$ and is rescaled to be a probability measure. Thus $U_\mu = \gamma$ where $\gamma = 1/\mathrm{cap}(E)$ $\mu$-almost everywhere on $E$ (since sets of zero capacity have zero measure) and is continuous off $E$, but since $U_\mu$ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on $E$.

Note that $U_\mu$ has a (negative) logarithmic pole at $\infty$.

Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\overline{z}))$. This has logarithmic poles at 0 and $\infty$.

**Claim:** there are constants $C_1, C_2$ so that for $z \in \partial K$,

$$v(z) + C_1 \leq 0, \qquad v(z) + C_2 \geq 0.$$

Here we use the assumption that $\partial K \subset \{r < |z| < R\}$. Then for $z \in \partial K$ and $w \in E$, we have

$$1 - R \leq |z - w| \leq 1 + r$$

$$\log(1 - R) \leq \log|z - w| \leq \log(1 + r)$$

$$\frac{2}{1+r} \leq U_\mu(z) = \int_E \log \frac{2d\mu(w)}{|z-w|} \leq \log \frac{2}{1-r}$$

$$\frac{2}{1+r} + \frac{2}{(1/r)+1} \leq v(z) \leq \log \frac{2}{1-r} + \log \frac{2}{(1/R)-1}$$

$$-C_2(r, R) \leq v(z) \leq -C_1(r, R).$$

This proves the claim.

Thus by the maximum principle,

$$\frac{v(z)+C_1}{\gamma+C_1} \leq h(z) \leq \frac{v(z)+C_2}{\gamma+C_2}.$$

To prove this note that

(1) For $z \in E$, $h(z) = 1$ and $v(z) = \cap(E)$, so LHS $=$ RHS $= 1 = h(z)$.

(2) For $z \in \partial K$, we have LHS $\leq 0 = h(z)$

(3) For $z \in \partial K$, we have RHS $\geq 0 = h(z)$

Thus LHS $\leq h \leq$ RHS everywhere by the maximum principle.

Since we have equality on $E$, we get on $E$

$$\frac{\partial}{\partial n}(\frac{v(z)+C_1}{\gamma+C_1}) \leq \frac{\partial h}{\partial n} \leq \frac{\partial}{\partial n}(\frac{v(z)+C_2}{\gamma+C_2}).$$

When we integrate over $E$, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$\begin{aligned} -\int_E \frac{\partial}{\partial n}\frac{v(z)+C_1}{\gamma+C_1}ds &= \frac{1}{\gamma+C_1}\int_{\mathbb{D}} \Delta(v)dxdy \\ &= \frac{\pi}{\gamma+C_1} \end{aligned}$$

because $v$ is harmonic except for a $\frac{1}{2}\log\frac{1}{|z|}$ pole at the origin.

A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma + C_1} \leq D(h) = M(\mathcal{F}) \leq \frac{\pi}{\gamma + C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n} ds$. Hence

$$\gamma + C_1 \leq \pi\lambda(\mathcal{F}) \leq \gamma + C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals. $\square$

**Lemma 6.2.** *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin. Let $\mathcal{F}$ be the path family connecting $K$ and $E$ in $\mathbb{D} \setminus K$. Fix an admissible metric $\rho$ for $\mathcal{F}$ and for each $z \in \mathbb{T}$, define $f(z) = \inf \int_\gamma \rho ds$ where the infimum is over all paths in $\mathbb{D}$ that connect $K$ to $z$. Then $f$ is lower semi-continuous.*

*Proof.* Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned}
\int_{2^{-n-1}}^{2^{-n}} [\int_{|z|=r} \rho ds]^2 dr &\leq \int_{2^{-n-1}}^{2^{-n}} \int_0^{2\pi} r^2 \rho^2 dr d\theta \\
&\leq \pi 2^{-n} \int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dx dy \\
&= o(2^{-n}).
\end{aligned}$$

Therefore we can choose circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z - z_0| < 2^{-n}\}$ of $\mathbb{D}$ centered at $z_0$ and with $\rho$-length $\epsilon_n$ tending to 0. By taking a subsequence we may assume $\sum \epsilon_n < \infty$. Now choose $z_n \to z_0$ with

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We want to show that there is a path connecting $K$ to $z_0$ whose $\rho$-length is as close to $\alpha$ as we wish. Passing to a subsequence we may assume $z_n$ is separated from $K$ by $\gamma_n$. Let $c_n$ be the infimum of $\rho$-lengths of paths connecting $\gamma_n$ and $\gamma_{n+1}$. By considering a path connecting $K$ to $z_n$, we see that $\sum_1^n c_k \leq f(z_n)$, for all $n$ and hence $\sum_1^\infty c_n \leq \alpha$.

Next choose $\epsilon > 0$ and choose $n$ so that we can connect $K$ to $z_n$ (and hence to $\gamma_n$) by a path of $\rho$-length less than $\alpha + \epsilon$. We can then connect $\gamma_n$ to $z_0$ by a infinite concatenation of arcs of $\gamma_k$, $k > n$ and paths connecting $\gamma_k$ to $\gamma_{k+1}$ that have total length $\sum_n^\infty (\epsilon_n + c_n) = o(1)$. Thus $K$ can be connected to $z_0$ by a path of $\rho$-length as close to $\alpha$ as we wish. $\square$

**Corollary 6.3.** *Suppose $E \subset \mathbb{T}$ is compact and $\epsilon > 0$. Then there is a finite collection of closed intervals $F$ so that $E \subset F$ and*

$$\lambda(\mathcal{F}_E) \leq \lambda(\mathcal{F}_F) + \epsilon.$$

*Proof.* By Lemma 6.2,

$$V = \{z \in \mathbb{T} : f(z) > r = (\frac{M(\mathcal{F}_E) + \epsilon}{M(\mathcal{F}_E) + 2\epsilon})^{1/2}\},$$

is open, and therefore we can choose a set $F$ of the desired form inside $V$. Then $\rho/r$ is admissible for $\mathcal{F}_F$, so

$$M(\mathcal{F}_F) \leq \int (\frac{\rho}{r})^2 dxdy = \frac{M(\mathcal{F}_E) + \epsilon}{M(\mathcal{F}_E) + 2\epsilon} \int \rho^2 dxdy \leq M(\mathcal{F}_E) + \epsilon.$$

Thus an inequality in the opposite direction holds for extremal length. $\square$

**Corollary 6.4.** *Pfluger's theorem holds for all compact subsets of* $\mathbb{T}$.

*Proof.* Suppose $E$ is compact and choose sets $E_n \searrow E$ that are finite unions of closed intervals. We have proven both

$$\lambda(\mathcal{F}_{E_n}) \to \lambda(\mathcal{F}_E),$$

(Corollary 6.3) and

$$\operatorname{cap}(E_n) \to \operatorname{cap}(E),$$

(Lemma 5.6), so the inequalities in in Pfluger's theorem extend to $E$. $\square$

# 7. Harmonic Measure

Suppose $\Omega$ is a planar domain bounded by a Jordan curve, $z \in \Omega$ and $E \subset \partial\Omega$ is Borel. Suppose $f : \mathbb{D}\Omega$ is conformal and $f(0) = z$ (by the Riemann mapping theorem there is always such a map). By Carathéodory's theorem, $f$ extends continuously (even homeomorphically) to the boundary, so $f^{-1}(E) \subset \mathbb{T}$ is also Borel.

We define "the harmonic measure of the set $E$ for the domain $\Omega$, with respect to the point $z$" as

$$\omega(z, E, \Omega) = |E|/2\pi,$$

where $|E|$ denotes the Lebesgue 1-dimensional measure of $E$.

This depends on the choice of the Riemann map $f$, but any two maps, both sending 0 to $z$, will differ only by a pre-composition with a rotation. Thus the two possible pre-images of $E$ differ by a rotation and hence have the same Lebesgue measure.

If we fix $E$ and $\Omega$, then $\omega(z, E, \Omega)$ is a harmonic function of $z$, giving rise the name "harmonic measure".

Since we always have $0 \leq \omega(z, E, \Omega) \leq 1$, Harnack's theorem implies that if $E$ has harmonic measure with respect to one point $z$ in $\Omega$ then it has zero harmonic measure with respect to all points.

If $\partial\Omega$ is merely locally connected, then Carathéodory's theorem still implies that the Riemann map $f$ has a continuous extension to the boundary, so the same definition of harmonic measure works.

For general simply connected domains the conformal maps extends radially almost everywhere and this can be used to define harmonic measure.

**Lemma 7.1.** *For any compact $E \subset \mathbb{T}$,*
$$\operatorname{cap}(E) \geq \frac{1}{1 + \log 2 + \pi + \log \frac{1}{|E|}} \geq \frac{1}{5 + \log \frac{1}{|E|}}.$$
*If $E \subset \mathbb{T}$ has positive Lebesgue measure, then it has positive capacity. In particular, if $E \subset \mathbb{T}$ is an arc, then*
$$\operatorname{cap}(E) \leq \frac{1}{\log 4 + \log \frac{1}{|E|}} \leq \frac{1}{1 + \log \frac{1}{|E|}}.$$
*For arcs of small measure, the two bounds are comparable.*

*Proof.* If $\mu$ is Lebesgue measure restricted to $E$, then clearly the corresponding potential function is less than potential function of an arc $I$ of the same measure evaluated at the center $x$ of that arc. Since $\frac{2}{\pi}t \leq |x-y| \leq t$ if the arclength between $x, y \in \mathbb{T}$ is $t$, this value is at most

$$\int_I \log \frac{2}{|x-y|} dy \leq 2 \int_0^{|E|/2} \log \frac{\pi}{t} dt = |E| \log \frac{2}{|E|} + (1+\pi)|E|$$

If we normalize the measure to have mass one, then we get

$$U_\mu \leq \log \frac{2}{|E|} + 1 + \pi = \log \frac{1}{|E|} + 1 + \log 2 + \pi.$$

If $E$ is an arc, then the center $x$ of the arc is at most distance $|E|/2$ from any other point of the arc, and so

$$U_\mu(x) \geq \log \frac{2}{|E|/2} = \log \frac{4}{|E|} = \log \frac{1}{|E|} + \log 4,$$

for any probability measure supported on $E$. This gives the desired estimate. $\square$

**Theorem 7.2.** *Suppose* $\Omega$ *is a Jordan domain,* $z_0 \in \Omega$ *with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and* $E \subset \partial\Omega$. *Let* $\Gamma$ *be the family of curves in* $\Omega$ *which connects* $D(z_0, 1/2)$ *to* $E$. *Then*

$$\omega(z_0, E, \Omega) \leq C \exp(-\pi\lambda(\Gamma)).$$

*If* $E \subset \partial\Omega$ *is an arc then the two sides are comparable.*

*Proof.* Let $f : \mathbb{D} \to \Omega$ be conformal. By Koebe's $\frac{1}{4}$-theorem (Theorem 2.3), the disk $D(z, \frac{1}{2})$ in $\Omega$ maps to a smooth region $K$ in the unit disk that contains the origin, and $\partial K$ is uniformly bounded away from both the origin and $\mathbb{T}$.

Thus by Pfluger's theorem applied to the curve family $\Gamma_X$ connecting $K$ and the compact set $X = f^{-1}(E)$,

$$\frac{1}{\operatorname{cap}(X)} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq \frac{1}{\operatorname{cap}(X)} + C_2(K),$$

for constants $C_1, C_2$ that are bounded independent of all our choices.

By Lemma 7.1 the right-hand side of

$$1 + \log 4 + \log \frac{1}{|X|} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq 1 + \log 2 + \log \frac{1}{|X|} + C_2(K).$$

holds in general, and the left-hand side also holds if $X$ is an interval.

Multiply by $-1$ and exponentiate to get

$$\frac{|X|}{2e^{1+\pi+C_2}} \leq \exp(-\pi\lambda(\Gamma_X)) \leq \frac{|X|}{4e^{C_1}}$$

under the same assumptions. Now use $\omega(z, E, \Omega) = \omega(0, X, \mathbb{D}) = |X|/2\pi$ to deduce the result. $\square$

One of the most famous and most useful applications of this result is

**Corollary 7.3** (Ahlfors distortion theorem)**.** *Suppose* $\Omega$ *is a Jordan domain,* $z_0 \in \Omega$ *with* $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ *and* $x \in \partial\Omega$*. For each* $0 < t < 1$ *let* $\ell(t)$ *be the length of* $\Omega \cap \{|w - x| = t\}$*. Then there is an absolute* $C < \infty$*, so that*

$$\omega(z_0, D(x,r), \Omega) \leq C \exp(-\pi \int_r^1 \frac{dt}{\ell(t)}).$$

*Proof.* Let $K$ be the disk of radius $1/2$ around $z_0$ and let $\Gamma$ be the family of curves in $\Omega$ which connects $D(x,r) \cap \partial\Omega$ to $K$. Define a metric $\rho$ by $\rho(z) = 1/\ell(t)$ if $z \in C_t = \{z \in \Omega : |x - z| = t\}$ and $\ell(t)$ is the length of $C_t$. Any curve $\gamma \in \Gamma$ has $\rho$-length at least

$$L = \int_r^{1/2} \frac{dt}{\ell(t)},$$

and

$$A = \iint_\Omega \rho^2 dxdy \geq \int_r^{1/2} \int_{C_r \cap \Omega} \ell(z)^{-2} r dr d\theta = \int \ell(z)^{-1} dr = L.$$

Therefore $\lambda(\Gamma) \geq A/L^2 = 1/L$, and this proves the result. $\square$

**Corollary 7.4** (Beurling's estimate)**.** *There is a $C < \infty$ so that if $\Omega$ is simply connected, $z \in \Omega$ and $d = \mathrm{dist}(z, \partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x, rd), \Omega) \leq Cr^{1/2}$$

*Proof.* Apply Corollary 7.3 at $x$ and use $\theta(t) \leq 2\pi t$ to get

$$\exp(-\pi \int_{rd}^{d} \frac{dt}{\theta(t)t}) \leq C \exp(-\frac{1}{2} \log r) \leq C\sqrt{r}.$$

$\square$

**Corollary 7.5.** *There is an $R < \infty$ so that for any $\Omega$ is a Jordan domain and any $z \in \Omega$*

$$\omega(z, \partial\Omega \setminus D(z, R\mathrm{dist}(z, \partial\Omega), \Omega)) \leq 1/2.$$

*Proof.* Rescale so $z = 1$ and $\mathrm{dist}(z, \partial\Omega) = 1$. Then apply $w \to 1/w$ which fixes $z$ and maps $\partial\Omega \setminus D(z, R)$ into $D(0, 1/R - 1)$. Then Lemma 7.4 implies the result if $R \geq 4C^2 + 1$ ($C$ is as in Lemma 7.4). $\square$

**Corollary 7.6.** *For any Jordan domain and any $\epsilon > 0$,*

$$\omega(z, \partial\Omega \cap D(z, (1+\epsilon)\text{dist}(z, \partial\Omega)), \Omega) > C\epsilon,$$

*for some fixed $C > 0$.*

*Proof.* Renormalize so $z = 0$ and 1 is a closest point of $\partial\Omega$ to $z$. By Corollary 7.5, the set $E = \partial\Omega \cap D(0, 1+\epsilon)$ has harmonic measure at least $1/2$ from the point $1-\epsilon/R$. Since $\omega(z, E, \Omega)$ is a positive, harmonic function on $\mathbb{D}$, Harnack's inequality says it is larger than $C\epsilon/R$ at the origin. $\square$

This is a weak version of the Beurling projection theorem which says that the sharp lower bound is given by the slit disk $D(0, 1+\epsilon) \setminus [1, 1+\epsilon)$. The harmonic measure of the slit in this case can be computed as an explicit function of $\epsilon$ because this domain can be mapped to the disk by sequence of elementary functions.

**Lemma 7.7.** *Suppose* $\Gamma$ *is a closed Jordan curve dividing the sphere into two simply connected domains* $\Omega_1, \Omega_2$. *Let* $z_i \in \Omega_i$ *satisfy* $\mathrm{dist}(z_i, \partial\Omega_1)$ *for* $i = 1, 2$. *Then there is a* $C < \infty$ *so that for any disk* $D$,

$$\omega_1(D)\omega_2(D) \leq C|D|^2.$$

*Proof.* Let $x \in \Gamma$ and let $\theta_i(t)$ be the function corresponding to $\Omega_i$ for $i = 1, 2$. The multiplying the estimates for each domain gives

$$\omega_1(D)\omega_2(D) \leq C \exp(-\pi \int_{|D|}^{1} (\frac{1}{\theta_1(t)} + \frac{1}{\theta_2(t)})dt).$$

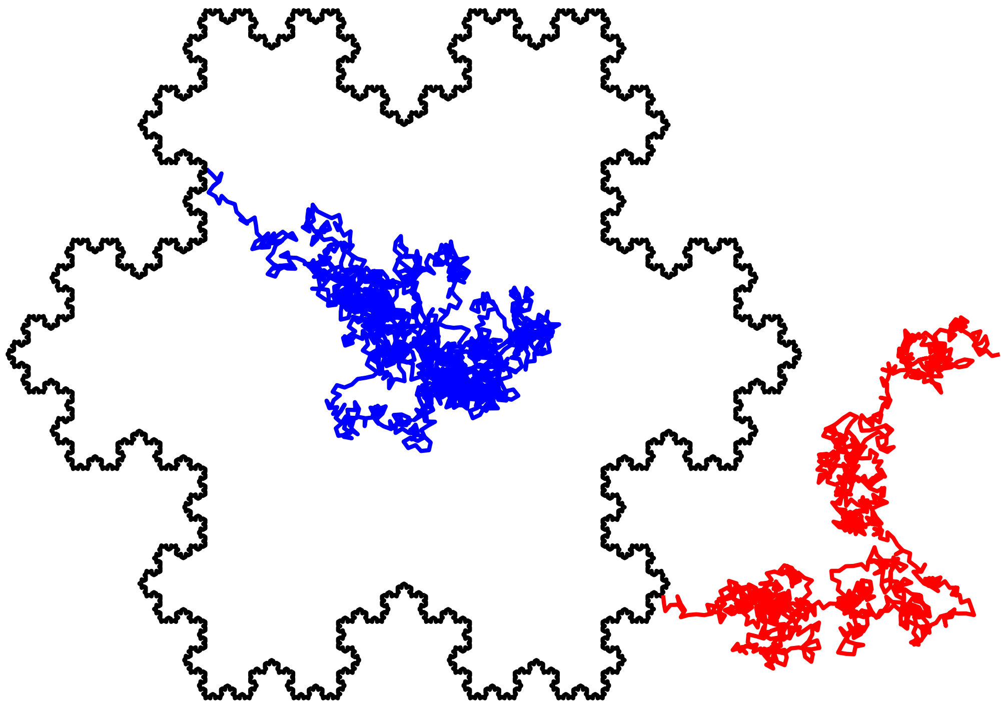
Since $\Omega_1$ and $\Omega_2$ are disjoint, $\theta_1 + \theta + 2 \leq 2\pi t$ and so a simple calculus exercise shows that $\theta_1^{-1} + \theta_2^{-1} \geq 2/\pi t$. Thus

$$\omega_1(D)\omega_2(D) \leq C \exp(-\pi \int_{|D|}^{1} \frac{2\pi t}{d} t) = C|D|^2,$$

as desired. $\square$

**Theorem 7.8.** *Suppose* $z_1 \in \Omega_1$, $z_2 \in \Omega_2$ *and let* $\omega_1, \omega_2$ *denote the corresponding harmonic measures. Then* $\omega_1$ *and* $\omega_2$ *are mutually absolutely continuous on the set of tangent points of* $\Gamma$ *and are mutually singular on the rest of* $\Gamma$*. In particular,* $\omega_1 \perp \omega_2$ *iff the tangent points have zero linear measure.*

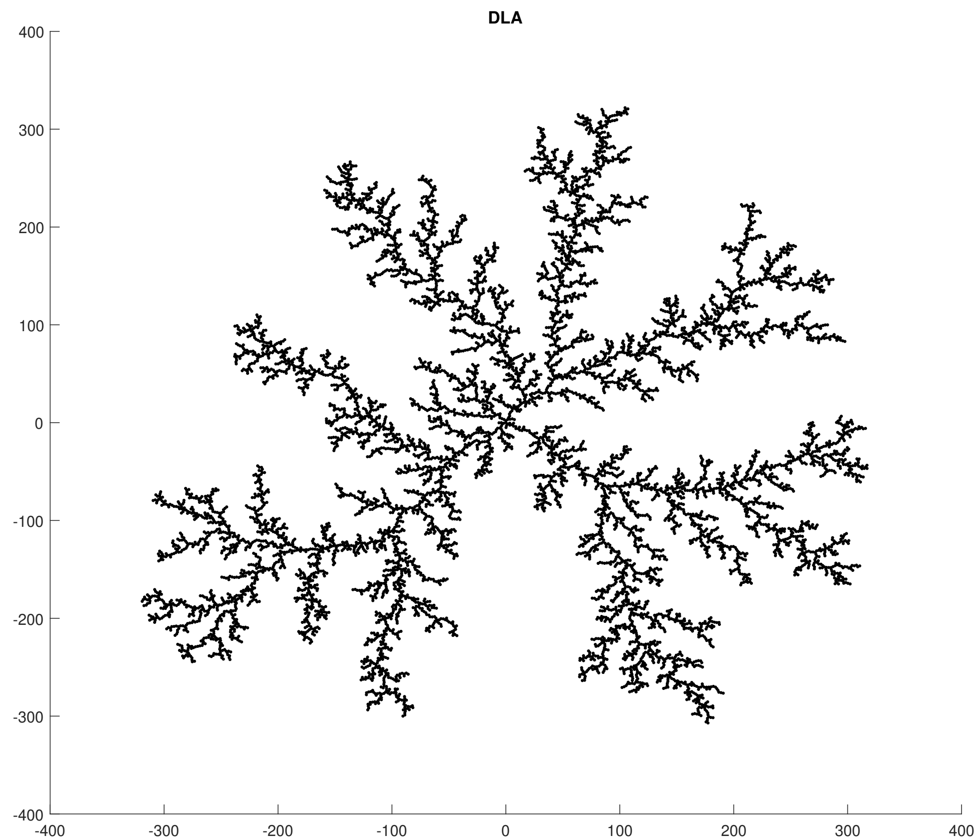
For a fractal curve, inside and outside harmonic measures are singular.



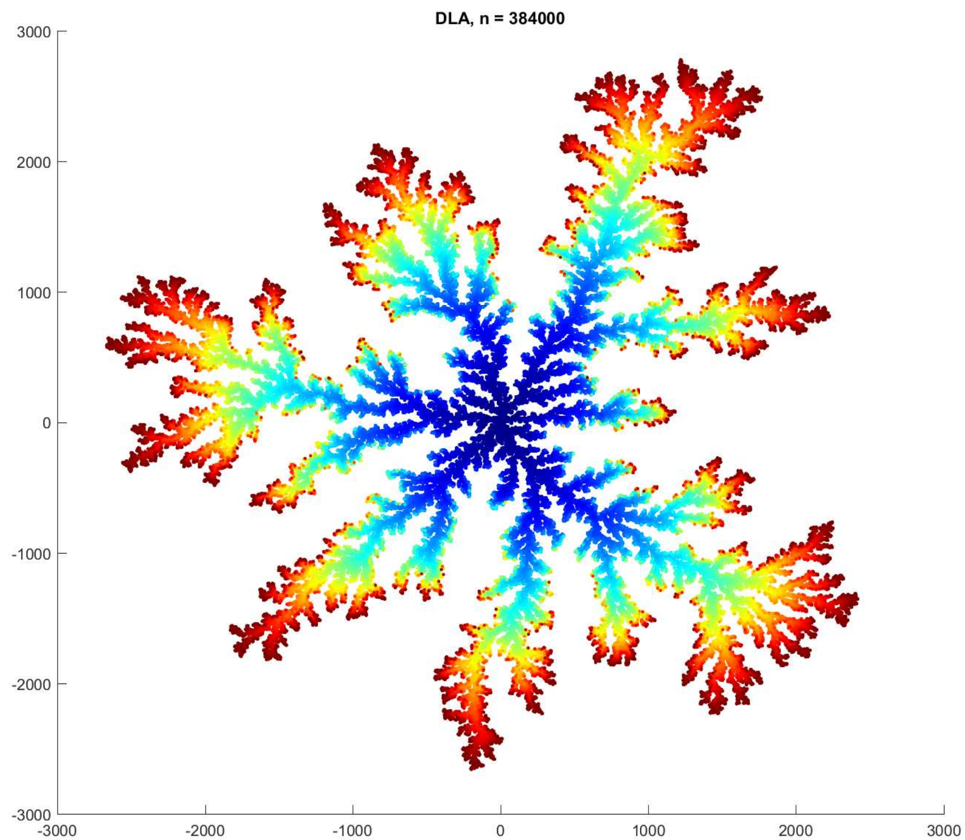
$\omega_1 \perp \omega_2$ iff tangents points have zero length.

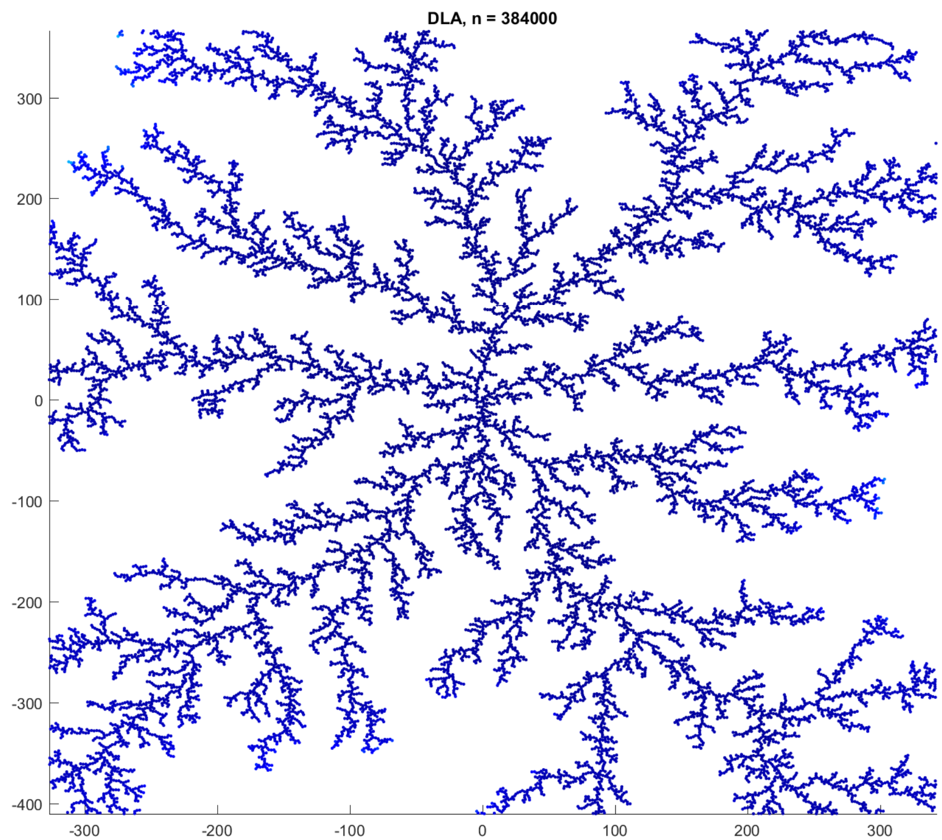
# 8. Kesten's theorem on growth of DLA

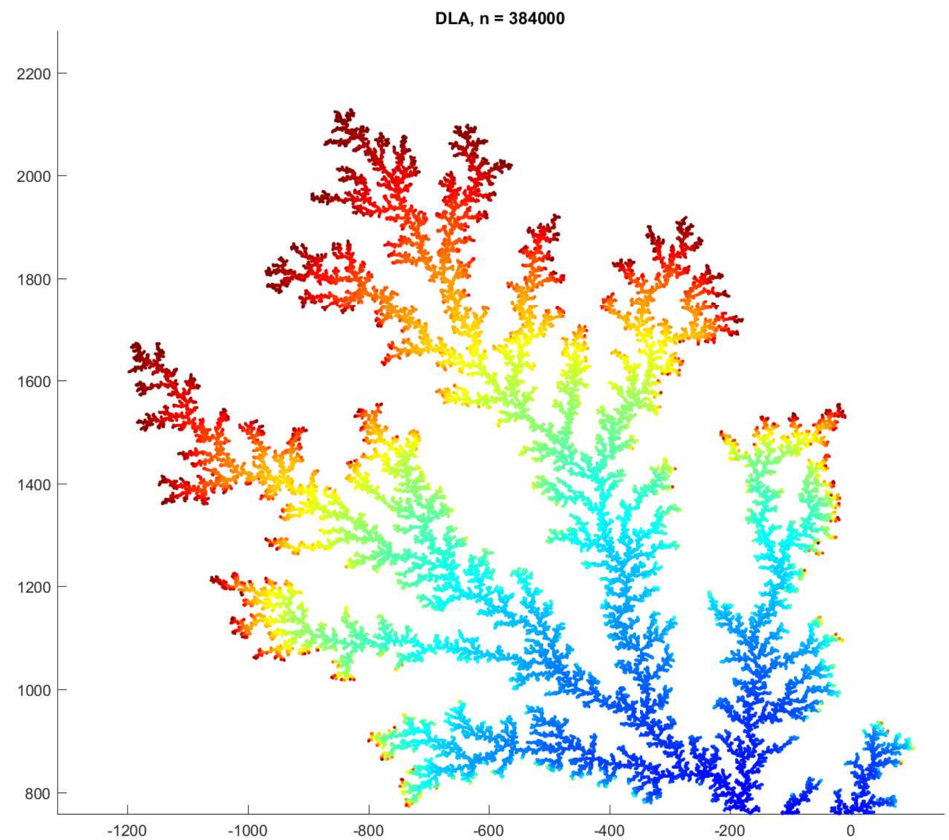
Start with a unit disk centered at the origin. Imagine another unit disk, whose center moves as a Brownian motion starting near infinity unit the it hits the first disk and the stops. Now send in another random disk until it hits one of the first two. Continue in this way until $n$ disks have accumulated to form a connected set.

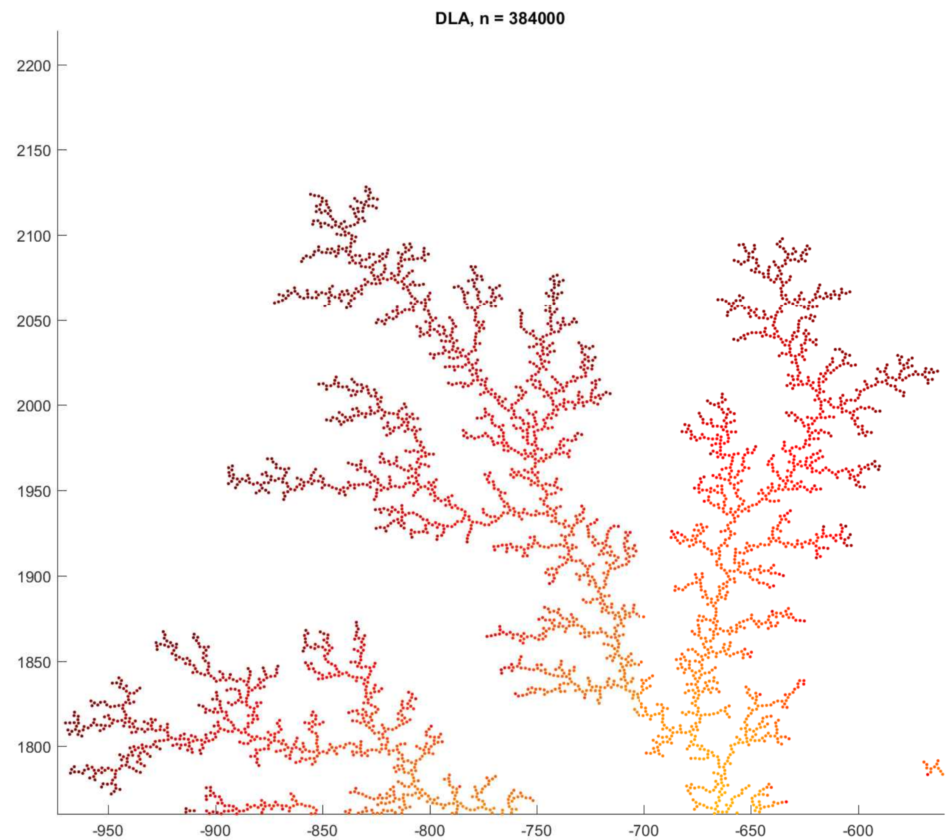


DLA
400
300
200
100
0
-100
-200
-300
-400
-400
-300
-200
-100
0
100
200
300
400


DLA, n = 384000
3000
2000
1000
0
-1000
-2000
-3000
-3000
-2000
-1000
0
1000
2000
3000


DLA, n = 384000
300
200
100
0
-100
-200
-300
-400
-300
-200
-100
0
100
200
300


DLA, n = 384000
2200
2000
1800
1600
1400
1200
1000
800
-1200
-1000
-800
-600
-400
-200
0


DLA, n = 384000
2200
2150
2100
2050
2000
1950
1900
1850
1800
-950
-900
-850
-800
-750
-700
-650
-600

As discussed at beginning of the course, it is easy to show that The diameter of DLA grows less than n and greater than $\sqrt{n}$.

Harry Kesten proved:

**Theorem 8.1.** *Almost surely, the diameter of DLA(n) is $O(n^{2/3})$.*

No non-trivial lower bound is yet known.

*Proof.* The first step is to make the definition of DLA a little more precise. A moving disk will hit a set $E$ when the center is precisely distance 1 from that set. In our case, the set is a union of $n$ unit disks centered at a finite set of points $P_n = \{p_1, \ldots, p_n\}$.

Thus the process of adding the next disk by letting it wander by Brownian motion, is precisely the same as choosing a point $p_{n+1}$ on the set

$$E_n = \{z : \mathrm{dist}(z, P_n) = 2\},$$

with respect to harmonic measure at $\infty$ for the domain $\Omega_n$ that is the unbounded complementary component of $E_n$. Since $E_n$ is, by definition, a connected set, $\Omega_n$ is simply connected and will be bounded by a finite number of circular arcs.

Almost surely (with probability one) $\Omega_n$ will be the entire complement of $E_n$.

Otherwise, we must have chosen a disk that made contact with two or more earlier disks. But there are only a finite number of points on $E_k$ where this happens, and finite sets have harmonic measure zero (e.g., Beurling's theorem), so the probability of making such a choice is zero.

Thus, almost surely, each disk in the cluster (except the one at the origin) hits exactly one previously chosen disk, although it may be hit by several (at most four, almost surely) later ones.

Consider

$$\operatorname{rad}(n) = \max\{|p| : p \in P_n\},$$

which measures the size of the DLA cluster in terms of a disk around the origin, and its inverse

$$\operatorname{exit}(m) = \max\{n : \operatorname{rad}(n) \leq m\},$$

which measures how soon the cluster grows beyond a given radius.

The theorem is stated in terms of an upper bound for $\operatorname{rad}(n)$, but is equivalent to a lower bound for $\operatorname{exit}(m)$:

$$\liminf_{m\to\infty} \frac{\operatorname{exit}(m)}{m^{3/2}} \geq \beta, \tag{8.1}$$

holds almost surely for some constant $\beta > 0$.

More precisely, we define

$$V_m = \{\operatorname{exit}(m) \leq \beta m^{3/2}\},$$

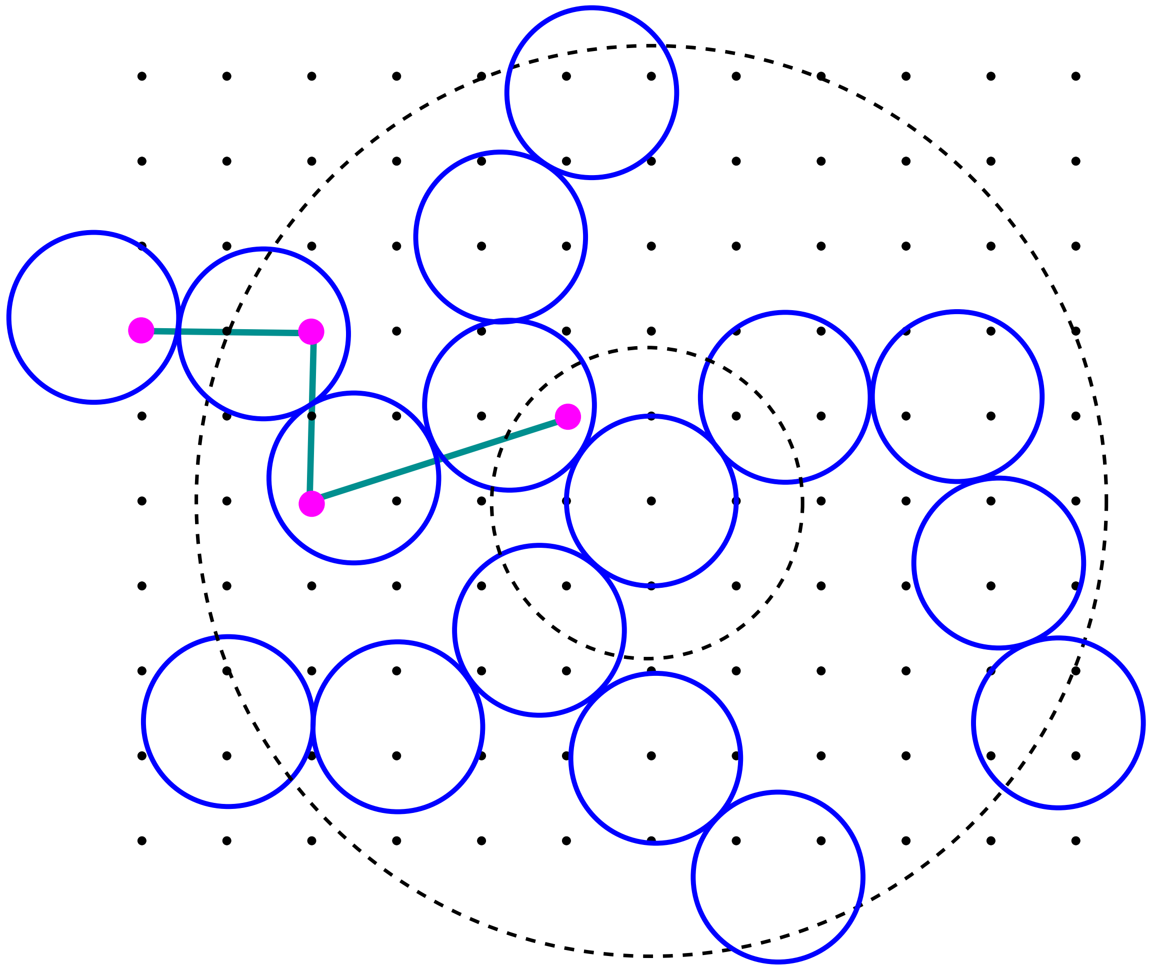
and we will prove that $\sum_m \mathbb{P}(V_m) < \infty$ if $\beta > 0$ is small enough.

The Borel-Cantelli lemma then implies that the probability that $V_m$ occurs infinitely often is zero. Thus almost surely $V_m$ only occurs finitely often, which gives (8.1).

We estimate the probability of $V_m$ by placing these events inside larger events and estimating those. If $V_m$ occurs, it means that the DLA cluster contains a path of at most $\beta m^{3/2}$ disks $\{D_1, \dots D_N\}$ that starts at the origin and ends with a disk that hits the circle $\{|z| = m\}$.

Moreover, every $D_{j+1}$, $j = 1, \dots N - 1$ was selected after $D_j$ in the growth process. Otherwise suppose $D_{j+1}$ is the first counterexample in the path. Then $D_{j+1}$ is the unique earlier disk hit by $D_j$, so $D_{j-1}$, which also touches $D_j$, must have been chosen later than $D_j$, making $D_j$ a counterexample too.

Every unit disk contains a point in the lattice $\mathbb{Z} \times \mathbb{Z}$, so for each path of unit disks as above, we can choose a sequence of lattice points $\mathbf{z} = \{z_1, \dots z_N\}$ such $z_j \in D_j$, $j = 1, \dots z_N$ and $|z_j - z_{j+1}| \leq 4$ since the union of two touching unit disks has diameter 4.



We say that sequence of distinct lattice points $\{z_1, \ldots, z_k\}$ is $m$-admissible if

$$|z_1| \leq m/2, \quad |z_k| \geq m, \quad |z_j - z_{j+1}| \leq 4.$$

Note that there are at most $m^2 80^{k-1}$ $m$-admissible sequences of length $k$ :

- there are $m^2$ possible choices for $z_1$,
- each following choice is made from a $9 \times 9$ square, omitting the center.

Moreover, the length of an $m$-admissible sequence is at least $m/8$ since the first and last points are at least distance $m/2$ apart.

Given an $m$-admissible sequence $\mathbf{z}$ of length $k$, we define $W_m(\mathbf{z})$ to be the set of DLA clusters so that:

(1) the cluster contains at most $\beta m^{3/2}$ disks,

(2) the cluster contains the sequence $\mathbf{z}$, and

(3) the disk containing $z_{j+1}$ was chosen after the disk containing $z_j$.

By our comments above each cluster in $V_m$ contained in the event $W_m(\mathbf{z})$ for some $m$-admissible sequence of length $k \leq \beta m^{3/2}$.

Thus all of $V_m$ is contained $W_m$, the union of $W_m(\mathbf{z})$ over all $m$-admissible sequences of length at most $\beta m^{3/2}$.

We claim that if $\mathbf{z}$ has length $k$, then

$$\mathbb{P}(W_m(\mathbf{z})) \leq (C\beta)^k. \tag{8.2}$$

We will finish the proof of the theorem assuming this is true, and then prove the estimate.

Given (8.2)

$$\begin{aligned}
\mathbb{P}(W_m) &\leq \sum_{\mathbf{z}} \mathbb{P}(W(\mathbf{z})) \\
&\leq \#(m - \text{admissible } \mathbf{z}) \cdot (C\beta)^k \\
&\leq m^2 80^{k-1} (C\beta)^k \\
&\leq m^2 (80C\beta)^k \\
&\leq m^2 (80C\beta)^{m/4},
\end{aligned}$$

since $k \geq m/4$. Thus if $\beta < 1/80C$,

$$\sum_m \mathbb{P}(V_m) \leq \sum_m \mathbb{P}(W_m) \leq \sum_m m^2 (80C\beta)^{m/4} < \infty.$$

This completes the proof of Theorem 8.1, except for the proof of (8.2).

First we explain the general idea for proving (8.2). We will make it precise later.

Suppose we have already grown a cluster that contains the points $z_1, \ldots, z_j$. How long do we have to wait before the cluster contains $z_{j+1}$? We must add a disk within distance 4 of the disk containing $z_j$.

Since the cluster has diameter at least $m/2$, by Beurling's estimate (Lemma 7.4) the probability of choosing such a disk is less that $C/\sqrt{m}$.

Therefore the expected number of disks we add before covering $z_{j+1}$ is at least $\sqrt{m}/C$. This has to happen $k$ times, so we expect that $k\sqrt{m}/C$ disks need to be added to the cluster before the whole sequence $\mathbf{z}$ is covered.

Since $k \geq m/8$, we therefore expect to need about $m^{3/2}/C$ disks to be added. However, clusters in the event $W_m(\mathbf{z})$ only use $\beta m^{3/2}$ disks to cover $\mathbf{z}$.

If $\beta$ is small compared to $1/C$, this event should have small probability.

To make this idea precise, let $D_1, \ldots$ be an enumeration of the disks in the cluster, in the order they are added. Suppose $z_j$ is contained in disk $D_{k(j)}$ and let $w(j) = k(j+1) - k(j)$; the "waiting time" between covering $z_j$ and $z_{j+1}$.

Then

$$\mathbb{P}(w(j) > t) \geq (1-p)^t,$$

where $p \leq C/\sqrt{m}$ is probability of hitting disk containing $z_j$ (this is where we use Beurling's estimate).

Therefore $w(j)$ is bounded below by a geometric random variable (the same one for each $j$), and $\sum_j w(j)$ will be bounded below by the corresponding sum of geometric variables. We estimate this distribution using:

**Lemma 8.2.** *Suppose $X_1, \dots X_n$ are independent geometric random variables, i.e., $\mathbb{P}(X_j = s) = p(1-p)^{s-1}$ for some $0 < p < 1/2$, and $Y = \sum_{j=1}^n X_j$. If $a \geq 2p$, then*

$$\mathbb{P}(Y \leq an/p) \leq (2e^2a)^n.$$

*Proof.* Define the moment generating function of the random variable $Y$ as the expected value of $\exp(tY)$. If $X$ is a geometric random variable, then

$$\mathbb{E}(e^{tX}) \;=\; \sum_{j=1}^{\infty} e^{tj} p(1-p)^{j-1} = pe^t \sum_{j=0}^{\infty} (e^t(1-p))^j = \frac{pe^t}{1-e^t(1-p)}.$$

Since $Y$ is a sum of independent copies of $X$,

$$\mathbb{E}(e^{tY}) \;=\; \prod_{j=1}^{\infty} \mathbb{E}(e^{tX}) = \left[\frac{pe^t}{1-e^t(1-p)}\right]^n.$$

By Chebyshev's inequality

$$\mathbb{P}\left(Y < \frac{\ln \lambda}{-t}\right) = \mathbb{P}(e^{-tY} > \lambda) \leq \frac{1}{\lambda}\mathbb{E}(e^{-tY}).$$

Set $\lambda = \exp(-ant/p)$ to get

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\leq \exp(ant/p)\mathbb{E}(e^{-tY}) \\
&= \frac{\exp(ant/p)e^{-nt}p^n}{(1 - e^{-t}(1-p))^n} \\
&= \frac{\exp(ant/p)p^n}{(e^t - (1-p))^n}
\end{aligned}$$

Now set $t = \ln(a(1-p)/(a-p)$ and this becomes

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\leq \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(\frac{a(1-p)}{a-p} - (1-p))^n} \\
&\leq \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{a}{a-p} - 1)^n} \\
&\leq \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{p}{a-p})^n} \\
&\leq \left(\frac{a(1-p)}{a-p}\right)^{an/p} \left(\frac{a-p}{1-p}\right)^n.
\end{aligned}$$

Using $p < 1/2$ and $a \geq 2p$, we get $a \leq 2(a-p)$ and $1-p > 1/2$, so

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\leq \left(\frac{a(1-p)}{a-p}\right)^{an/p} (2a)^n \\
&\leq \left(\frac{a}{a-p}\right)^{an/p} (2a)^n \\
&\leq \left(1 + \frac{p}{a-p}\right)^{an/p} (2a)^n \\
&\leq \left(1 + \frac{p}{a-p}\right)^{2(a-p)n/p} (2a)^n \\
&\leq (e^2 2a)^n,
\end{aligned}$$

since $(1 + \frac{1}{x})^x \leq e$. $\square$

To finish the proof of (8.2), apply Lemma 8.2 with $a = \beta k/p \geq C_1\beta m^{3/2}$

$$\begin{aligned}
\mathbb{P}(W_m) &\leq \mathbb{P}(\sum_{j=1}^{k} w(j) < \beta m^{3/2}) \\
&\leq \mathbb{P}(\sum_{j=1}^{k} X_j < C_1\beta k/p) \\
&\leq (2e^2 C_1\beta)^k = (C_2\beta)^k,
\end{aligned}$$

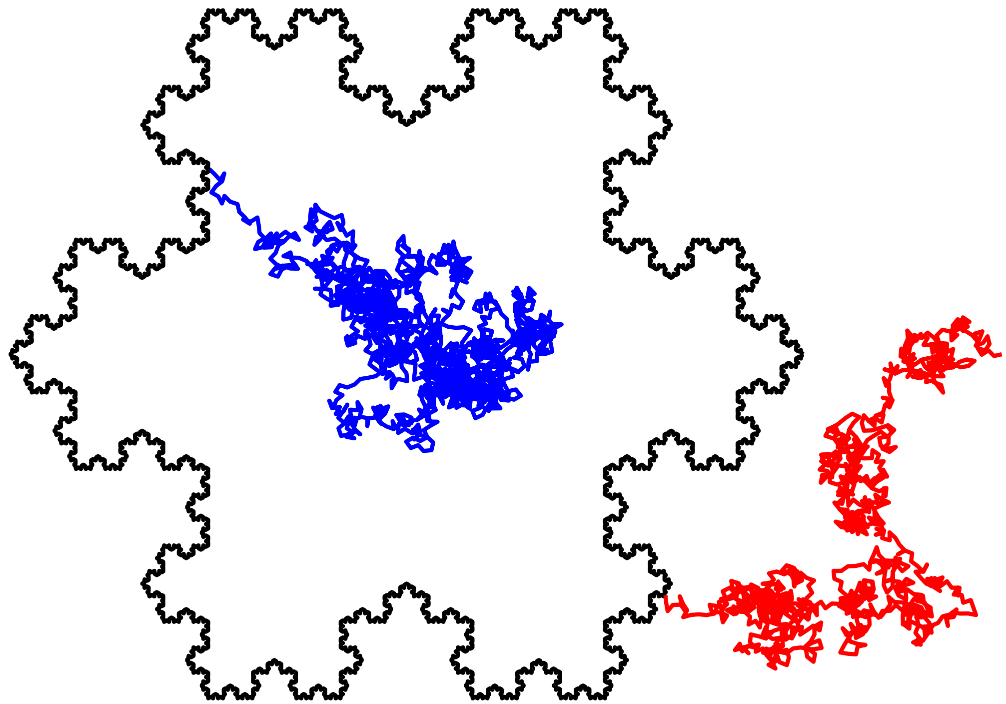
as desired. This completes the proof of (8.2) and hence of Theorem 8.1. □

This completes the proof of Kesten's theorem.

# 9. Makarov's upper bound

**Theorem 9.1.** *Suppose* $z_1 \in \Omega_1$, $z_2 \in \Omega_2$ *and let* $\omega_1, \omega_2$ *denote the corresponding harmonic measures. Then* $\omega_1$ *and* $\omega_2$ *are mutually absolutely continuous on the set of tangent points of* $\Gamma$ *and are mutually singular on the rest of* $\Gamma$. *In particular,* $\omega_1 \perp \omega_2$ *iff the tangent points have zero linear measure.*

For a fractal curve, inside and outside harmonic measures are singular.



$\omega_1 \perp \omega_2$ iff tangents points have zero length.

*Proof.* We will only sketch the proof, using three "well known" facts:

- the F. and M. Riesz theorem,
- McMillan's twist point theorem,
- and Makarov's $\dim(\omega) \leq 1$ theorem.

**Theorem 9.2** (F. and M. Riesz Theorem, Version 2). *Suppose that* $\Phi$ *is a univalent map of* $\mathbb{D}$ *onto a simply connected domain* $\Omega$ *with rectifiable boundary. Suppose* $E \subset \mathbb{T}$. *Then* $\mathcal{H}^1(E) = 0$ *iff* $\mathcal{H}^1(\Phi(E)) = 0$. *In other words, harmonic measure on* $\partial\Omega$ *is mutually absolutely continuous to* 1*-dimensional Hausdorff measure.*

**Theorem 9.3** (McMillan's Twist Point Theorem)**.** *If $\Omega$ is a simply connected domain then almost every point on $\partial\Omega$ (with respect to harmonic measure) is either an inner tangent point or a twist point.*

We also use the following result of Makarov:

**Lemma 9.4.** *Suppose* $\Omega$ *is simply connected and let* $\omega$ *be harmonic measure with respect to some point in* $\Omega$*. If* $T \subset \partial\Omega$ *denotes the set of inner tangents then there is an* $F \subset \partial\Omega \setminus T$ $\omega(F) = \omega(\partial\Omega \setminus T)$ *such that for any* $M > 0$ *there is a disjoint covering of* $F$ *by disks* $\{D_j\}$ *with* $\omega(D_j) \geq M|D_j|$*.*

**Corollary 9.5.** *Harmonic measure on a simply connected domain gives full measure to a set of* $\sigma$*-finite length.*

**Corollary 9.6.** *Harmonic measure on a simply connected domain has dimension* $\leq 1$*.*

Divide $\Gamma$ into:

(1) Tangent points,

(2) Twist points,

(3) Everything else.

The F and M Riesz theorem implies harmonic measures are mutually absolutely continuous on (1).

McMillan's twist point theorem says that (3) has zero harmonic measure from both sides.

We need to show is that the measures are singular on the twist points.

Choose a large $n$ and by the first lemma choose disjoint disks $\{D_j^n\}$ so that

$$\omega_1(D_j^n) \geq n|D_j^n|,$$

$$\omega(\cup_j D_j^n) = \omega_1(\text{twist points}).$$

Then if $F = \cap_n \cup_{k>n} \cup_j D_j^k$, we have

$$\omega_1(F) = \omega_1(\text{twist points}),$$

but by the product estimate,

$$\begin{aligned} \omega_2(F) &\leq \sum_j \frac{C|D_j^n|^2}{\omega_1(D_j^n)} \leq \frac{C}{n}\sum_j |D_j| \\ &\leq \frac{C}{n^2}\sum_j \omega(D_j) \leq \frac{C}{n^2} \to 0. \end{aligned}$$

Thus the measures are singular on the twist points. $\square$

Some historical background to Makarov's theorem.

**Theorem 9.7.** *Suppose $\Omega$ is a Jordan domain and $E \subset \partial\Omega$ has zero $\frac{1}{2}$-Hausdorff measure. Then $E$ has zero harmonic measure in $\Omega$.*

*Proof.* Since dilations do not change dimension or harmonic measure, we can rescale so that $\Omega$ contains a unit disk centered at some point $z$. Harnack's inequality says that if $\omega$ is zero at one point, it is zero everywhere. Hence it suffices to show $E$ has harmonic measure zero with respect to $z$.

By definition, the hypothesis means that for any $\epsilon > 0$, the set $E$ can be covered by open disks $\{D(x_j, r_j)\}$ that satisfy $\sum_j r_j^{1/2} \leq \epsilon$. By Beurling's estimate, this implies

$$\omega(z, E, \Omega) \leq \sum_j \omega(z, D_j, \Omega) \leq O(\sum_j r_j^{1/2}) = O(\epsilon).$$

$\square$

This result was not improved until Lennart Carleson showed in a tour de force that the $\frac{1}{2}$ could be replaced by some $\alpha > \frac{1}{2}$.That result was not improved until Makarov showed it holds for all $\alpha < 1$.

I plan to prove Makarov's theorem later in the course. Even though we have not defined harmonic measure for multiply connected domains, it is clear that no analog is possible in that case: if the boundary of $\Omega$ is a Cantor set of dimension $\alpha$, then it must have full harmonic measure, even if $\alpha$ is small.

However, Peter Jones and Tom Wolff showed that harmonic measure has dimension $\leq 1$ for any planar domain. The optimal upper bound in $\mathbb{R}^n$ remains unknown (but it is $> n-1$).

**Corollary 9.8.** *If $\Omega$ is Jordan domain, then harmonic measure is singular to area measure.*

*Proof.* By the Lebesgue density theorem, at Lebesgue almost every point $z$ of a set $E$ of positive area, all small enough disks satisfy

$$\text{area}(E \cap D(z,r)) \geq (1-\epsilon)\text{area}(D(z,r)),$$

for all $r < r_0$. In particular we must have $\ell(t) \leq \frac{\epsilon}{t}$ on a set of measure at least $r/4$ in $[r/2, r]$. Thus by the Ahlfors distortion theorem

$$\omega(D(z, r_0 2^{-n}) \leq C \exp(-\pi \int_{2^{-n}r_0}^{r_0} \frac{dt}{\epsilon t}) \leq C 2^{-\pi n/\epsilon}.$$

This is much less than $(2^{-n} r_0)$ if $n$ is large. Thus almost every point of $\partial\Omega$ can be covered by arbitrarily small disks so that $\omega(D(z_j, r_j)) = o(r_j^2)$. Use Vitali's covering theorem to take a disjoint cover of a set of full harmonic measure, and we deduce that harmonic measure gives full mass to set of zero area. $\square$

**Corollary 9.9.** *There is an $\epsilon > 0$ so that harmonic measure on a planar Jordan domain always gives full measure to a set of Hausdorff dimension at most $2 - \epsilon$.*

*Proof.* Fix a large integer $b$ and consider the $b$-adic squares in the plane. Take one such square $Q$ that intersects $\partial\Omega$ and consider its $b^2$ children squares. We claim that if $b$ is large enough, then at least one of them has harmonic measure that is less than $(2b^2)^{-1}$ times the harmonic measure of $Q$. If there is a subsquare that misses $\partial\Omega$, then its harmonic measure is zero, and the claim is true.

Therefore we may assume every subsquare hits $\partial\Omega$. Suppose $Q$ has side length 1 and define a finite sequence of squares $S_k$, concentric with $Q$ and with side lengths $\frac{1}{b}, \frac{3}{b}, \frac{6}{b}, \dots, 1$. If $z \in \partial S_k$, then $\mathrm{dist}(z, \partial\Omega) \leq \sqrt{2}/b$ and $\mathrm{dist}(z, S_{k-1}) > 3/b$, so by Corollary 7.6,

$$\max_{z \in \partial S_k} \omega(z, \partial\Omega \cap S_{k-1}, \Omega \setminus S_{k-1}) < 1 - \delta,$$

for some uniform $\delta > 0$ (independent of $k$ and $b$).

By the maximum principle and induction,

$$\omega(S_1) \leq (1-\delta)^{b/3},$$

and this is less than $1/(2b^2)$ if $b$ is large enough. This prove the claim, that $\omega$ deviates from the uniform distribution on the sub-squares by a fixed amount.

The rest is standard. The deviation from uniformity implies that the entropy

$$h(\mu) = -\sum_{k=1}^{b^2} \omega(Q_j) \log_b \omega(Q_j),$$

is strictly less than 2, the maximum that occurs when every square has equal measure. The strong law of large numbers and Billingsley's lemma now imply that $\omega$ has dimension strictly less than 2, with a bound that depends on $b$, but not on $\Omega$. $\square$

Jean Bourgain proved this holds for general domains in higher dimensions, with a $\delta$ that depends only on the dimension. We shall see later that the bound $\dim(\omega) \leq 1$ holds in the plane.

A gap in Bourgain's proof was recently found (and fixed) by Mathew Badger and Alyssa Genschaw, "Hausdorff dimension of caloric measure", preprint 2021.