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Conformal maps and conformal invariants

This is a book about fractals that all have some sort of invariance under conformal maps. A fundamental tool for understanding such sets are conformal invariants, i.e., numerical values that can be associated to a certain geometric configurations and that remain unchanged (or at least change in predictable ways) under the application of conformal or holomorphic maps. There are three conformal invariants that will be particularly important through the book: extremal length, harmonic measure and hyperbolic distance. Of these, extremal length is the most important because it can be defined in many situations and estimated by direct geometric arguments. The other two are defined on the disk and then transferred to other domains by a conformal map. In this chapter, we introduce extremal length, hyperboli distance and harmonic measure, and derive a famous estimate for the latter, due to Arne Beurling, using the former. As a reward for our efforts we will deduce a growth bound, due to Harry Kesten, for diffusion limited aggregation (DLA), one of the most appealing, and most challenging, conformal fractals.

1.1 Extremal length

Our first conformal invariant is extremal length. Consider a positive function ρ on a domain Ω. We think of ρ as analogous to $|f'|$ where f is a conformal map on Ω. Just as the image area of a set E can be computed by integrating $\int_E |f'|^2 dxdy$, we can use ρ to define areas by $\int_E \rho^2 dxdy$. Similarly, just as we can define $\ell(f(\gamma)) = \int_\gamma |f'(z)| ds$, we can define the ρ-length of a curve γ by $\int_\gamma \rho ds$. For this to make sense, we need γ to be locally rectifiable (so the arclength measure ds is defined) and it is convenient to assume that ρ is Borel (so that its restriction to any curve γ is also Borel and hence measurable for length measure on γ).

Suppose Γ is a family of locally rectifiable paths in a planar domain Ω and ρ is a non-negative Borel function on Ω. We say ρ is **admissible** for Γ if

$$\ell(\Gamma) = \ell_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \int_\gamma \rho ds \geq 1.$$

In this case we write $\rho \in \mathscr{A}(\Gamma)$. We define the **modulus** of the path family Γ as

$$\mathrm{Mod}(\Gamma) = \inf_\rho \int_M \rho^2 dxdy,$$

where the infimum is over all admissible ρ for Γ. The **extremal length** of Γ is defined as

$$\lambda(\Gamma) = 1/M(\Gamma).$$

Note that if the path family Γ is contained in a domain Ω, then we need only consider metrics ρ are zero outside Ω. Otherwise, we can define a new (smaller) metric by setting $\rho = 0$ outside Ω; the new metric is still admissible, and a smaller integral than before. Therefore $M(\Gamma)$ can be computed as the infimum over metrics which are only nonzero inside Ω.

Modulus and extremal length satisfy several useful properties that we list as a series of lemmas.

Lemma 1.1.1 (Conformal invariance) *If Γ is a family of curves in a domain Ω and f is a one-to-one holomorphic mapping from Ω to Ω' then $M(\Gamma) = M(f(\Gamma))$.*

Proof This is just the change of variables formulas

$$\int_\gamma \rho \circ f |f'| ds = \int_{f(\gamma)} \rho ds,$$

$$\int_\Omega (\rho \circ f)^2 |f'|^2 dxdy = \int_{f(\Omega)} \rho dxdy.$$

These imply that if $\rho \in \mathscr{A}(f(\Gamma))$ then $|f'| \cdot \rho \circ f \in \mathscr{A}(f(\Gamma))$, and thus by taking the infimum over such metrics we get $M(f(\Gamma)) \leq M(\Gamma)$ Note that there might be admissible metrics for $f(\Gamma)$ that are not of this form, possibly giving a strictly small modulus. However, by switching the roles of Ω and Ω' and replacing f by f^{-1} we see equality does indeed hold. □

Lemma 1.1.2 (Monotonicity) *If Γ_0 and Γ_1 are path families such that every $\gamma \in \Gamma_0$ contains some curve in Γ_1 then $M(\Gamma_0) \leq M(\Gamma_1)$ and $\lambda(\Gamma_0) \geq \lambda(\Gamma_1)$.*

Proof The proof is immediate since $\mathscr{A}(\Gamma_0) \supset \mathscr{A}(\Gamma_1)$. □

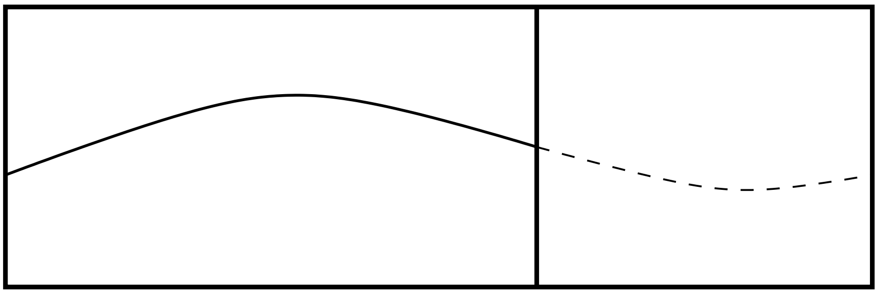


Figure 1.1.1 The Monotone rule: each curve of the first family contains a curve of the second family.

Lemma 1.1.3 (Grötsch Principle) *If Γ_0 and Γ_1 are families of curves in disjoint domains then $M(\Gamma_0 \cup \Gamma_1) = M(\Gamma_0) + M(\Gamma_1)$.*

Proof Suppose ρ_0 and ρ_1 are admissible for Γ_0 and Γ_1. Take $\rho = \rho_0$ and $\rho = \rho_1$ in their respective domains. Then it is easy to check that ρ is admissible for $\Gamma_0 \cup \Gamma_1$ and, since the domains are disjoint, $\int \rho^2 = \int \rho_1^2 + \int \rho_2^2$. Thus $M(\Gamma_0 \cup \Gamma_1) \leq M(\Gamma_0) + M(\Gamma_1)$. By restricting an admissible metric ρ to each domain, a similar argument proves the other direction. □

The Grötsch principle and the monotonicity combine to give

Corollary 1.1.4 (Parallel Rule) *Suppose Γ_0 and Γ_1 are path families in disjoint domains $\Omega_0, \Omega_1 \subset \Omega$ that connect disjoint sets E, F in $\partial\Omega$. If Γ is the path family connecting E and F in Ω, then*

$$M(\Gamma) \geq M(\Gamma_0) + M(\Gamma_1).$$

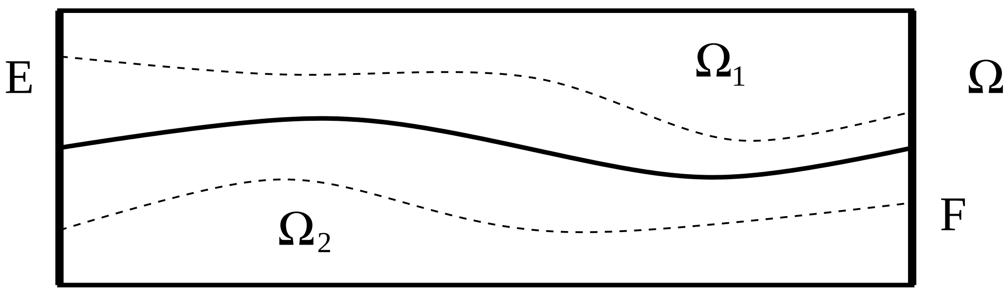


Figure 1.1.2 The Parallel Rule: curves connecting two boundary sets in the whole domain and in two disjoint subdomains.

Lemma 1.1.5 (Series Rule) *If Γ_0 and Γ_1 are families of curves in disjoint domains and every curve of $\mathscr{F}$ contains both a curve from both Γ_0 and Γ_1, then $\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1)$.*

Proof If $\rho_j \in \mathscr{A}(\Gamma_j)$ for $j = 0, 1$, then $\rho_t = (1-t)\rho_0 + t\rho_1$ is admissible for Γ. Since the domains are disjoint we may assume $\rho_0\rho_1 = 0$. Integrating ρ^2 then shows

$$M(\Gamma) \leq (1-t)^2 M(\Gamma_0) + t^2 M(\Gamma_1),$$

for each t. To find the optimal t set $a = M(\Gamma_1)$, $b = M(\Gamma_0)$, differentiate the right hand side above, and set it equal to zero

$$2at - 2b(1-t) = 0.$$

Solving gives $t = b/(a+b)$ and plugging this in above gives

$$M(\mathscr{F}) \leq t^2 a + (1-t^2)b = \frac{b^2 a a^2 b}{(a+b)^2}$$
$$= \frac{ab(a+b)}{(a+b)^2} = \frac{ab}{a+b} = \frac{1}{\frac{1}{a} + \frac{1}{b}}$$

or

$$\frac{1}{M(\Gamma)} \geq \frac{1}{M(\Gamma_0)} + \frac{1}{M(\Gamma_1)},$$

which, by definition, is the same as

$$\lambda(\Gamma) \geq \lambda(\Gamma_0) + \lambda(\Gamma_1). \quad \square$$

Next we actually compute the modulus of some path families. The fundamental example is to compute the modulus of the path family connecting opposite sides of a $a \times b$ rectangle; this serves as the model of almost all modulus estimates. So suppose $R = [0,b] \times [0,a]$ is a b wide and a high rectangle and Γ consists of all rectifiable curves in R with one endpoint on each of the sides of length a.

Lemma 1.1.6 $\mathrm{Mod}(\Gamma) = a/b$.

Proof Then each such curve has length at least b, so if we let ρ be the constant $1/b$ function on R we have

$$\int_\gamma \rho\, ds \geq 1,$$

for all $\gamma \in \Gamma$. Thus this metric is admissible and so

$$\mathrm{Mod}(\Gamma) \leq \iint_T \rho^2 dxdy = \frac{1}{b^2} ab = \frac{a}{b}.$$

To prove a lower bound, we use the well known Cauchy-Schwarz inequality:

$$(\int fgdx)^2 \le (\int f^2dx)(\int g^2dx).$$

To apply this, suppose ρ is an admissible metric on R for γ. Every horizontal segment in R connecting the two sides of length a is in Γ, so since γ is admissible,

$$\int_0^b \rho(x,y)dx \ge 1,$$

and so by Cauchy-Schwarz

$$1 \le \int_0^b (1\cdot\rho(x,y))dx \le \int_0^b 1^2 dx \cdot \int_0^b \rho^2(x,y)dx.$$

Now integrate with respect to y to get

$$a = \int_0^a 1dy \le b\int_0^a\int_0^b \rho^2(x,y)dxdy,$$

or

$$\frac{a}{b} \le \iint_R \rho^2 dxdy,$$

which implies $\mathrm{Mod}(\Gamma) \ge \frac{b}{a}$. Thus $\mathrm{Mod}(\Gamma) = \frac{b}{a}$. □

Another useful computation is the modulus of the family of path connecting the inner and out boundaries of the annulus $A = \{z : r < |z| < R\}$.

Lemma 1.1.7 *If $A = \{z : r < |z| < R\}$ then the modulus of the path family connecting the two boundary components is $2\pi/\log\frac{R}{r}$. More generally, if Γ is the family of paths connecting $r\mathbb{T}$ to a set $E \subset R\mathbb{T}$, then $M(\Gamma) \ge |E|/\log\frac{R}{r}$.*

Proof By conformal invariance, we can rescale and assume $r = 1$. Suppose ρ is admissible for Γ. Then for each $z \in E \subset \mathbb{T}$,

$$1 \le (\int_1^R \rho ds)^2 \le (\int_1^R \frac{ds}{s})(\int_1^R \rho^2 sds) = \log R\int_1^R \rho^2 sds$$

and hence we get

$$\int_0^{2\pi}\int_1^R \rho^2 sdsd\theta \ge \int_E\int_1^R \rho^2 sdsd\theta \ge |E|\int_1^R \rho^2 sds \ge \frac{|E|}{\log R}.$$

When $E = \mathbb{T}$ we prove the other direction by taking $\rho = (s\log R)^{-1}$. This is an admissible metric and

$$\mathrm{Mod}(\Gamma) \le \int_0^{2\pi}\int_1^R \rho^2 sdsd\theta = \frac{2\pi}{(\log R)^2}\int_1^R \frac{1}{s}ds = \frac{2\pi}{\log R}. \quad \square$$

Given a Jordan domain Ω and two disjoint closed sets $E, F \subset \partial\Omega$, the **extremal distance** between E and F (in Ω) is the extremal length of the path family in Ω connecting E to F (paths in Ω that have one endpoint in E and one endpoint in F). The series rule is a sort of "reverse triangle inequality" for extremal distance. See Figure 1.1.3.

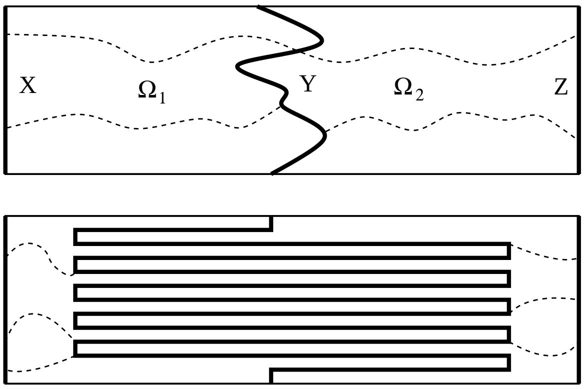


Figure 1.1.3 The series rule says that the extremal distance from X to Z in the rectangle is greater than the sum the extremal distance from X to Y in Ω_1 plus the extremal distance from Y to Z in Ω_2. The bottom figure show a more extreme case where the extremal distance between opposite sides of the rectangle is much larger than either of the other two terms.

Extremal distance can be particularly useful when both E and F are connected. In this case, their complement in $\partial\Omega$ also consists of two arcs, and the extremal distance between these is the reciprocal of the extremal distance between E and F. This holds because of conformal invariance, the fact that it is true for rectangles and an applications of the Riemann mapping theorem (we can always map Ω to a rectangle, so that E and F go to opposite sides (See Exercise 1.1).

Obtaining an upper bound for the modulus of a path family usually involves choosing a metric; every metric gives an upper bound. Giving a lower bound usually involves a Cauchy-Schwarz type argument, which can be harder to do in general cases. However, in the special case of extremal distance between arcs $E, F \subset \partial\Omega$, a lower bound for the modulus can also be computed by giving a upper bound for the reciprocal separating family. Thus estimates of both types can be given by producing metrics (for different families) and this is often the easiest thing to do.

If γ is a path in the plane let $\bar{\gamma}$ be its reflection across the real line and let

$$\gamma_u = \gamma \cap \mathbb{H}_u, \quad \gamma_\ell = \gamma \cap \mathbb{H}_l, \quad \gamma_+ = \gamma_u \cup \overline{\gamma_\ell},$$

where $\mathbb{H}_u = \{x+iy : y > 0\}$, $\mathbb{H}_l = \{x+iy : y < 0\}$ denote the upper and lower half-planes. For a path family Γ, define $\overline{\Gamma} = \{\bar{\gamma} : \gamma \in \Gamma\}$ and $\Gamma_+ = \{\gamma_+ : \gamma \in \Gamma\}$.

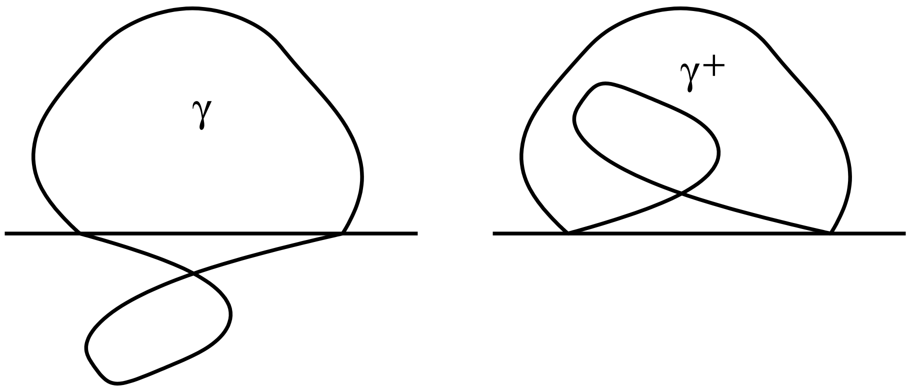


Figure 1.1.4 The curves γ and γ_+

Lemma 1.1.8 (Symmetry Rule) *If* $\Gamma = \overline{\Gamma}$ *then* $M(\Gamma) = 2M(\Gamma_+)$.

Proof We start by proving $M(\Gamma) \leq 2M(\Gamma_+)$. Given a metric ρ admissible for γ_+, define $\sigma(z) = \max(\rho(z), \rho(\bar{z}))$. Then for any $\gamma \in \Gamma$,

$$\begin{aligned}
\int_\gamma \sigma ds &= \int_{\gamma_u} \sigma(z) ds + \int_{\gamma_\ell} \sigma(z) ds \\
&\geq \int_{\gamma_u} \rho(z) ds + \int_{\gamma_\ell} \rho(\bar{z}) ds \\
&= \int_{\gamma_u} \rho(z) ds + \int_{\overline{\gamma_\ell}} \rho(z) ds \\
&\geq \int_{\gamma_+} \rho ds \\
&\geq \inf_{\gamma \in \Gamma} \int_\gamma \rho ds.
\end{aligned}$$

Thus if ρ admissible for Γ_+, then σ is admissible for Γ. Since $\max(a,b)^2 \leq a^2 + b^2$, integrating gives

$$M(\Gamma) \leq \int \sigma^2 dxdy \leq \int \rho^2(z) dxdy + \int \rho^2(\bar{z}) dxdy \leq 2 \int \rho^2(z) dxdy.$$

Taking the infimum over admissible ρ's for Γ_+ makes the right hand side equal to $2M(\Gamma_+)$, proving $\mathrm{Mod}(\Gamma) \leq 2\mathrm{Mod}(\Gamma_+)$.

For the other direction, given ρ define $\sigma(z) = \rho(z) + \rho(\bar{z})$ for $z \in \mathbb{H}_u$ and $\sigma = 0$ if $z \in \mathbb{H}_l$. Then

$$\begin{aligned}\int_{\gamma_+} \sigma ds &= \int_{\gamma_+} \rho(z) + \rho(\bar{z}) ds \\ &= \int_{\gamma_u} \rho(z) ds + \int_{\gamma_u} \rho(\bar{z}) ds + \int_{\gamma_ell} \rho(z) + \int_{\gamma_\ell} \rho(\bar{z}) ds \\ &= \int_{\gamma} \rho(z) ds + \int_{\overline{\gamma}} \rho(\bar{z}) ds \\ &= 2 \inf_{\rho} \int_{\gamma} \rho ds.\end{aligned}$$

Thus if ρ is admissible for Γ, $\frac{1}{2}\sigma$ is admissible for Γ_+. Since $(a+b)^2 \leq 2(a^2 + b^2)$, we get

$$\begin{aligned}M(\Gamma_+) &\leq \int (\frac{1}{2}\sigma)^2 dxdy \\ &= \frac{1}{4} \int_{\mathbb{H}_u} (\rho(z) + \rho(\bar{z}))^2 dxdy \\ &\leq \frac{1}{2} \int_{\mathbb{H}_u} \rho^2(z) dxdy + \int_{\mathbb{H}_u} \rho^2(\bar{z}) dxdy \\ &= \frac{1}{2} \int \rho^2 dxdy.\end{aligned}$$

Taking the infimum over all admissible ρ's for Γ gives $\frac{1}{2}M(\Gamma)$ on the right hand side, proving the lemma. □

Lemma 1.1.9 *Let $\mathbb{D}^* = \{z : |z| > 1\}$ and $\Omega_0 = \mathbb{D}^* \setminus [R, \infty)$ for some $R > 1$. Let $\Omega = \mathbb{D}^* \setminus K$, where K is a closed, unbounded, connected set in $\mathbb{D}^*$ which contains the point $\{R\}$. Let Γ_0, Γ denote the path families in these domains with separate the two boundary components. Then $M(\Gamma_0) \leq M(\Gamma)$.*

Proof We use the symmetry principle we just proved. The family Γ_0 is clearly symmetric (i.e., $\Gamma = \overline{\Gamma}$, so $M(\Gamma_0^+) = \frac{1}{2}M(\Gamma_0)$. The family Γ may not be symmetric, but we can replace it by a larger family that is. Let Γ_R be the collection of rectifiable curves in $\mathbb{D}^* \setminus \{R\}$ which have zero winding number around $\{R\}$, but non-zero winding number around 0. Clearly $\Gamma \subset \Gamma_R$ and Γ_R is symmetric so $M(\Gamma) \geq M(\Gamma_R) = 2M(\Gamma_R^+)$. Thus all we have to do is show $M(\Gamma_R^+) = M(\Gamma_0^+)$. We will actually show $\Gamma_R^+ = \Gamma_0^+$. Since $\Gamma_0 \subset \Gamma_R$ is obvious, we need only show $\Gamma_R^+ \subset \Gamma_0^+$.

Suppose $\gamma \in \Gamma_R$. Since γ has non-zero winding around 0 it must cross both

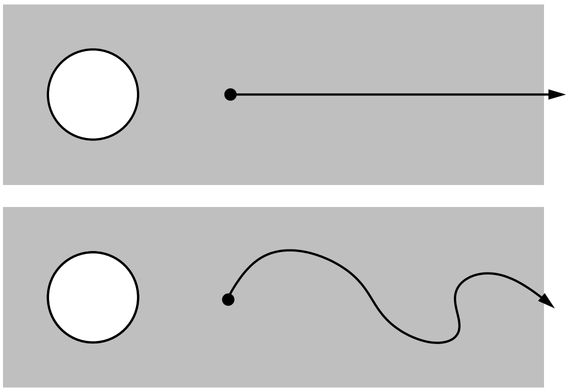


Figure 1.1.5 The topological annulus on top has smaller modulus than any other annulus formed by connecting R to ∞.

the negative and positive real axes. If it never crossed $(0,R)$ then the winding around 0 and R would be the same, which false, so γ must cross$(0,R)$ as well. Choose points $z_- \in \gamma \cap (-\infty,0)$ and $z_+ \in \gamma \cap (0,R)$. These points divide γ into two subarcs γ_1 and γ_2. Then $\gamma_+ = (\gamma_1)_+ \cup (\gamma_2)_+$. But if we reflect $(\gamma_2)_+$ into the lower half-plane and join it to $(\gamma_1)_+$ it forms a closed curve γ_0 that is in Γ_0 and $(\gamma_0)_+ = \gamma_+$. Thus $\gamma_+ \in (\Gamma_0)_+$, as desired. $\square$

Let $\Omega_{\varepsilon,R} = \{z : |z| > \varepsilon\} \setminus [R,\infty)$. Note that $\Omega_{1,R}$ is the domain considered in the previous lemma (e.g., see the top of Figure 1.1.5). We can estimate the moduli of these domains using the Koebe map

$$k(z) = \frac{z}{(1+z)^2} = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 - \dots,$$

which conformal maps the unit disk to $\mathbb{R}^2 \setminus [\frac{1}{4},\infty)$ and satisfies $k(0) = 0$, $k'(0) = 1$. Then $k^{-1}(\frac{1}{4R}z)$ maps $\Omega_{\varepsilon,R}$ conformally to an annular domain in the disk whose outer boundary is the unit circle and whose inner boundary is trapped between the circle of radius $\frac{\varepsilon}{4R}(1 \pm O(\frac{\varepsilon}{R}))$. Thus the modulus of $\Omega_{\varepsilon,R}$ is

$$2\pi \log \frac{4R}{\varepsilon} + O(\frac{\varepsilon}{R}). \tag{1.1.1}$$

Next we prove the Koebe $\frac{1}{4}$-theorem for conformal maps. The standard proof of Koebe's $\frac{1}{4}$-theorem uses Green's theorem to estimate the power series coefficients of conformal map (proving the Bieberbach conjecture for the second coefficient). However here we will present a proof, due to Mateljevic [**?**], that uses the symmetry property of extremal length.

Theorem 1.1.10 (The Koebe $\frac{1}{4}$ Theorem) *Suppose f is holomorphic, 1-1 on $\mathbb{D}$ and $f(0) = 0$, $f'(0) = 1$. Then $D(0, \frac{1}{4}) \subset f(\mathbb{D})$.*

Proof Recall that the modulus of a doubly connected domain is the modulus of the path family that separates the two boundary components (and is equal to the extremal distance between the boundary components). Let $R = \text{dist}(0, \partial f(\mathbb{D}))$. Let $A_{\varepsilon,r} = \{z : \varepsilon < |z| < r\}$ and note that by conformal invariance

$$2\pi \log \frac{1}{\varepsilon} = M(A_{\varepsilon,1}) = M(f(A_{\varepsilon,1})).$$

Let $\delta = \min_{|z|=\varepsilon} |f(z)|$. Since $f'(0) = 1$, we have $\delta = \varepsilon + O(\varepsilon^2)$. Note that $f(A_{\varepsilon,1}) \subset f(\mathbb{D}) \setminus D(0, \delta)$, so

$$M(f(A_{\varepsilon,1})) \leq M(f(\mathbb{D}) \setminus D(0, \delta)).$$

By Lemma 1.1.9 and Equation (1.1.1),

$$M(f(\mathbb{D}) \setminus D(0, \delta)) \leq M(\Omega_{\delta,R}) = 2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}).$$

Putting these together gives

$$2\pi \log \frac{4R}{\delta} + O(\frac{\delta}{R}) \geq 2\pi \log \frac{1}{\varepsilon},$$

or

$$\log 4R - \log(\varepsilon + O(\varepsilon^2)) + O(\frac{\varepsilon}{R}) \geq -\log \varepsilon,$$

and hence

$$\log 4R \geq -O(\frac{\varepsilon}{R}) + \log(1 + O(\varepsilon)).$$

Taking $\varepsilon \to 0$ shows $\log 4R \geq 0$, or $R \geq \frac{1}{4}$. □

1.2 Logarithmic capacity

Logarithmic capacity associates a non-negative number to each Borel subset of the unit circle. Applying a Möbius transformation can change this value, so it is not a conformal invariant, but it will act as an intermediate between extremal and harmonic measure (a conformal invariant that will be defined later).

Suppose μ is a positive, finite Borel measure on $\mathbb{C}$ and define its potential function as

$$U_\mu(z) = \int \log \frac{2}{|z - w|} d\mu(w), z \in \mathbb{C}.$$

and its energy integral by

$$I(\mu) = \iint \log \frac{2}{|z-w|} d\mu(z) d\mu(w) = \int U_\mu(z) d\mu(z).$$

We put the "2" in the numerator so that the integrand is non-negative when $z, w \in \mathbb{T}$, however, this is a non-standard usage.

Lemma 1.2.1 *U_μ is lower semi-continuous, i.e.,*

$$\liminf_{z \to z_0} U_\mu(z) \geq U_\mu(z_0).$$

Proof Fatou's lemma. □

Recall that $\mu_n \to \mu$ weak-* if $\int f d\mu_n \to \int f d\mu$ for every continuous function f of compact support.

Lemma 1.2.2 *If $\{\mu_n\}$ are positive measures and $\mu_n \to \mu$ weak*, then* $\liminf_n U_{\mu_n}(z) \geq U_\mu(z)$.

Proof If we replace $\varphi = \log \frac{2}{|z-w|}$ by the continuous kernel $\varphi_r = \max(r, \varphi)$ in the definition of U to get U^r, then weak convergence implies

$$\lim_n U^r_{\mu_n}(z) \nearrow U^r_\mu(z).$$

Moreover, the convergence is increasing since the measures positive. So for any $\varepsilon > 0$ we can choose N so that $n > N$ implies

$$U^r_{\mu_n}(z) \geq U^r_\mu(z) - \varepsilon.$$

As $r \to \infty$ $U^r \to U$ (by the monotone convergence theorem), so for r large enough and $n > N$ we have

$$U_{\mu_n}(z) \geq U^r_{\mu_n}(z) \geq U_\mu(z) - 2\varepsilon.$$

which proves the result. □

Lemma 1.2.3 *If $\mu_n \to \mu$ weak*, then* $\liminf_n I(\mu_n) \geq I(\mu)$.

Proof The proof is almost the same as for the previous lemma, except that we have to know that if $\{\mu_n\}$ converges weak*, then so does the product measure $\mu_n \times \mu_n$. However, weak convergence of $\{\mu_n\}$ implies convergence of integrals of the form

$$\iint f(x) g(y) d\mu_n(x) d\mu_n(y).$$

and Stone-Weierstrass theorem implies that the finite sums of such product functions are dense in all continuous function on the product space. Since weak-* convergent sequences are bounded, the product measures $\mu_n \times \mu_n$ also have uniformly bounded masses, and hence convergence on a dense set of continuous functions of compact support implies convergence on all continuous functions of compact support. This, together with the fact that weak* convergent sequences are bounded ([**?**]), implies that $\mu_n \times \mu_n$ converges weak*. □

Suppose E is Borel and μ is a positive measure that has its closed support inside E. We say μ is admissible for E if $U_\mu \leq 1$ on E and we define the **logarithmic capacity** of E as

$$\text{cap}(E) = \sup\{\|\mu\| : \mu \text{ is admissible for } E\}$$

and we write $\mu \in \mathscr{A}(E)$. We define the **outer capacity** (or exterior capacity) as

$$\text{cap}^*(E) = \inf\{\text{cap}(V) : E \subset V, V \text{open}\}.$$

We say that a set E is **capacitable** if $\text{cap}(E) = \text{cap}^*(E)$.

The logarithmic kernel can be replaced by other functions, e.g., $|z-w|^{-\alpha}$, and there is a different capacity associated to each one. To be precise, we should denote logarithmic capacity as $\text{cap}_{\log}$ or logcap, but to simplify notation we simply use “cap” and will often refer to logarithmic capacity as just “capacity”. Since we do not use any other capacities in these notes, this abuse should not cause confusion.

WARNING: The logarithmic capacity that we have defined is **NOT** the same as is used in other texts such as Garnett and Marshall’s book [**?**], but is related to what they call the Robin’s constant of E, denoted $\gamma(E)$. The exact relationship is $\gamma(E) = \frac{1}{\text{cap}(E)} - \log 2$. Garnett and Marshall [**?**] define the logarithmic capacity of E as $\exp(-\gamma(E))$. The reason for doing this is that the logarithmic kernel $\log \frac{1}{|z-w|}$ takes both positive and negative values in the plane, so the potential functions for general measures and the Robin’s constant for general sets need not be non-negative. Exponentiating takes care of this. Since we are only interested in computing the capacity of subsets of the circle, taking the extra “2” in the logarithm gave us a non-negative kernel on the unit circle, and we defined a corresponding capacity in the usual way. Since the kernel is the logarithm, we feel justified in calling the corresponding capacity the logarithmic capacity, despite the divergence with usual usage.

POSSIBLE ALTERNATES : Robin’s capacity, conformal capacity, circular capacity.

Lemma 1.2.4 *Compact sets are capacitable.*

Proof Since $\text{cap}(E) \leq \text{cap}^*(E)$ is obvious, we only have to prove the opposite direction. Set $U_n = \{z : \text{dist}(z,E) < 1/n\}$ and choose a measure μ_n supported in U_n with $\|\mu_n\| \geq \text{cap}(U_n) - 1/n$. Let μ be a weak accumulation point of $\{\mu_n\}$ and note

$$U_\mu(z) = \int \log \frac{2}{|z-w|} d\mu(w) \leq \int \log \frac{2}{|z-w|} d\mu_n(w) \leq 1$$

so μ is admissible in the definition of cap(E). Thus

$$\text{cap}(E) \geq \limsup \|\mu_n\| = \lim \text{cap}(U_n) = \lim \text{cap}(U_n) = \text{cap}^*(E).$$

□

It is also true that all Borel sets are capacitable. Indeed, this holds for all analytic sets (i.e., continuous images of complete separable topological spaces). See Appendix B of [**?**].

It is clear from the definitions that logarithmic capacity is monotone

$$E \subset F \quad \Rightarrow \quad \text{cap}(E) \leq \text{cap}(F). \tag{1.2.1}$$

and satisfies the regularity condition

$$\text{cap}(E) = \sup\{\text{cap}(K) : K \subset E, K\text{compact}\}. \tag{1.2.2}$$

Lemma 1.2.5 (Sub-additive) *For any sets* $\{E_n\}$*,*

$$\text{cap}(\cup E_n) \leq \sum \text{cap}(E_n). \tag{1.2.3}$$

Proof We can write any $\mu = \sum \mu_n$ as a sum of mutually singular measures so that μ_n gives full mass to E_n. We can then restrict each μ_n to a compact subset K_n of E_n so that $\mu_n(K_n) \geq (1-\varepsilon)\mu(E_n)$. These restrictions are admissible for each E_n and hence

$$\sum \text{cap}(E_n) \geq \sum \mu_n(K_n) \geq (1-\varepsilon) \sum \mu_n(E_n) = (1-\varepsilon)\|\mu\|.$$

Taking $\varepsilon \to 0$ proves the result. □

Corollary 1.2.6 *A countable union of zero capacity sets has zero capacity.*

Corollary 1.2.7 *Outer capacity is also sub-additive.*

Proof Given sets $\{E_n\}$ choose open sets $V_n \supset E_n$ so that $\text{cap}(V_n) \leq \text{cap}^*(E_n) + \varepsilon 2^{-n}$. By the sub-additivity of capacity

$$\text{cap}^*(\cup E_n) \leq \text{cap}(\cup V_n) \leq \sum \text{cap}(V_n) \leq \varepsilon + \sum \text{cap}^*(E_n).$$

Taking $\varepsilon \to$ proves the result. □

Although capacity informally "measures" the size of a set, it is not additive, and hence not a measure. See Exercise 1.4.

Lemma 1.2.8 *If E is compact, there exists an admissible μ that attains the maximum mass in the definition of capacity and $U_\mu(z) = 1$ everywhere on E, except possible a set of capacity zero.*

Proof Let μ_n be a sequence of measures on E so that $\|\mu_n\| \to \mathrm{cap}(E)$ and $U_n = U_{\mu_n}$ is bounded above by 1 on E (such a sequence exists by the definition of logarithmic capacity). By Lemma 1.2.2, U_μ is also bounded above by 1. Also, by a standard property of weak* convergence $\|\mu\| \le \liminf_n \|\mu_n\| = \mathrm{cap}(E)$ ([**?**]), and by Lemma 1.2.3,

$$I(\mu) \le \liminf_n I(\mu_n) \le \liminf_n \|\mu_n\| = \mathrm{cap}(E),$$

so we must have $I(\mu) = \mathrm{cap}(E)$.

First we claim that $U_\mu \ge 1$ except possibly on a set of zero capacity. Otherwise let $T \subset E$ be a set of positive capacity on which $U_\mu < 1 - \varepsilon$ and let σ be a non-zero, positive measure on T which potential bounded by 1. Define

$$\mu_t = (1-t)\mu + t\sigma.$$

This is a measure on E so that

$$\begin{aligned} I(\mu_t) &\le \int \log \frac{1}{|z-w|}((1-t)d\mu + td\sigma)((1-t)d\mu + td\sigma) \\ &\le (1-t)^2 I(\mu) + 2t \int U_\mu d\sigma + t^2 I(\sigma) \\ &\le I(\mu) - 2tI(\mu) + 2t \int U_\mu d\sigma + O(t^2) \\ &\le I(\mu) - 2tI(\mu) + 2t(1-\varepsilon)\|\sigma\| + O(t^2) \\ &< I(\mu), \end{aligned}$$

if $t > 0$ is small enough. This contradicts minimality of μ.

Next we show that $U_\mu \le 1$ everywhere on the closed support of μ. By the previous step we know $U_\mu \ge 1$ except on capacity zero, hence except on a set of μ-measure zero. If there is a point z in the support of μ such that $U_\mu(z) > 1$, then by lower semi-continuity of potentials, U_μ is $> 1 + \varepsilon$ on some neighborhood of z and this neighborhood has positive μ measure (since z is in the support of μ) and thus $I(\mu) = \int U_\mu d\mu > \|\mu\|$, a contradiction. □

The following makes a connection between logarithmic capacity and extremal length. Eventually, this will become a connection between extremal length and harmonic measure.

If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with 0 in the interior of K. Let K^* be the reflection of K across $\mathbb{T}$. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals, let Ω be the connected component of $\mathbb{C} \setminus (E \cup$

$K \cup K^*$) that has E on its boundary. Let $h(z)$ be the harmonic function in Ω with boundary values 0 on K and K^* and boundary value 1 on E. By the usual theory of the Dirichlet problem (e.g. [**?**]), all boundary points are regular (since all boundary components are non-degenerate continua) and hence h extends continuously to the boundary with the correct boundary values. Moreover, h is symmetric with respect to $\mathbb{T}$, and this implies its normal derivative on $\mathbb{T}\setminus E$ is 0. Let $D(h) = \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy$.

Lemma 1.2.9 *With notation as above, $M(\Gamma_E) = D(h)$.*

Proof Clearly $|\nabla h|$ is an admissible metric for Γ_E, so

$$M(\Gamma_E) \le D(h) \equiv \int_{\mathbb{D}\setminus K} |\nabla h|^2 dxdy.$$

Thus we need only show the other direction.

Green's theorem states that

$$\iint_{\Omega} u\Delta v - v\Delta u dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds. \tag{1.2.4}$$

Using this and the fact that $h = 1$ on E, we have

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = -\int_{\mathbb{T}} \frac{\partial h}{\partial n} ds = -\int_{E} \frac{\partial h}{\partial n} ds = -\int_{E} h\frac{\partial h}{\partial n} ds.$$

and

$$\begin{aligned}\int_{\partial K} \frac{\partial h}{\partial n} ds &= -\frac{1}{2}\int_{E} \frac{\partial (h^2)}{\partial n} ds \\ &= \frac{1}{2}\int_{\mathbb{T}\setminus E} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\partial K} \frac{\partial (h^2)}{\partial n} ds + \frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.\end{aligned}$$

The first term is zero because h has normal derivative zero on $\mathbb{T}\setminus E$, and hence the same is true for h^2. The second term is zero because h is zero on K and so $\frac{\partial (h^2)}{\partial n} h^2 = 2h\frac{\partial h}{\partial n} = 0$. To evaluate the third term, we use the identity

$$\begin{aligned}\Delta(h^2) &= 2h_x \cdot h_x + 2h \cdot h_{xx} + 2h_y \cdot h_y + 2h \cdot h_{yy} \\ &= 2h\Delta h + 2\nabla h \cdot \nabla h \\ &= 2h \cdot 0 + 2|\nabla h|^2 \\ &= 2|\nabla h|^2,\end{aligned}$$

to deduce

$$\frac{1}{2}\int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Therefore,

$$\int_{\partial K} \frac{\partial h}{\partial n} ds = \int_{\mathbb{D}\setminus K} \Delta(h^2) dxdy.$$

Thus the tangential derivative of h's harmonic conjugate has integral $D(h)$ around ∂K and therefore $2\pi h/D(h)$ is the real part of a holomorphic function g on $\mathbb{D}\setminus K$. Then $f = \exp(g)$ maps $\mathbb{D}\setminus K$ into the annulus

$$A = \{z : 1 < |z| < \exp(2\pi/D(h))\}$$

with the components of E mapping to arcs of the outer circle and the components of $\mathbb{T}\setminus E$ mapping to radial slits. The path family Γ_E maps to the path family connecting the inner and outer circles without hitting the radial slits, and our earlier computations show the modulus of this family is $D(h)$. □

Theorem 1.2.10 (Pfluger's theorem) *If $K \subset \mathbb{D}$ is a compact connected set with smooth boundary with* 0 *in the interior of K. Then there are constants C_1, C_2 so that following holds. For any $E \subset \mathbb{T}$ that is a finite union of closed intervals,*

$$\frac{1}{\text{cap}(E)} + C_1 \le \pi\lambda(\Gamma_E)) \le \frac{1}{\text{cap}(E)} + C_2,$$

where Γ_E is the path family connecting K to E. The constants C_1, C_2 can be chosen to depend only on $0 < r < R < 1$ if $\partial K \subset \{r \le |z| \le R\}$.

Proof Using Lemma 1.2.9, we only have to relate $D(h)$ to the logarithmic capacity of E. Let μ be the equilibrium probability measure for E. We know in general that $U_\mu = \gamma$ where $\gamma = 1/\text{cap}(E)$ almost everywhere on E (since sets of zero capacity have zero measure) and is continuous off E, but since U_μ is harmonic in $\mathbb{D}$ and equals the Poisson integral of its boundary values, we can deduce $U_\mu = \gamma$ everywhere on E. Let $v(z) = \frac{1}{2}(U_\mu(z) + U_\mu(1/\bar{z})$. Then since ∂K has positive distance from 0, there are constants C_1, C_2 so that

$$v + C_1 \le 0, \qquad v + C_2 \ge 0,$$

on ∂K. Note that $C_1 \ge -\gamma$ by the maximum principle and $C_2 \ge 0$ trivially. Moreover, since μ is a probability measure supported on the unit circle, given $0 < r < R < 1$, U_μ is uniformly bounded on both the annulus $\{r \le |z| \le R\}$ and its reflection across the unit circle, since these both have bounded, but positive distance from the unit circle. This proves that C_1, C_2 can be chosen to depend on only these numbers, as claimed in the final statement of the theorem.

The following inequalities are easy to check on K, K^* and E,

$$\frac{v(z) + C_1}{\gamma + C_1} \le h(z) \le \frac{v(z) + C_2}{\gamma + C_2}.$$

and hence hold on Ω by the maximum principle. Since we have equality on E, we also get

$$\frac{\partial}{\partial n}\left(\frac{v(z)+C_1}{\gamma+C_1}\right) \le \frac{\partial h}{\partial n} \le \frac{\partial}{\partial n}\left(\frac{v(z)+C_2}{\gamma+C_2}\right)$$

for $z \in E$. When we integrate over E, the middle term is $-D(h)$ (we computed this above) and by Green's theorem

$$\begin{aligned} -\int_E \frac{\partial}{\partial n}\frac{v(z)+C_1}{\gamma+C_1}ds &= \frac{1}{\gamma+C_1}\int_{\mathbb{D}} \Delta(v)dxdy \\ &= \frac{\pi}{\gamma+C_1} \end{aligned}$$

because v is harmonic except for a $\frac{1}{2}\log\frac{1}{|z|}$ pole at the origin. A similar computation holds for the other term and hence

$$\frac{\pi}{\gamma+C_1} \le D(h) = M(\Gamma_E) \le \frac{\pi}{\gamma+C_2},$$

since $D(h) = \int_E \frac{\partial h}{\partial n} ds$. Hence

$$\gamma+C_1 \le \pi\lambda(\Gamma_E) \le \gamma+C_2.$$

This completes the proof of Pfluger's theorem for finite unions of intervals. □

Next we prove Pfluger's theorem for all compact subsets of $\mathbb{T}$. First we need a continuity property of extremal length. Recall that an extended real-valued function is lower semi-continuous if all sets of the form $\{f > \alpha\}$ are open.

Lemma 1.2.11 *Suppose $E \cap \mathbb{T}$ is compact, $K \subset \mathbb{D}$ is compact, connected and contains the origin, and Γ_E is the path family connecting K and E in $\mathbb{D}\setminus K$. Fix an admissible metric ρ for Γ_E and for each $z \in \mathbb{T}$, define $f(z) = \inf\int_\gamma \rho ds$ where the infimum is over all paths in Γ_E that connect K to z. Then f is lower semi-continuous.*

Proof Suppose $z_0 \in \mathbb{T}$ and use Cauchy-Schwarz to get

$$\begin{aligned} \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r}\rho ds\right)^2 dr &\le \int_{2^{-n-1}}^{2^{-n}} \left(\int_{|z-z_0|=r}\rho^2 ds\right) dr \left(\int_{|z-z_0|=r} 1 ds\right) dr \\ &\le \int_{2^{-n-1}}^{2^{-n}} r\int_0^{2\pi} \rho^2 r d\theta dr \\ &\le \pi 2^{-n}\int_{2^{-n-1}<|z-z_0|<2^{-n}} \rho^2 dxdy \\ &= o(2^{-n}). \end{aligned}$$

Therefore we can choose circular cross-cuts $\{\gamma_n\} \subset \{z : 2^{-n-1} < |z-z_0| <$

$2^{-n}\}$ of $\mathbb{D}$ centered at z_0 and with ρ-length ε_n tending to 0. By taking s subsequence we may assume $\sum \varepsilon_n < \infty$. Now choose $z_n \to z_0$ with

$$f(z_n) \to \alpha \equiv \liminf_{z \to z_0} f(z).$$

We want to show that there is a path connecting K to z_0 whose ρ-length is as close to α as we wish. Passing to a subsequence we may assume z_n is separated from K by δ_n. Let c_n be the infimum of ρ-lengths of paths connecting γ_n and γ_{n+1}. By considering a path connecting K to z_n, we see that $\sum_1^n c_k \leq f(z_n)$, for all n and hence $\sum_1^\infty c_n \leq \alpha$.

Next choose $\varepsilon > 0$ and choose n so that we can connect K to z_n (and hence to γ_n) by a path of ρ-length less than $\alpha + \varepsilon$. We can then connect γ_n to z_0 by a infinite concatenation of arcs of γ_k, $k > n$ and paths connecting γ_k to γ_{k+1} that have total length $\sum_n^\infty (\varepsilon_n + c_n) = o(1)$. Thus K can be connected to z_0 by a path of ρ-length as close to α as we wish. □

Corollary 1.2.12 *Suppose $E \subset \mathbb{T}$ is compact and $\varepsilon > 0$. Then there is a finite collection of closed intervals F so that $E \subset F$ and*

$$\lambda(\Gamma_E) \leq \lambda(\Gamma_F) + \varepsilon,$$

where the path families are defined as above.

Proof Choose an admissible ρ so that $\int \rho^2 dxdy \leq M(\Gamma_E) + \varepsilon$. Set

$$r = \left(\frac{M(\Gamma_E) + \varepsilon}{M(\Gamma_E) + 2\varepsilon}\right)^{1/2}$$

By Lemma 1.2.11 $V = \{z \in \mathbb{T} : f(z) > r\}$ is open, and therefore we can choose a set F of the desired form inside V. Then ρ/r is admissible for Γ_F, so

$$M(\Gamma_F) \leq \int \left(\frac{\rho}{r}\right)^2 dxdy = \frac{M(\Gamma_E) + 2\varepsilon}{M(\Gamma_E) + \varepsilon} \int \rho^2 dxdy \leq M(\Gamma_E) + 2\varepsilon.$$

Thus an inequality in the opposite direction holds for extremal length. □

Corollary 1.2.13 *Pfluger's theorem holds for all compact sets in $\mathbb{T}$.*

Proof Suppose E is compact. Using Corollary 1.2.12 and Lemma 1.2.4 we can choose nested sets $E_n \searrow E$ that are finite unions of closed intervals and satisfy

$$\lambda(\mathscr{F}_{E_n}) \to \lambda(\mathscr{F}_E),$$

and

$$\text{cap}(E_n) \to \text{cap}(E).$$

Thus the inequalities in Pfluger's theorem extend to E. □

1.3 Hyperbolic distance

We start on the disk, and then extend to simply connected domains via the Riemann mapping theorem and to general planar domains via the uniformization theorem.

The **hyperbolic metric** on $\mathbb{D}$ is given by $d\rho(z) = |dz|/(1-|z|^2)$. This means that the hyperbolic length of a rectifiable curve γ in $\mathbb{D}$ is defined as

$$\ell_\rho(\gamma) = \int_\gamma \frac{|dz|}{1-|z|^2}, \tag{1.3.1}$$

and the hyperbolic distance between two points $z, w \in \mathbb{D}$ is the infimum of the lengths of paths connecting them (we shall see shortly that there is an explicit formula for this distance in terms of z and w). In many sources, there is a "2" in the numerator of (1.3.1), but we follow [**?**], where the definition is as given in (1.3.1). For most applications this makes no difference, but the reader is warned that some of our formulas may differ by a factor of 2 from the analogous formulas in some papers and books.

We define the **hyperbolic gradient** of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ as

$$D_H^H f(z) = |f'(z)| \frac{1-|z|^2}{1-|f(z)|^2}.$$

More generally, given a map f between metric spaces (X,d) and (Y,ρ) we define the gradient at a point z as

$$D_d^\rho f(z) = \limsup_{x \to z} \frac{\rho(f(z), f(x))}{d(x,z)}.$$

The use of the word "gradient" is not quite correct; a gradient is usually a vector indicating both the direction and magnitude of the greatest change in a function. We use the term in a sense more like the term "upper gradient" that occurs in metric measure theory to denote a function $\rho \geq 0$ that satisfies

$$|f(b) - f(a)| \leq \int_\gamma \rho\, ds,$$

for any curve γ connecting a and b. I hope that the slight abuse of the term will not be confusing.

In these notes, the most common metrics we will use are the usual Euclidean metric on $\mathbb{C}$, the spherical metric

$$\frac{ds}{1+|z|^2},$$

on the Riemann Sphere, S^2 and the hyperbolic metric on the disk or on some other hyperbolic planar domain. To simplify notation, we use E, S and H to denote whether we are taking a gradient with respect to Euclidean, spherical or hyperbolic metrics. For example if $f : U \to V$, the symbol $D_H^H f$ means that we are taking a gradient from the hyperbolic metric on U to the hyperbolic metric on V (assuming the domains are clear from context; otherwise we write D_U^V or $D_{\rho_U}^{\rho_V}$ if we need to be very precise.)

In this notation, the spherical derivative of a function, usually denoted

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2},$$

is written $D_E^S f(z)$ since it is a limit of quotients where the numerator is measured in the spherical metric and the denominator is measured in the Euclidean metric. Similarly D_H^S denotes a gradient measuring expansion from a hyperbolic to the spherical metric. This particular gradient is important in the theory of normal families (e.g., see Montel's theorem in [**?**]). Another variation we will use is $D_{\mathbb{D}}^E f$. If this is bounded on the disk, then f is a Lipschitz function from the hyperbolic metric on the disk to the Euclidean metric on the plane. Such functions are called Bloch functions.

A **linear fractional transformation** is a map of the form

$$z \to \frac{a+bx}{c+dz},$$

where $a,b,c,d \in \mathbb{C}$. These exactly the 1-to-1, holomorphic maps of the Riemann sphere to itself. Such maps are also called **Möbius transformations**.

Lemma 1.3.1 *Möbius transformations of $\mathbb{D}$ to itself are isometries of the hyperbolic metric.*

Proof When f is a Möbius transformation of the disk we have

$$f(z) = \frac{z-a}{1-\bar{a}z}, \qquad f'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus

$$\begin{aligned}
D_H^H f(z) &= \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|f(z)|^2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}\frac{1-|z|^2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2-|z-a|^2} = \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z)(1-a\bar{z})-(z-a)(\bar{z}-\bar{a})} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1-\bar{a}z-a\bar{z}+|az|^2)-(|z|^2-a\bar{z}-z\bar{a}+|a|^2)} \\
&= \frac{(1-|a|^2)(1-|z|^2)}{(1+|az|^2-|z|^2-|a|^2)} = 1.
\end{aligned}$$

Note that

$$\ell_\rho(f(\gamma)) \leq \int_\gamma D_H^H f(z) \frac{|dz|}{1-|z|^2}.$$

Thus Möbius transformations multiply hyperbolic length by at most one. Since the inverse also has this property, we see that Möbius transformation preserve hyperbolic length. □

The segment $(-1,1)$ is clearly a geodesic for the hyperbolic metric and since isometries take geodesics to geodesics, we see that geodesics for the hyperbolic metric are circles orthogonal to the boundary.

On the disk it is convenient to define the pseudo-hyperbolic metric

$$T(z,w) = |\frac{z-w}{1-\bar{w}z}|.$$

The hyperbolic metric between two points can then be expressed as

$$\rho(w,z) = \frac{1}{2}\log\frac{1+T(w,z)}{1-T(w,z)}. \tag{1.3.2}$$

On the upper half-plane the corresponding function is

$$T(z,w) = |\frac{z-w}{w-\bar{z}}|,$$

and ρ is related as before.

Lemma 1.3.2 (Schwarz's Lemma) *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$ then $|f'(0)| \leq 1$ with equality iff f is a rotation. Moreover, $|f(z)| \leq |z|$ for all $|z| < 1$, with equality for $z \neq 0$ iff f is a rotation.*

Proof Define $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$. This is a holomorphic function since if $f(z) = \sum a_n z^n$ then $a_0 = 0$ and so $g(z) = \sum a_n z^{n-1}$ has a convergent power series expansion. Since $\max_{|z|=r} |g(z)| \leq \frac{1}{r} \max_{|z|=r} |f| \leq \frac{1}{r}$. By the maximum principle $|g| \leq \frac{1}{r}$ on $\{|z| < r\}$. Taking $r \nearrow 1$ shows $|g| \leq 1$ on $\mathbb{D}$ and equality anywhere implies g is constant. Thus $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$ and equality implies f is a rotation. □

In terms of the hyperbolic metric this says that

$$\rho(f(0), f(z)) = \rho(0, f(z)) \leq \mathbb{H}_r(0,z),$$

which shows the hyperbolic distance from 0 to any point is non-increasing. For an arbitrary holomorphic self-map of the disk f and any point $w \in \mathbb{D}$ we can always choose Möbius transformations τ, σ so that $\tau(0) = w$ and $\sigma(f(w)) = 0$, so that $\sigma \circ f \circ \tau(0) = 0$. Since Möbius transformations are hyperbolic isometries, this shows

Corollary 1.3.3 *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic then $\rho(f(w), f(z)) \le \rho(w,z)$.*

Lemma 1.3.4 *If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.*

Proof We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\}_1^\infty \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E (e.g., see [**?**]). Thus

$$|f(z) - f_n(z_n) \le |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \le |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis. □

A planar domain Ω is called **hyperbolic** if $\mathbb{C} \setminus \Omega$ has at least two points.

Theorem 1.3.5 *Every hyperbolic plane domain Ω is holomorphically covered by $\mathbb{D}$ (i.e., there is a locally 1-to-1, holomorphic covering map from $\mathbb{D}$ to Ω).*

We will prove this in three steps: bounded domains, simply connected domains and finally the general case.

Uniformization for bounded domains If Ω is bounded, then by a translation and rescaling, we may assume $\Omega \subset \mathbb{D}$ and $0 \in \Omega$. We will define a sequence of domains $\{\Omega_n\}$ with $\Omega_0 = \Omega$ and covering maps $p_n : \Omega_n \to \Omega_{n-1}$ such that $p(0) = 0$. We will show that Ω_n contains hyperbolic disks centered at 0 of arbitrarily large radius and that the covering map $q_n = p_1 \circ \cdots \circ p_n : \Omega_n \to \Omega_0 = \Omega$ converges uniformly on compacta to a covering map $q : \mathbb{D} \to \Omega$.

If $\Omega_0 = \mathbb{D}$ we are done, since the identity map will work. In general assume that we have $q_n : \Omega_n \to \Omega_0$ and that there is a point $w \in \mathbb{D} \setminus \Omega_n$. Let τ and σ be Möbius transformations of the disk to itself so that $\tau(w) = 0$, choose a square root α of $\tau(0)$ and choose σ so $\sigma(\alpha) = 0$. Then $p_{n+1}(z) = \sigma(\sqrt{\tau(z)})$ and let Ω_{n+1} be the component of $U = p_{n+1}^{-1}(\Omega_n)$ that contains the origin (the set U will have one or two components; two if w is in a connected component of $\mathbb{D} \setminus \Omega_n$ that is compact in $\mathbb{D}$, and one otherwise). Since σ and τ are hyperbolic isometries and $\sqrt{z}$ expands the hyperbolic metric, we see that Ω_{n+1} contains a larger hyperbolic ball around 0 than Ω_n did.

More precisely, suppose $\mathrm{dist}(\partial\Omega_n, 0) < r < 1$ for all n. Since $f(z) = z^2$ maps the disk to itself, it strictly contracts the hyperbolic metric; a more explicit computation shows

$$D_H^H f(z) = |2z| \frac{1 - |z|^2}{1 - |z|^4} = \frac{2|z|}{1 + |z|^2} < 1.$$

Thus $g(z) = \sqrt{z}$ is locally an expansion of the hyperbolic metric, at least on a subdomain $W \subset \mathbb{D}$ where it has a well defined branch. For $z \neq 0$,

$$D_H^H g(z) = |\frac{1}{2\sqrt{z}}| \frac{1-|z|^2}{1-|z|} \geq \frac{1+|z|}{2\sqrt{z}}. \tag{1.3.3}$$

Then (1.3.3) says that

$$D_H^H p_n(0) = D_H^H \sqrt{z}(\tau(0)) > \frac{1+r}{2\sqrt{r}} > 1,$$

since $|\tau(0)| = |w| < r$. Hence $D_H^H q_n(0)$ increases by this much at every step. But $D_H^H q_n(0) \leq 1$, which is a contradiction. Thus $d_n \to 1$.

Thus $\{q_n\}$ is a sequence of uniformly bounded holomorphic functions on the disk. By Montel's theorem, there a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $q : \mathbb{D} \to \Omega$. It is non-constant since it has non-zero gradient at the origin; moreover, by Hurwitz's theorem (see [**?**]), q' never vanishes on $\mathbb{D}$ since it is the locally uniform limit of the sequence $\{q_n'\}$, and these functions never vanish since they are all derivatives of locally univalent covering maps. Next we show that q is a covering map $\mathbb{D} \to \Omega$.

Fix $a \in \Omega$ and let $d = \text{dist}(a, \partial\Omega)$. Since Ω is bounded, this is finite. Let $D = D(a,d) \subset \Omega$. Since q_n is a covering map, every branch of q_n^{-1} is 1-to-1 holomorphic map of D into $\mathbb{D}$ and hence each q_n is a contraction from the hyperbolic metric on D to the hyperbolic metric on $\mathbb{D}$. Thus every preimage of $\frac{1}{2}D$ has uniformly bounded hyperbolic diameter.

Now fix a point $b \in q^{-1}(a)$. Since $q_n(b) \to q(b) = a$, $q_n(b) \in \frac{1}{2}D$ for n large enough, so there is branch of q_n^{-1} that contains b. Since these branches are uniformly bounded holomorphic functions, by Montel's theorem we can pass to a subsequence so that they converge to a holomorphic function g from $\frac{1}{2}D$ into $\mathbb{D}$. Moreover,

$$q(g(z)) = \lim_n q_n(q_n^{-1}(z)) = z,$$

by Lemma 1.3.4. □

This proves the existence of a covering map for bounded domains Ω. If Ω is bounded and simply connected, then we have proved the Riemann mapping theorem for Ω. For unbounded simply connected domains we use the following argument.

Riemann mapping theorem It suffices to show any simply connected planar domain, except for the plane itself, can be conformally mapped to a bounded domain. If the domain Ω is bounded, there is nothing to do. If Ω. omits a disk

$D(x,r)$ then the map $z \to 1/(z-x)$ conformal maps Ω to a bounded domain. Otherwise, translate the domain so that 0 is on the boundary and consider a continuous branch of $\sqrt{z}$. The image is a 1-1, holomorphic image of Ω, but does not contain both a point and its negative. Since the image contains some open ball, it also omits an open ball and hence can be mapped to a bounded domain by the previous case. □

The final step is to deduce the uniformization theorem for all hyperbolic plane domains (we have only proved it for bounded domains so far). It suffices to show that any hyperbolic plane domain has a covering map from some bounded domain W, for then we can compose the covering maps $\mathbb{D} \to W$ and $W \to \Omega$. We can reduce to the following special case:

Theorem 1.3.6 *There is a holomorphic covering map from $\mathbb{D}$ to $\mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$*

Proof Let

$$\Omega = \{z = x+iy : y > 0, 0 < x < 1, |z - \frac{1}{2}| > \frac{1}{2}\} \subset \mathbb{H}_u.$$

This is simply connected and hence can be conformally mapped to $\mathbb{H}_u$ with $0, 1, \infty$ each fixed. We can then use Schwarz reflection to extend the map across the sides of Ω. Every such reflection of Ω stays in $\mathbb{H}_u$ maps to either the lower or upper half-planes. Continuing this forever gives a covering map from a simply connected subdomain U of $\mathbb{H}_u$ to W. Since U is simply connected and not the whole plane (it is a subset of $\mathbb{H}_u$) it is conformally equivalent to $\mathbb{D}$ and hence a covering $q : \mathbb{D} \to W$ exists. (Actually $U = \mathbb{H}_u$, but we do not need this stronger result. See Exercise 1.8.) □

Uniformization of general planar domains Let $q : \mathbb{D} \to \mathbb{C}^{**} = \mathbb{C} \setminus \{0,1\}$. be a covering map of the twice punctured plane. If $\{a,b\} \in \mathbb{C} \setminus \Omega$ then $h(z) = bq(z) + a$ is a covering map from $U = h^{-1}(\Omega) \subset \mathbb{D}$ to Ω. Any connected component of U shows that Ω has a covering from a bounded plane domain, finishing the proof. □

We can now define a hyperbolic metric ρ on any hyperbolic domain using the covering map $p : \mathbb{D} \to \Omega$. The function ρ should be defined so that p is locally an isometry, i.e.,

$$\begin{aligned} 1 &= D_{\mathbb{D}}^{\Omega} p(w) \\ &= D_{\mathbb{D}}^{E} \mathrm{Id}(w) \cdot D_{E}^{E} p(w) \cdot D_{E}^{\rho_\Omega} \mathrm{Id}(p(w)) \\ &= \frac{1}{\rho_{\mathbb{D}}(w)} \cdot |p'(w)| \cdot \rho_\Omega(z) \end{aligned}$$

and so we take

$$\rho_\Omega(z) = \frac{|p'(w)|}{1-|w|^2} = |p'(w)|\rho_{\mathbb{D}}(w)$$

where $p(w) = z$. Different choices of p and w give the same value for $\rho_\Omega(z)$ since they differ by an isometry of $\mathbb{D}$. Thus every hyperbolic planar domain has a hyperbolic metric.

We want to give some useful estimates for ρ_Ω in terms of more geometric quantities, such as the quasi-hyperbolic metric, defined as

$$\widetilde{\rho}_\Omega(z)ds = \frac{ds}{\mathrm{dist}(z,\partial\Omega)}.$$

For simply connected domains, ρ and $\widetilde{\rho}$ are boundedly equivalent; for more general domains this can fail, but some useful estimates are still available.

The first observation is that if $f : U \to V$ is conformal and $\rho_U(z)ds$ and $\rho_V(z)ds$ are the densities of the hyperbolic metrics on U and V then

$$\rho_V(f(z)) = \rho_U(z)/|f'(z)|.$$

Applying this to the map $\tau(z) = (z+1)/(z-1)$ that maps the right half-plane $\mathbb{H}_r = \{x+iy : x > 0\}$ to the unit disk $\mathbb{D}$, we see that the hyperbolic density for the half-plane is

$$\rho_{\mathbb{H}_r}(z) = |\tau'(z)|\rho_{\mathbb{D}}(\tau(z)) = \frac{2}{|z-1|^2}\frac{1}{1-|\tau(z)|^2} = \frac{1}{2x} = \frac{1}{2\,\mathrm{dist}(z,\partial\mathbb{H}_r)}.$$

Thus the hyperbolic density on a half-plane is approximately the same as the quasi-hyperbolic metric. Using Koebe's theorem (Lemma 1.1.10) we can deduce that that this is true for any simply connected domain.

Lemma 1.3.7 *For simply connected domains, the hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent, i.e.,*

$$d\rho_\Omega \le d\widetilde{\rho}_\Omega \le 4d\rho_\Omega. \tag{1.3.4}$$

Proof Using Koebe's theorem,

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \le \rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z),\partial\Omega} = \frac{1}{\mathrm{dist}(f(z),\partial\Omega} = \widetilde{\rho}(f(z)),$$

which is one half of the result. The other half is similar:

$$\rho_\Omega(f(z)) = \frac{\rho_{\mathbb{D}}(z)}{|f'(z)|} \ge \frac{1}{4}\rho_{\mathbb{D}}(z)\frac{1-|z|^2}{\mathrm{dist}(f(z),\partial\Omega)} = \frac{1}{4}\widetilde{\rho}(f(z)).$$

□

Corollary 1.3.8 *If $f : \Omega \to \Omega'$ is conformal, then*

$$\frac{\mathrm{dist}(f(z), \partial\Omega')}{4\,\mathrm{dist}(z, \partial\Omega)} \le |f'(z)| \le \frac{4\,\mathrm{dist}(f(z), \partial\Omega')}{\mathrm{dist}(z, \partial\Omega)}.$$

Proof Write $f = g \circ h^{-1}$ where $g : \mathbb{D} \to \Omega'$ and $h : \mathbb{D} \to \Omega$ and use the chain rule and Koebe's theorem. □

The following is immediate from Schwarz's lemma.

Corollary 1.3.9 *If $U \subset V$ are both hyperbolic, then $\rho_U \ge \rho_V$.*

Proof If $\Pi_U : \mathbb{D} \to U$ and $\Pi_V : \mathbb{D} \to V$ are the covering maps then the inclusion map $U \to V$ can be lifted to conformal map $\mathbb{D} \to \Pi_V^{-1}(U) \subset \mathbb{D}$. Applying Schwarz's lemma to this map (and using the fact that the projections are local isometries) gives the result. □

Lemma 1.3.10 *If $f : \mathbb{D} \to \Omega$ is conformal with $f'(0) = 1$, then $|f''(0)| \le 200$.*

Proof We can assume $f(0) = 0$. Then $\partial\Omega \cap \overline{\mathbb{D}} \ne \emptyset$, otherwise $|f'(0)| > 1$, so for $z \in \mathbb{D} \cap \Omega$, $\mathrm{dist}(z, \partial\Omega) \le 1 + |z|$. Thus on $\Omega \cap \mathbb{D}$,

$$\rho_\Omega(z) \ge \frac{1}{4}\widetilde{\rho}_\Omega(z) \ge \frac{1}{4(1+r)} \ge \frac{1}{8}.$$

Therefore $|f(z)| \le 1$ on the ball of hyperbolic radius $1/8$ around the origin, which is the same as the Euclidean ball of radius $\frac{1}{2}\log\frac{9}{7} > .1$. By the Cauchy estimate $|f''(0)| \le 200$. □

In fact, the correct bound is not 200, but 4; we have only given a quick proof of a weaker result. See Exercise **??** for how to derive the sharp estimate.

Corollary 1.3.11 *If $f : \mathbb{D} \to \Omega$ is conformal then $\varphi(z) = \log|f'(z)|$ is Lipschitz from the hyperbolic metric to the Euclidean metric, with bound that is independent of f.*

Proof We want to bound $D_H^E \varphi$ uniformly on the disk, but by pre-composing Möbius transformations, it suffices to bound $|\varphi'(0)|$ uniformly in f. By the Cauchy estimate for derivatives, it suffices to show $|\varphi(z) - \varphi(0)|$ is uniformly bounded on a uniform neighborhood of the origin, or equivalently, that $|f'(z)/f'(0)|$ is uniformly bounded on such a neighborhood. Let $d = \mathrm{dist}(f(z), \partial\Omega)$. Then every point in the Euclidean ball $D = D(f(z), d/2)$ is at most distance $3d/2$ from $\partial\Omega$, so integrating over paths from $f(z)$ to ∂D, we see that every point in ∂D is at least $\widetilde{\rho}$-distance $1/3$ from $f(z)$. By Lemma 1.3.7, every boundary point is at least hyperbolic distance $1/12$ from $f(z)$. Thus $U = f^{-1}(D)$ contains a hyperbolic disk of radius $1/12$ around the origin and on this disk (applying Corollary 1.3.8 twice),

$$|f'(z)| \le 4\frac{\mathrm{dist}(f(z),\partial\Omega)}{1-|z|} \le 8\,\mathrm{dist}(f(0),\partial\Omega) \le 32|f'(0)|,$$

as desired. □

Again, this is not sharp; for a proof of the optimal bound, see Exericse 1.23.

The Lipschitz holomorphic functions from the disk with its hyperbolic metric to the plane with its Euclidean metric is called the Bloch class and is a Banach space with the norm

$$\|\varphi\|_{\mathscr{B}} = |\varphi(0)| + \sup_{|z|<1} |\varphi'(z)|(1-|z|^2).$$

In a later chapter, we shall see that Lemma 1.3.11 leads to an intimate connection between conformal maps and martingales that allows various results from probability theory about the latter to be directly to the former, e.g., Makarov's law of the iterated logarithm.

1.4 Boundary continuity

The boundary of a simply connected domain need not be a Jordan curve, nor even locally connected, and such examples arise naturally in complex dynamics as the Fatou components of various polynomials and entire functions. However, this makes little difference to the study of harmonic measure. In this section we show that, from the point view of harmonic measure, it is always enough to consider regions with locally connected boundaries.

Lemma 1.4.1 *Suppose Q is a quadrilateral with opposite pairs of sides E,F and C,D. Assume*

1. *E and F can be connected in Q by a curve σ of diameter $\le \varepsilon$,*
2. *any curve connecting C and D in Q has diameter at least* 1.

Then the modulus of the path family connecting E and F in Q is larger than $M(\varepsilon)$ where $M(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Proof Define a metric on Q by $\rho(z) = \frac{1}{2}|z-a|^{-1}/\log(1/2\varepsilon)$ for $\varepsilon < |z-a| < 1/2$. Any curve γ connecting C and D must cross σ and since γ has diameter ≥ 1 it must leave the annulus where ρ is non-zero. This shows that the modulus of the path family in Q separating E and F is small, hence the modulus of the family connecting them is large. □

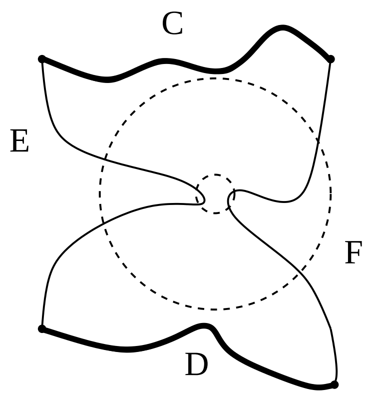


Figure 1.4.1 Proof of Lemma 1.4.1.

The following fundamental fact says that hyperbolic geodesics are almost the same as Euclidean geodesics.

Theorem 1.4.2 (Gehring-Hayman inequality) *There is an absolute constant $C < \infty$ to that the following holds. Suppose $\Omega \subset \mathbb{C}$ is hyperbolic and simply connected. Given two points in Ω, let γ be the hyperbolic geodesic connecting these two points and let σ be any other curve in Ω connecting them. Then $\ell(\gamma) \leq C\ell(\sigma)$.*

Proof Let $f : \mathbb{D} \to \Omega$ be conformal, normalized so that γ is the image of $I = [0, r] \subset \mathbb{D}$ for some $0 < r < 1$. Without loss of generality we may assume $r = r_N 1 - 2^{-N}$ for some N. Let

$$Q_n = \{z \in \mathbb{D} : 2^{-n-1} < |z - 1| < 2^{-n}\},$$

and let

$$\gamma_n = \{z \in \mathbb{D} : |z - 1| = 2^{-n}\},$$

$$z_n = \gamma_n \cap [0, 1).$$

Let $Q'_n \subset Q_n$ be the sub-quadrilateral of points with $|\arg(1 - z)| < \pi/6$. Each of these has bounded hyperbolic diameter and hence by Koebe's theorem its image is bounded by four arcs of diameter $\simeq d_n$ and opposite sides are $\simeq d_n$ apart. In particular, this means that any curve in $f(Q_n)$ separating $f(\gamma_n)$ and $f(\gamma_{n+1})$ must cross $f(Q'_n)$ and hence has diameter $\gtrsim d_n$. Since Q_n has bounded modulus, so does $f(Q_n)$ and so Lemma 1.4.1 says that the shortest curve in $f(Q_n)$ connecting γ_n and γ_{n+1} has length $\ell_n \simeq d_n$. Thus any curve γ in

Q connecting γ_n and γ_{n+1} has length at least ℓ_n, and so

$$\ell(\gamma) = O(\sum d_n) = O(\sum \ell_n) \leq O(\ell(\sigma)). \quad \square$$

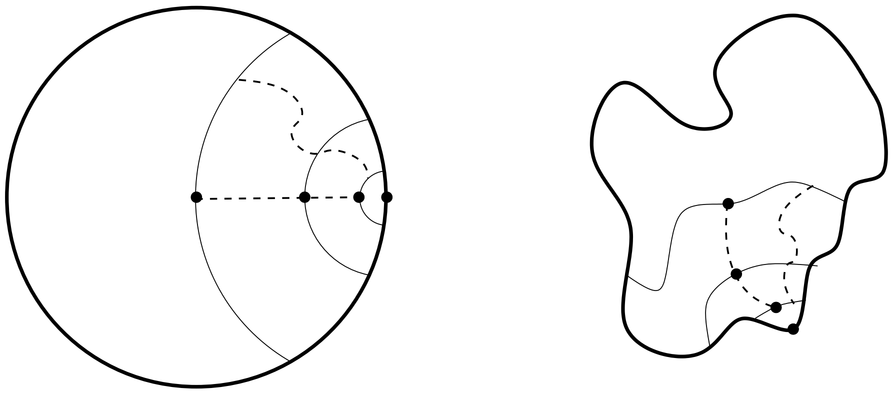


Figure 1.4.2 Proof of the Gehring-Hayman inequality.

If $f : \mathbb{D} \to \Omega$ is conformal define

$$a(r) = \text{area}(\Omega \setminus f(r \cdot \mathbb{D})).$$

If Ω has finite area (e.g., if it is bounded), then clearly $a(r) \searrow 0$ as $r \nearrow 1$.

Lemma 1.4.3 *There is a $C < \infty$ so that the following holds. Suppose $f : \mathbb{D} \to \Omega$ and $\frac{1}{2} \leq r < 1$. Let $E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(rx)| \geq \delta$ for some $r < s < 1\}$. Then the extremal length of the path family $\mathscr{P}$ connecting $D(0,r)$ to E is bounded below by $\delta^2/Ca(r)$.*

Proof Let $z = f(sx)$ and suppose $w \in f(D(0,r))$. By the Gehring-Hayman estimate, the length of any curve from w to z is at least $1/C$ times the length of the hyperbolic geodesic γ between them. But this geodesic has a segment γ_0 that lies within a uniformly bounded distance of the geodesic γ_1 from $f(rx)$ to z. By the Koebe distortion theorem γ_0 and γ_1 have comparable Euclidean lengths, and clearly the length of γ_1 is at least δ. Thus the length of any path from $f(D(0,r))$ to $f(sx)$ is at least δ/C. Now let $\rho = C/\delta$ in $\Omega \setminus f(D(0,r))$ and 0 elsewhere. Then ρ is admissible for $f(\mathscr{P})$ and $\iint \rho^2 dxdy$ is bounded by $C^2 a(r)/\delta^2$. Thus $\lambda(\mathscr{P}) \geq \frac{\delta^2}{C^2 a(r)}$. $\square$

Lemma 1.4.4 *Suppose $f : \mathbb{D} \to \Omega$ is conformal, and for $R \geq 1$,*

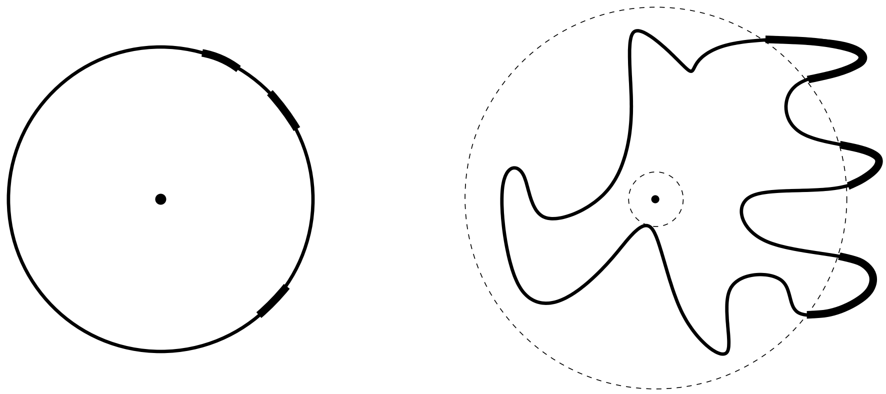
$$E_R = \{x \in \mathbb{T} : |f(x) - f(0)| \geq R \operatorname{dist}(f(0), \partial\Omega)\}.$$

Then E_R has capacity $O(1/\log R)$ if R is large enough.

Proof Assume $f(0)=0$ and $\text{dist}(0,\partial\Omega)=1$ and let $\rho(z)=|z|^{-1}/\log R$ for $z\in\Omega\cap\{1<|z|<R\}$. Then ρ is admissible for the path family Γ connecting $D(0,1/2)$ to $\partial\Omega\setminus D(0,R)$ and $\iint\rho^2dxdy\leq 2\pi/\log R$. By definition $M(\Gamma)\leq 2\pi/\log R$ and $\lambda(\Gamma)\geq(\log R)/2\pi$. By the Koebe distortion theorem $f^{-1}(D(0,1/2))$ is contained in a compact subset of $\mathbb{D}$, independent of Ω. By Pfluger's theorem (Theorem 1.2.10),

$$\cap(E_r)\leq\frac{2}{-2C_2+\log R},$$

which proves the result. □



Corollary 1.4.5 *If $f:\mathbb{D}\to\Omega$ is conformal, then f has radial limits except on a set of zero capacity (and hence has finite radial limits a.e. on $\mathbb{T}$).*

Proof Let $E_{r,\delta}\subset\mathbb{T}$ be the set of $x\in\mathbb{T}$ so that $\text{diam}(f(rx,x))>\delta$, and let $E_\delta=\cap_{0<r<1}E_{r,\delta}$. If f does not have a radial limit at $x\in\mathbb{T}$, then $x\in E_\delta$ for some $\delta>0$, and this has zero capacity by Lemma 1.4.3. Taking the union over a sequence of δ's tending to zero proves the result. The set where f has a radial limit ∞ has zero capacity by Lemma 1.4.4, so we deduce f has finite radial limits except on zero capacity. □

Combining the last two results proves

Corollary 1.4.6 *Given $\varepsilon>0$ there is a $C<\infty$ so that the following holds. If $f:\mathbb{D}\to\Omega$ is conformal, $z\in\mathbb{D}$ and $I\subset\mathbb{T}$ is an arc that satisfies $|I|\geq\varepsilon(1-|z|)$ and $\text{dist}(z,I)\leq\frac{1}{\varepsilon}(1-|z|)$, then I contains a point w where f has a radial limit and $|f(w)-f(z)|\leq C\,\text{dist}(f(z),\partial\Omega)$.*

We can now prove:

Theorem 1.4.7 (Carathéodory) *Suppose that $f:\mathbb{D}\to\Omega$ is conformal, and that $\partial\Omega$ is compact and locally path connected (for every $\varepsilon>0$ there is a $\delta>0$ so that any two points of $\partial\Omega$ that are within distance δ of each other can be connected by a path in $\partial\Omega$ of diameter at most ε). Then f extends continuously to the boundary of $\mathbb{D}$.*

Proof Suppose $\eta>0$ is small. Since $\partial\Omega$ is compact $\Omega\setminus f(\{|z|<1-\frac{1}{n}\})$ has finite area that tends to zero as $n\nearrow\infty$. Thus if n is sufficiently large, this region contains no disk of radius η.

Choose $\{z_j\}$ to be n equally spaced points on the unit circle and using Lemma **??** choose interlaced points $\{w_j\}$ so that f has a radial limit $f(w_j)$ at w_j and this limit satisfies $|f(w_j)-f(rw_j)|\le C\eta$ where $r=1-1/n$. Then

$$\begin{aligned}|f(w_j)-f(w_{j+1})| &\le |f(w_j)-f(rw_j)| \\ &\quad +|f(rw_j)-f(rw_{j+1})| \\ &\quad\quad +|f(rw_{j+1})-f(w_{j+1})| \\ &\le C\delta,\end{aligned}$$

where the center term is bounded by Koebe's theorem and the other two by definition.

Fix $\varepsilon>0$ and choose $\delta>0$ as in the definition of locally connected. Thus if η is so small that $C\eta<\delta$, then the shorter arc of $\partial\Omega$ with endpoints $f(w_j)$ and $f(w_{j+1})$ can be connected in $\partial\Omega$ by a curve of diameter at most ε. Thus the image under f of the Carleson square with base I_j (the arc between w_j and w_{j+1}) has diameter at most $C\eta+\varepsilon$. This implies f has a continuous extension to the boundary. □

It is an inconvenient fact is that conformal maps do not have to extend continuously to the boundary. We noted above however, that radial do exist almost everywhere. Another convenient substitute for full continuity says that every conformal map is continuous on a subdomain of $\mathbb{D}$ whose boundary hits "most of" $\partial\mathbb{D}$. The precise statement requires a new definition.

Given a compact set $E\subset\mathbb{T}$ we will now define the associated "sawtooth" region W_E Suppose $\{I_n\}$ are the connected components of $\mathbb{T}\setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n and which makes angle θ with I_n (so $\gamma_n(0)=I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E(\theta)=\mathbb{D}\setminus\cup_n C_n(\theta)$. Let $W_E=W_E(\pi/8)$ (and let $W_E^*\subset\overline{\mathbb{D}}^c$ be its reflection across $\mathbb{T}$).

If $f:\mathbb{D}\to\Omega$ and $0<r<1$, then define

$$d_f(r)=\sup\{|f(z)-f(w)|:|z|=|w|=r \text{ and } |z-w|\le 1-r\}. \quad (1.4.1)$$

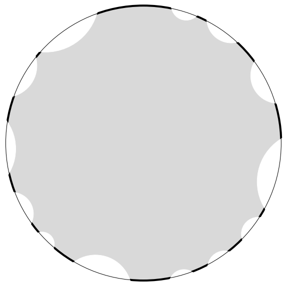


Figure 1.4.3 The sawtooth domain W_E

If $\partial\Omega$ is bounded in the plane, then it is easy to see this goes to zero as $r \nearrow 1$, since otherwise any neighborhood of $\partial\Omega$ would contain infinitely many disjoint disks of a fixed, positive size by Koebe's theorem (Theorem 1.1.10).

Lemma 1.4.8 *Suppose $f : \mathbb{D} \to \Omega \subset S^2$ is conformal. Then for any $\varepsilon > 0$ there is a compact set $X \subset \mathbb{T}$ with* $\text{cap}(\mathbb{T} \setminus X) < \varepsilon$ *such that f is continuous on $\overline{W_X}$.*

Proof By applying a square root and a Möbius transformation, we may assume that $\partial\Omega$ is bounded in the plane. Given $r < 1$ let

$$E(\delta, r) = \{x \in \mathbb{T} : |f(sx) - f(tx)| > \varepsilon \text{ for some } r < s < t < 1\}$$

and note that by Pfluger's theorem (Theorem 1.2.10) and Lemma 1.4.3

$$\text{cap}(E(\delta, r)) \leq \exp(-\pi\varepsilon^2 / Ca(r)),$$

where $a(r) = \text{area}(f(\mathbb{D}) \setminus f(r \cdot \mathbb{D}))$, as before. Moreover, this set is open since f is continuous at the points sx and tx. Fix $\varepsilon > 0$, take $\varepsilon_n = 2^{-n}$, and choose r_n so close to 1 that $\text{cap}(E_n) \equiv \text{cap}(E(\varepsilon_n, r_n)) \leq \varepsilon 2^{-n}$. If we define $X = \mathbb{T} \setminus \cup_{n>1} E_n$, then X is closed and $\mathbb{T} \setminus X$ has capacity $\leq \varepsilon$ by subadditivity.

To show f is continuous at every $x \in \overline{W_X}$, we want to show that $|x - y|$ small implies $|f(x) - f(y)|$ is small. We only have to consider points $x \in \partial W_X \cap \mathbb{T}$. First suppose $y \in \partial W_X \cap \mathbb{T}$. Choose the maximal n so that $s = |x - y| \leq 1 - r_n$. Then $x, y \notin E_n$, so

$$|f(x) - f(y)| \leq |f(x) - f(sx)| + |f(sx) - f(sy)| + |f(sy) - f(y)|.$$

The first and last terms on the right are $\leq \varepsilon_{n-1}$ by the definition of X. The middle term is at most $d_f(1-s)$ (defined in (1.4.1), which tends to 0 as $s \to 0$. Thus $|f(x)-f(y)|$ is small if $|x-y|$ is.

Now suppose $x \in \partial W_X \cap \mathbb{T}$, $y \in \partial W_X \setminus \mathbb{T}$. From the definition of W_X it is easy to see there is a point $w \in \partial W_X \cap \mathbb{T}$ such that $|w-y| \le 2(1-|y|) \le 2|x-y|$. For the point w we know by the argument above that $|f(x)-f(w)|$ is small. On the other hand,

$$|f(y)-f(w)| \le |f(y)-f(|y|w)| + |f(|y|w)-f(w)|.$$

The first term is bounded by $Cd_f(|y|)$ and the second is small since $w \notin E_n$. Thus $|f(x)-f(y)|$ is small depending only on $|x-y|$. Hence f is continuous on $\overline{W_X}$. □

1.5 Harmonic measure

Suppose Ω is a planar domain bounded by a Jordan curve, $z \in \Omega$ and $E \subset \partial\Omega$ is Borel. Suppose $f : \mathbb{D}\Omega$ is conformal and $f(0) = z$ (by the Riemann mapping theorem there is always such a map). By Carathéodory's theorem, f extends continuously (even homeomorphically) to the boundary, so $f^{-1}(E) \subset \mathbb{T}$ is also Borel. We define "the harmonic measure of the set E for the domain Ω, with respect to the point z" as

$$\omega(z, E, \Omega) = |E|/2\pi,$$

where $|E|$ denotes the Lebesgue 1-dimensional measure of E. This depends on the choice of the Riemann map f, but any two maps, both sending 0 to z, will differ only by a pre-composition with a rotation. Thus the two possible pre-images of E differ by a rotation and hence have the same Lebesgue measure. If we fix E and Ω, then $\omega(z, E, \Omega)$ is a harmonic function of z (Exercise 1.12), giving rise the name "harmonic measure". Since we always have $0 \le \omega(z, E, \Omega) \le 1$, we can deduce that if E has harmonic measure with respect to one point z in Ω then it has zero harmonic measure with respect to all points (Exercise 1.13). If $\partial\Omega$ is merely locally connected, then Carathéodory's theorem still implies that the Riemann map f has a continuous extension to the boundary, so the same definition of harmonic measure works.

Theorem 1.4.8 allows us to define harmonic measure on a general simply connected proper subdomain of $\mathbb{C}$ by

$$\omega(z, E, \Omega) = \sup_n \omega(z, E \cap \partial\Omega_n, \Omega_n),$$

where $f : \mathbb{D} \to \Omega$ is conformal with $f(0) = z$, $\Omega_n = f(W_{F_n})$ and $\{F_n\}$ are nested, increasing compact sets with measure tending to $|\mathbb{T}|$ chosen using Lemma 1.4.8 so that f is continuous on each $\overline{W_{F_n}}$. It is easy to verify that this definition does not depend on any of the choice involved.

In general, we can not assume that Ω_n in the previous paragraph is a Jordan domain. For example, if $\Omega = \mathbb{D} \setminus [0,1)$ is a slit disk, then any approximating domains will have to hit both sides of the slit in nearly full harmonic measure, and thus $\partial\Omega$ will contain self-intersections. However, if we are willing to give up approximation of the whole boundary, and only approximate sets of positive measure, then we can do this with Jordan subdomains. This will be discussed in Section **??**, after we have proven the Moore triod theorem and the F. and M. Riesz Theorem.

We want estimate harmonic measure in terms of extremal length. We have already seen how to relate extremal length to logarithmic capacity, and the following relates the latter to harmonic measure:

Lemma 1.5.1 *For any compact $E \subset \mathbb{T}$,*

$$\mathrm{cap}(E) \geq \frac{1}{1 + \log 2 + \pi + \log \frac{1}{|E|}}.$$

If $E \subset \mathbb{T}$ has positive Lebesgue measure, then it has positive capacity. In particular, if $E \subset \mathbb{T}$ is an arc, then

$$\mathrm{cap}(E) \leq \frac{1}{\log 4 + \log \frac{1}{|E|}}.$$

For arcs of small measure, the two bounds are comparable.

Proof If μ is Lebesgue measure restricted to E, then clearly the corresponding potential function is less than potential function of an arc I of the same measure evaluated at the center x of that arc. Since $\frac{2}{\pi} t \leq |x-y| \leq t$ if the arclength between $x, y \in \mathbb{T}$ is t, this value is at most

$$\int_I \log \frac{2}{|x-y|} dy \leq 2 \int_0^{|E|/2} \log \frac{\pi}{t} dt = |E| \log \frac{2}{|E|} + (1+\pi)|E|$$

If we normalize the measure to have mass one, then we get

$$U_\mu \leq \log \frac{2}{|E|} + 1 + \pi = \log \frac{1}{|E|} + 1 + \log 2 + \pi.$$

If E is an arc, then the center x of the arc is at most distance $|E|/2$ from any other point of the arc, and so

$$U_\mu(x) \geq \log \frac{2}{|E|/2} = \log \frac{4}{|E|} = \log \frac{1}{|E|} + \log 4,$$

for any probability measure supported on E. This gives the desired estimate. □

The following is the fundamental estimate for harmonic measure, from which all other estimates flow (at least, all the ones that we will use).

Theorem 1.5.2 *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ and $E \subset \partial\Omega$. Let Γ be the family of curves in Ω which connects $D(z_0, 1/2)$ to E. Then*

$$\omega(z_0, E, \Omega) \leq C \exp(-\pi\lambda(\Gamma)).$$

If $E \subset \partial\Omega$ is an arc then the two sides are comparable.

Proof Let $f : \mathbb{D} \to \Omega$ be conformal. By Koebe's $\frac{1}{4}$-theorem (Theorem 1.1.10), the disk $D(z, \frac{1}{2})$ in Ω maps to a smooth region K in the unit disk that contains the origin, and ∂K is uniformly bounded away from both the origin and the unit circle. Thus by Pfluger's theorem applied to the curve family Γ_X connecting K and the compact set $X = f^{-1}(E)$,

$$\frac{1}{\operatorname{cap}(X)} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq \frac{1}{\operatorname{cap}(X)} + C_2(K),$$

for constants C_1, C_2 that are bounded independent of all our choices.

By Lemma 1.5.1 the right-hand side of

$$1 + \log 4 + \log \frac{1}{|X|} + C_1(K) \leq \pi\lambda(\Gamma_X) \leq 1 + \log 2 + \log \frac{1}{|X|} + C_2(K).$$

holds in general, and the left-hand side also holds if X is an interval. Multiply by -1 and exponentiate to get

$$\frac{|X|}{2e^{1+\pi+C_2}} \leq \exp(-\pi\lambda(\Gamma_X)) \leq \frac{|X|}{4e^{C_1}}$$

under the same assumptions. Now use $\omega(z, E, \Omega) = \omega(0, X, \mathbb{D}) = |X|/2\pi$ to deduce the result. □

One of the most famous and most useful applications of this result is

Corollary 1.5.3 (Ahlfors distortion theorem) *Suppose Ω is a Jordan domain, $z_0 \in \Omega$ with $\operatorname{dist}(z_0, \partial\Omega) \geq 1$ and $x \in \partial\Omega$. For each $0 < t < 1$ let $\ell(t)$ be the length of $\Omega \cap \{|w - x| = t\}$. Then there is an absolute $C < \infty$, so that*

$$\omega(z_0, D(x, r), \Omega) \leq C \exp(-\pi \int_r^1 \frac{dt}{\ell(t)}).$$

Proof Let K be the disk of radius $1/2$ around z_0 and let Γ be the family of curves in Ω which connects $D(x,r)\cap\partial\Omega$ to K. Define a metric ρ by $\rho(z) = 1/\ell(t)$ if $z \in C_t = \{z \in \Omega : |x-z| = t\}$ and $\ell(t)$ is the length of C_t. Any curve $\gamma \in \Gamma$ has ρ-length at least

$$L = \int_r^{1/2} \frac{dt}{\ell(t)},$$

and

$$A = \iint_\Omega \rho^2 dxdy \geq \int_r^{1/2} \int_{C_r\cap\Omega} \ell(z)^{-2} r dr d\theta = \int \ell(z)^{-1} dr = L.$$

Therefore

$$\lambda(\Gamma) \geq A/L^2 = 1/L,$$

and this proves the result. □

Corollary 1.5.4 (Beurling's estimate) *There is a $C < \infty$ so that if Ω is simply connected, $z \in \Omega$ and $d = \operatorname{dist}(z,\partial\Omega)$ then for any $0 < r < 1$ and any $x \in \partial\Omega$,*

$$\omega(z, D(x,rd), \Omega) \leq Cr^{1/2}$$

Proof Apply Corollary 1.5.3 at x and use $\theta(t) \leq 2\pi t$ to get

$$\exp(-\pi \int_{rd}^{d} \frac{dt}{\theta(t)t}) \leq C \exp(-\frac{1}{2}\log r) \leq C\sqrt{r}.$$

□

Corollary 1.5.5 *There is an $R < \infty$ so that for any Ω is a Jordan domain and any $z \in \Omega$*

$$\omega(z, \partial\Omega \setminus D(z, R\operatorname{dist}(z,\partial\Omega), \Omega) \leq 1/2.$$

Proof Rescale so $z = 1$ and $\operatorname{dist}(z,\partial\Omega) = 1$. Then apply $w \to 1/w$ which fixes z and maps $\partial\Omega \setminus D(z,R)$ into $D(0, 1/R-1)$. Then Lemma 1.5.4 implies the result if $R \geq 4C^2 + 1$ (C is as in Lemma 1.5.4). □

Corollary 1.5.6 *For any Jordan domain and any $\varepsilon > 0$,*

$$\omega(z, \partial\Omega \cap D(z, (1+\varepsilon)\operatorname{dist}(z,\partial\Omega)), \Omega) > C\varepsilon,$$

for some fixed $C > 0$.

Proof Renormalize so $z = 0$ and 1 is a closest point of $\partial\Omega$ to z. By Corollary 1.5.5, the set $E = \partial\Omega \cap D(0, 1+\varepsilon)$ has harmonic measure at least $1/2$ from the point $1-\varepsilon/R$. Since $\omega(z,E,\Omega)$ is a positive, harmonic function on $\mathbb{D}$, Harnack's inequality says it is larger than $C\varepsilon/R$ at the origin. □

This is a weak version of the Beurling projection theorem which says that the sharp lower bound is given by the slit disk $D(0,1+\varepsilon)\setminus[1,1+\varepsilon)$. The harmonic measure of the slit in this case can be computed as an explicit function of ε because this domain can be mapped to the disk by sequence of elementary functions.

Theorem 1.5.7 *Suppose Ω is a Jordan domain and $E \subset \partial\Omega$ has zero $\frac{1}{2}$-Hausdorff measure. Then E has zero harmonic measure in Ω.*

Proof Since dilations do not change dimension or harmonic measure, we can rescale so that Ω contains a unit disk centered at some point z. By Exercise 1.13, it suffices to show E has harmonic measure zero with respect to z.

By definition, the hypothesis means that for any $\varepsilon > 0$, the set E can be covered by open disks $\{D(x_j,r_j)\}$ that satisfy $\sum_j r_j^{1/2} \leq \varepsilon$. By Beurling's estimate, this implies

$$\omega(z,E,\Omega) \leq \sum_j \omega(z,D_j,\Omega) \leq O(\sum_j r_j^{1/2}) = O(\varepsilon).$$

□

This result was not improved until Lennart Carleson [**?**] showed in a tour de force that the $\frac{1}{2}$ could be replaced by some $\alpha > \frac{1}{2}$ in [**?**]. That result was not improved until Makarov showed it holds for all $\alpha < 1$ [**?**]. We will prove Makarov's theorem in Chapter **??**. Even though we have not defined harmonic measure for multiply connected domains, it is clear that no analog is possible in that case: if the boundary of Ω is a Cantor set of dimension α, then it must have full harmonic measure, even if α is small.

Corollary 1.5.8 *If Ω is Jordan domain, then harmonic measure is singular to area measure.*

Proof By the Lebesgue density theorem, at Lebesgue almost every point z of a set E of positive area, all small enough disks satisfy

$$\text{area}(E\cap D(z,r)) \geq (1-\varepsilon)\,\text{area}(D(z,r)),$$

for all $r < r_0$. In particular we must have $\ell(t) \leq \frac{\varepsilon}{t}$ on a set of measure at least $r/4$ in $[r/2,r]$. Thus by the Ahlfors distortion theorem

$$\omega(D(z,r_0 2^{-n}) \leq C\exp(-\pi\int_{2^{-n}r_0}^{r_0}\frac{dt}{\varepsilon t}) \leq C2^{-\pi n/\varepsilon}.$$

This is much less than $(2^{-n}r_0)$ if n is large. Thus almost every point of $\partial\Omega$ can be covered by arbitrarily small disks so that $\omega(D(z_j,r_j)) = o(r_j^2)$. Use Vitali's covering theorem to take a disjoint cover of a set of full harmonic measure, and we deduce that harmonic measure gives full mass to set of zero area. □

Corollary 1.5.9 *There is an* $\varepsilon > 0$ *so that harmonic measure on a planar Jordan domain always gives full measure to a set of Hausdorff dimension at most* $2 - \varepsilon$.

Proof Fix a large integer b and consider the b-adic squares in the plane. Take one such square Q that intersects $\partial\Omega$ and consider its b^2 children squares. We claim that if b is large enough, then at least one of them has harmonic measure that is less than $(2b^2)^{-1}$ times the harmonic measure of Q. If there is a subsquare that misses $\partial\Omega$, then its harmonic measure is zero, and the claim is true. Therefore we may assume every subsquare hits $\partial\Omega$. Suppose Q has side length 1 and define a finite sequence of squares S_k, concentric with Q and with side lengths $\frac{1}{b}, \frac{3}{b}, \frac{6}{b}, \ldots, 1$. If $z \in \partial S_k$, then $\mathrm{dist}(z, \partial\Omega) \le \sqrt{2}/b$ and $\mathrm{dist}(z, S_{k-1}) > 3/b$, so by Corollary **??** ,

$$\max_{z \in \partial S_k} \omega(z, \partial\Omega \cap S_{k-1}, \Omega \setminus S_{k-1}) < 1 - \delta,$$

for some uniform $\delta > 0$ (independent of k and b). By the maximum principle and induction,

$$\omega(S_1) \le (1 - \delta)^{b/3},$$

and this is less than $1/(2b^2)$ if b is large enough. This prove the claim, that ω deviates from the uniform distribution on the sub-squares by a fixed amount.

The rest is standard. The deviation from uniformity implies that the entropy

$$h(\mu) = -\sum_{k=1}^{b^2} \omega(Q_j) \log_b \omega(Q_j),$$

is strictly less than 2, the maximum that occurs when every square has equal measure (Exercise **??**). The strong law of large numbers and Billingsley's lemma now imply that ω has dimension strictly less than 2, with a bound that depends on b, but not on Ω. □

Jean Bourgain [**?**] proved this holds for general domains in higher dimensions, with a δ that depends only on the dimension. We shall see later that the bound $\dim(\omega) \le 1$ holds in the plane.

1.6 Diffusion Limited Aggregation

Start with a unit disk centered at the origin. Imagine another unit disk, whose center moves as a Brownian motion starting near infinity unit the it hits the first disk and the stops. Now send in another random disk until it hits one of the first two. Continue in this way until n disks have accumulated to form a connected set as illustrated in Figure 1.6.1.

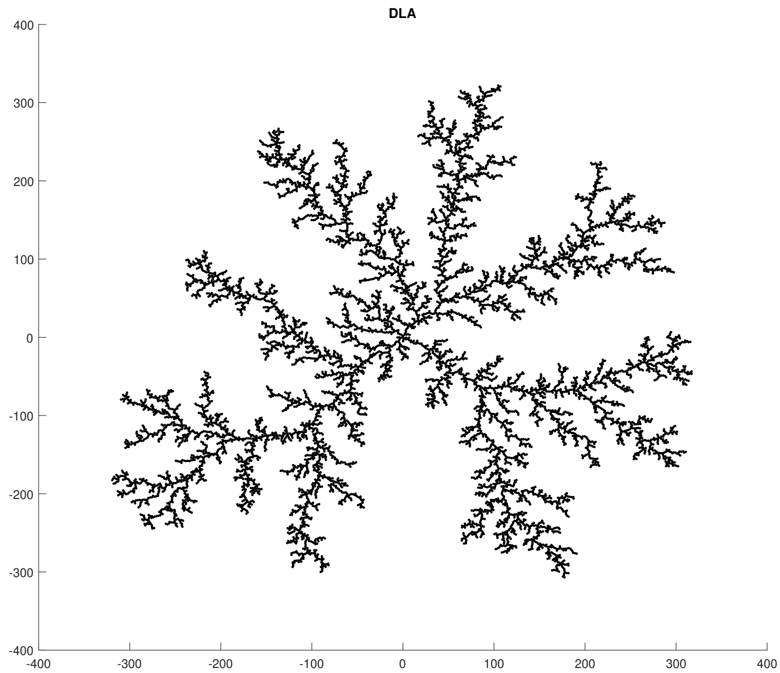


Figure 1.6.1 A diffusion limited aggregates, $n = 10,000$.

It is conjectured that these aggregates, properly rescaled, will have continuous limits that are fractals of dimension approximately 1.71 (based on large numerical simulations), but almost nothing is known rigorously. Indeed, the only rigorous result about DLA is the following upper bound due to Harry Kesten (see [**?**], [**?**], [**?**]), although our presentation follows the one in [**?**].

Theorem 1.6.1 *Almost surely, the diameter of DLA at the nth step is* $O(n^{2/3})$.

Proof The first step is to make the definition of DLA a little more precise. A moving disk will hit a set E when the center is precisely distance 1 from that set. In our case, the set is a union of n unit disks centered at a finite set of points $P_n = \{p_1, \ldots, p_n\}$. Thus the process of adding the next disk by letting

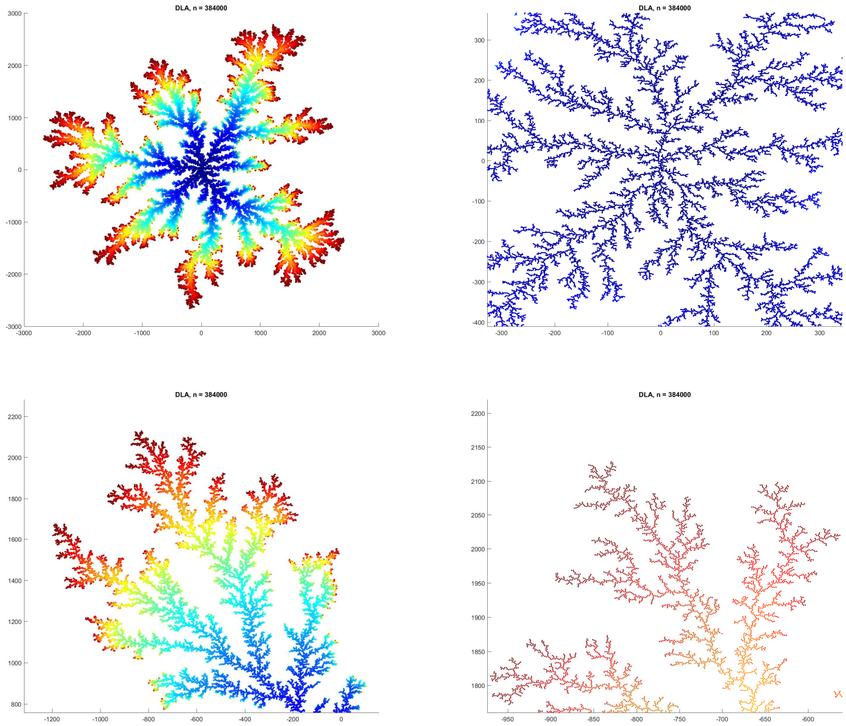


Figure 1.6.2 A diffusion limited aggregation with 384,000 disks. The disks are colored according to when they entered the cluster. Upper left is the full cluster. Upper right is an enlargement of the center. The bottom two pictures are successive enlargements of tip.

it wander by Brownian motion, is precisely the same as choosing a point p_{n+1} on the set

$$E_n = \{z : \text{dist}(z, P_n) = 2\},$$

with respect to harmonic measure at ∞ for the domain Ω_n that is the unbounded complementary component of E_n.. Since E_n is, by definition, a connected set, Ω_n is simply connected and will be bounded by a finite number of circular arcs.

Actually, almost surely Ω_n will be the entire complement of E_n. Otherwise, we must have chosen a disk that made contact with two or more earlier disks. But there are only a finite number of points on E_k where this happens, and finite sets have harmonic measure zero (e.g., Beurling's theorem), so the probability of making such a choice is zero. Thus, almost surely, each disk in the cluster (except the one at the origin) hits exactly one previously chosen disk, although it may be hit by several (at most four, almost surely) later ones.

Consider

$$\mathrm{rad}(n) = \max\{|p| : p \in P_n\},$$

which measures the size of the DLA cluster in terms of a disk around the origin, and its inverse

$$\mathrm{exit}(m) = \max\{n : \mathrm{rad}(n) \le m\},$$

which measures how soon the cluster grows beyond a given radius. The theorem is stated in terms of an upper bound for $\mathrm{rad}(n)$, but is equivalent to a lower bound for $\mathrm{exit}(m)$:

$$\liminf_{m\to\infty} \frac{\mathrm{exit}(m)}{m^{3/2}} \ge \beta, \tag{1.6.1}$$

holds almost surely for some constant $\beta > 0$. More precisely, we define

$$V_m = \{\mathrm{exit}(m) \le \beta m^{3/2}\},$$

and we will prove that $\sum_m \mathbb{P}(V_m) < \infty$. The Borel-Cantelli lemma then implies that the probability that V_m occurs infinitely often is zero. Thus almost surely V_m only occurs finitely often, which gives (1.6.1).

We estimate the probability of V_m by placing these events inside larger events and estimating those. If V_m occurs, it means that the DLA cluster contains a path of at most $\beta m^{3/2}$ disks $\{D_1, \dots D_N\}$ that starts at the origin and ends with a disk that hits the circle $\{|z| = m\}$. Moreover, every D_{j+1}, $j = 1, \dots N-1$ was selected after D_j in the growth process. Otherwise suppose $Dj+1$ is the first counterexample in the path. Then D_{j+1} is the unique earlier disk hit by D_j, so D_{j-1}, which also touches D_j, must have been chosen later than D_j, making D_j a counterexample too.

Every unit disk contains a point in the lattice $\mathbb{N} \times \mathbb{N}$, so for each path of unit disks as above, we can choose a sequence of lattice points $\mathbf{z} = \{z_1, \dots z_N\}$ such $z_j \in D_j$, $j = 1, \dots z_N$ and $|z_j - |z_{j+1}| \le 4$ since the union of two touching unit disks has diameter 4. We will say that sequence of distinct lattice points $\{z_1, \dots, z_k\}$ is m-admissible if

$$|z_1| \le m/2, \quad |z_k| \ge m, \quad |z_j - z_{j+1}| \le 4.$$

Note that there are at most $m^2 80^{k-1}$ m-admissible sequences of length k; there are m^2 possible choices for z_1, and each following choice is made from a 9×9 square, omitting the center. Moreover, the length of an m-admissible sequence is at least $m/8$ since the first and last points are at least distance $m/2$ apart.

Given an m-admissible sequence $\mathbf{z}$ of length k, we define $W_m(\mathbf{z})$ to be the set

of clusters so that:
(1) the cluster contains at most $\beta m^{3/2}$ disks,
(2) the cluster contains the sequence $\mathbf{z}$, and
(3) the disk containing z_{j+1} was chosen after the disk containing z_j. By our comments above each cluster in V_m contained in the event $W_m(\mathbf{z})$ for some m-admissible sequence of length $k \leq \beta m^{3/2}$. Thus all of V_m is contained W_m, the union of $W_m(\mathbf{z})$ over all m-admissible sequences of length at most $\beta m^{3/2}$.

We claim that if $\mathbf{z}$ has length k, then

$$\mathbb{P}(W_m(\mathbf{z})) \leq (C\beta)^k. \tag{1.6.2}$$

We will finish the proof of the theorem assuming this is true, and then prove the estimate. Given (1.6.2)

$$\begin{aligned}
\mathbb{P}(W_m) &\leq \sum_{\mathbf{z}} \mathbb{P}(W(\mathbf{z})) \\
&\leq \#(m - \text{admissible } \mathbf{z}) \cdot (C\beta)^k \\
&\leq m^2 80^{k-1} (C\beta)^k \\
&\leq m^2 (80C\beta)^k \\
&\leq m^2 (80C\beta)^{m/4},
\end{aligned}$$

since $k \geq m/4$. Thus

$$\sum_m \mathbb{P}(V_m) \leq \sum_m \mathbb{P}(W_m) \leq \sum_m m^2 (80C\beta)^{m/4} < \infty,$$

if we choose $\beta < 1/80C$. This completes the proof of Theorem 1.6.1, except for the proof of (1.6.2).

First we explain the general idea for proving (1.6.2). Suppose we have already grown a cluster that contains the points $z_1, \ldots, z_j$. How long do we have to wait before the cluster contains z_{j+1}? We must add a disk within distance 4 of the disk containing z_j. Since the cluster has diameter at least $m/2$, by Beurling's estimate (Lemma 1.5.4) the probability of choosing such a disk is less that $C/\sqrt{m}$. Therefore the expected number of disks we add before covering z_{j+1} is at least $\sqrt{m}/C$. This has to happen k times, so we expect that $k\sqrt{m}/C$ disks need to be added to the cluster before the whole sequence $\mathbf{z}$ is covered. Since $k \geq m/8$, we therefore expect to need about $m^{3/2}/C$ disks to be added. However, clusters in the event $W_m(\mathbf{z})$ only use $\beta m^{3/2}$ disks to cover $\mathbf{z}$. If β is small compared to $1/C$, this event should have small probability.

To make this idea precise, let $D_1, \ldots$ be an enumeration of the disks in the cluster, in the order they are added. Suppose z_j is contained in disk $D_{k(j)}$ and let $w(j) = k(j+1) - k(j)$; this is the time we "wait" between covering z_j and

z_{j+1}. Then

$$\mathbb{P}(w(j) > t) \geq (1-p)^t,$$

where $p \leq C/\sqrt{m}$. Therefore $w(j)$ is bounded below by a geometric random variable (the same one for each j), and $\sum_j w(j)$ will be bounded below by the corresponding sum of geometric variables. We estimate this distribution using:

Lemma 1.6.2 *Suppose $X_1, \dots X_n$ are independent geometric random variables, i.e., $\mathbb{P}(X_j = s) = p(1-p)^{s-1}$ for some $0 < p < 1/2$, and $Y = \sum_{j=1}^n X_j$. If $a \geq 2p$, then*

$$\mathbb{P}(Y \leq an/p) \leq (2e^2 a)^n.$$

Proof As usual, we define the moment generating function of the random variable Y as the expected value of $\exp(tY)$. If X is a geometric random variable, then

$$\mathbb{E}(e^{tX}) = \sum_{j=1}^{\infty} e^{tj} p(1-p)^{j-1} = pe^t \sum_{j=0}^{\infty} (e^t(1-p))^j = \frac{p}{1-e^t(1-p)}.$$

Since Y is a sum of independent copies of X,

$$\mathbb{E}(e^{tY}) = \prod_{j=1}^{\infty} \mathbb{E}(e^{tX}) = \left[\frac{p}{e}^t 1 - e^t(1-p)\right]^w.$$

By Chebyshev's inequality

$$\mathbb{P}(Y < \frac{\ln \lambda}{-t}) = \mathbb{P}(e^{-tY} > \lambda) \leq \frac{1}{\lambda} \mathbb{E}(e^{-tY}).$$

Set $\lambda = \exp(-ant/p)$ to get

$$\mathbb{P}(Y < an/p) \leq \exp(ant/p)\mathbb{E}(e^{-tY}) = \frac{\exp(ant/p)e^{-nt}p^n}{(1-e^{-t}(1-p))^n} = \frac{\exp(ant/p)p^n}{(e^t-(1-p))^n}$$

Now set $t = \ln(a(1-p)/(a-p)$ and this becomes

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\le \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(\frac{(a(1-p)}{a-p} - (1-p))^n} \\
&\le \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{(a}{a-p} - 1)^n} \\
&\le \frac{p^n \left(\frac{a(1-p)}{a-p}\right)^{an/p}}{(1-p)^n(\frac{p}{a-p})^n} \\
&\le \left(\frac{a(1-p)}{a-p}\right)^{an/p} \left(\frac{a-p}{1-p}\right)^n .
\end{aligned}$$

Using $p < 1/2$ and $a \ge 2p$, we get $a \le 2(a-p)$ and $1-p > 1/2$, so

$$\begin{aligned}
\mathbb{P}(Y < an/p) &\le \left(\frac{a(1-p)}{a-p}\right)^{an/p} (2a)^n \\
&\le \left(\frac{a}{a-p}\right)^{an/p} (2a)^n \\
&\le \left(1 + \frac{p}{a-p}\right)^{an/p} (2a)^n \\
&\le \left(1 + \frac{p}{a-p}\right)^{2(a-p)n/p} (2a)^n \\
&\le (e^2 2a)^n,
\end{aligned}$$

since $(1+\frac{1}{x})^x \le e$. □

To finish the proof of (1.6.2), apply Lemma 1.6.2 with $a = \beta k/p \ge C_1 \beta m^{3/2}$

$$\begin{aligned}
\mathbb{P}(W_m) &\le \mathbb{P}(\sum_{j=1}^{k} w(j) < \beta m^{3/2}) \\
&\le \mathbb{P}(\sum_{j=1}^{k} X_j < C_1 \beta k/p) \\
&\le (2e^2 C_1 \beta)^k = (C_2 \beta)^k,
\end{aligned}$$

as desired. This completes the proof of (1.6.2) and hence of Theorem 1.6.1.

□

1.7 Notes

Diffusion limited aggregation was introduced by Witten and Sander in 1981. See [**?**], [**?**]. There have been numerous numerical simulations of DLA and heuristic arguments for estimating its growth and geometry, but after thirty years, Kesten's bound is the only rigorously provable thing we know about DLA.

Many variants of DLA have also been proposed and studied. See, for example, [**?**], [**?**], [**?**], [**?**],

Our discussion of DLA assumed disks were added by moving them continuously by Brownian motion until they made contact with the existing cluster. An alternative model is to use a random walk on a lattice. In this case, the DLA cluster is a connected collection of lattice sites. This is a common formulation of the problem and was the version used in Kesten's papers [**?**], [**?**], [**?**]. The bound and proof are essentially the same as we have given (indeed, our proof is modeled on the discrete proof given by Lawler in []), but one needs a discrete version of Beurling's harmonic measure estimate, Lemma 1.5.4. We choose to give the continuous version of DLA in order to make use of the classical version of Beurling's estimate, which we will also need for other applications in this book.

We have only considered DLA in two dimensions. It is known that in 3 dimensions, the diameter is almost surely $O(n^{1/2}(\log n)^{1/4})$ and in dimensions $d \geq 4$ it is $O(n^{2/(d+1)})$. See [**?**]. It seems unbelievable that there is no non-trivial lower bound for the diameter. The trivial bound in the plane is of order $n^{1/2}$, since no more than $O(n)$ disjoint unit disks can be packed into a disk of radius $\sqrt{n}$ region. However, so far as the authors know, there is no proof that

$$\lim_{n\to\infty} \frac{\operatorname{diam}(\mathrm{DLA}(n))}{\sqrt{n}} = \infty.$$

It also seems very likely that the bound $2/3$ in Kesten's theorem can be improved; the numerics indicate this and looking at the pictures quickly convinces one that we should be able to improve the square root estimate in Beurling's theorem, which is only sharp for line segments (and DLA does not look like a line segment!). Even more difficult questions include whether DLA has a continuous scaling limit, and what the dimension of such a limit might be.

Stas Smirnov has warned that graduate students and postdocs not be allowed to work on DLA. Apparently they are particularly susceptible to a debilitating condition known as "diffusion limited aggravation".

1.8 Exercises

Exercise 1.1 If Ω is a Jordan domain and $E, F \subset \partial\Omega$ are disjoint closed subarcs, then there is a conformal map of Ω to some rectangle so that E and F map to opposite sides.

Exercise 1.2 If Ω is a topological annulus bounded by two Jordan curves, show that it can be conformally mapped to a round annulus.

Exercise 1.3 Let $E \subset \mathbb{C}$ be a closed set and z a point not in E. Compute the modulus of the path family connecting E to $\{z\}$.

Exercise 1.4 Let $E_n \subset \mathbb{T}$ be defined by $\{z : \mathrm{Re}(z^n) > 0\}$. Show that $\mathrm{Cap}_{\log}(\mathrm{E_n}) \to \mathrm{Cap}_{\log}(\mathbb{T})$ as $n \to \infty$. Since $\mathbb{T} \setminus E_n$ clearly has the same capacity as E_n, this implies capacity is not additive.

Exercise 1.5 Show that the linear fractional transformations that map $\mathbb{D}$ 1-to-1, onto itself are exactly those of the form $z \to \lambda(z-a)/(1-\overline{a}z)$ where $|a| < 1$ and $|\lambda| = 1$.

Exercise 1.6 Show a hyperbolic ball in the disk is also a Euclidean ball, but the hyperbolic and Euclidean centers are different (unless they are both the origin). Compute the Euclidean center and radius of a hyperbolic ball of radius r centered at z in $\mathbb{D}$.

Exercise 1.7 Show that the only isometries of the hyperbolic disk are Möbius transformations and their reflections across $\mathbb{R}$.

Exercise 1.8 Show that the domain U constructed in the proof of Theorem 1.3.6 is equal to $\mathbb{H}_u$.

Exercise 1.9 If $\{f_n\}$ are holomorphic functions on a domain Ω that converge uniformly on compact sets to f and if $z_n \to z \in \Omega$, then $f_n(z_n) \to f(z)$.

Exercise 1.10 Suppose E is compact and supports a positive measure μ so that $\mu(D(x,r)) \leq \varphi(r)$, where

$$\sum_{n=0}^{\infty} n\varphi(2^{-n}) < \infty,$$

Then E has positive capacity.

Exercise 1.11 If $E \subset \mathbb{T}$ is compact and has positive Hausdorff dimension, then it has positive capacity.

Exercise 1.12 Suppose Ω is a planar Jordan domain and $E \subset \partial\Omega$ is Borel. Prove that $\omega(z, E, \Omega)$ is a harmonic function of z.

Exercise 1.13 Suppose Ω is a planar Jordan domain and $E \subset \partial\Omega$ is Borel. Show that if $\omega(z, E, \Omega) = 0$ for some $z \in \Omega$, then it is zero on all of Ω.

Exercise 1.14 If $\{p_k\}_{k=1}^n$ are non-negative numbers and $\sum_{k=1}^n p_k = 1$, show that $h = -\sum_{k=1}^n p_k \log p_k$ is maximized uniquely when $p_k = 1/n$ for all k.

Exercise 1.15 Suppose $g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$ is univalent in $\mathbb{D}$. Then $\sum_{n=0}^\infty n|b_n|^2 \leq 1$. In particular, $|b_1| \leq 1$. This is the area theorem.

Exercise 1.16 Use the area theorem to prove that if $\varphi(z) = z + \sum_{n=2}^\infty a_n z^n$ is univalent on the unit disk with $\varphi'(0) = 1$, then $|a_2| \leq 2$. This is the case $n = 2$ of the Bieberbach conjecture (later to become deBrange's theorem [], []).

Exercise 1.17 Use the previous exercise to give a second proof of the Koebe $\frac{1}{4}$-theorem.

Exercise 1.18 If f is conformal on the disk, and $\varphi = \log f'$, then $|\varphi'(z)| \leq 6/(1-|z|^2)$ for all $z \in \mathbb{D}$.

Exercise 1.19 If φ is conformal on $\mathbb{D}$ then

$$\frac{1-|z|}{(1+|z|)^3} \leq |\varphi'(z)| \frac{1+|z|}{(1-|z|)^3}.$$

This is the distortion theorem. See e.g., Theorem I.4.5 of [**?**].

Exercise 1.20 If φ is conformal on $\mathbb{D}$ then

$$\frac{|z|}{(1+|z|)^2} \leq |\varphi(z)| \frac{|z|}{(1-|z|)^2}.$$

This is the growth theorem. See e.g., Theorem I.4.5 of [**?**].

Exercise 1.21

Exercise 1.22

Exercise 1.23

Solutions (eventually move to end of book)

Solution 1.1 First map Ω to the disk by the Riemann mapping theorem. Then use a Möbius transformation to arrange for the images of E and F to be arcs centered at ± 1 and symmetric with respect to the real line. Then the Schwarz-Christoffel formula gives a map to the desired rectangle.

Solution 1.2 Use uniformization theorem to get covering by disk. Then use Riemann map to get covering by vertical strip with deck transformations being vertical translations. Then use exponential map to send strip to annulus and collapsing orbits to single points.

Solution 1.3 Take an annulus around the point that is disjoint from E, but has modulus close to zero, and use monotonicity.

Solution 1.4 The logarithmic capacity of the circle is $1/\log 2$. Compute the potential of Lebesgue measure restricted to E_n and show that it is bounded by $1/2\log 2 + o(1)$ Therefore approximately twice this measure is still admissible, which means the capacity of E_n is close to the capacity of the circle, if n is large..

Solution 1.8

Solution 1.9 We may assume $\{z_n\}$ are contained in some disk $D \subset \Omega$ around z. Let $E = \{z_n\} \cup \{z\}$. This is a compact set so it has a positive distance d from $\partial\Omega$. The points within distance $d/2$ of E form a compact set F on which the functions $\{f_n\}$ are uniformly bounded on E, say by M. By the Cauchy estimate the derivatives are bounded by a constant M' on E. Thus

$$|f(z) - f_n(z_n) \le |f(z) - f_n(z)| + |f_n(z) - f_n(z_n)| \le |f(z) - f_n(z)| + M'|z - z_n|,$$

and both terms on the right tend to zero by hypothesis.

Solution 1.10 The condition easily implies U_μ is bounded, hence $\mathrm{supp}(\mu)$ has positive capacity.

Solution 1.11 This follows from Frostman's theorem (Theorem **??**) since if $\dim(E) > 0$ then E supports a measure that satisfies $\mu(D(x,r)) = O(r^\varepsilon)$ for some $\varepsilon > 0$ and $\sum_n 2^{-\varepsilon n} < \infty$.

Solution 1.12 Show that $\omega(z, E, \mathbb{D})$ must agree with the Poisson integral of the indicator function of E (the function that is 1 on E and 0 off E). This holds because the derivative of a Möbius transformation of the disk to itself has absolute value equal to the Poisson kernel when restricted to the unit circle.

Solution 1.13 By the maximum principle, a harmonic function that attains a minimum or maximum is constant.

Solution 1.15 For $0 < r < 1$ let $D_r = \mathbb{C} \setminus g(D(0,r))$. If $z = g(w)$ and $w = e^{i\theta}$ then $dw = iwd\theta$, so by (**??**),

$$\mathrm{area}(D_r) = \iint_{D_r} dxdy = \frac{1}{2i}\int_{\partial D_r} \bar{z}dz = \frac{-1}{2i}\int_{\partial D(0,r)} \bar{g}(w)g'(w)dw.$$

To evaluate the right hand side note that

$$g(z) = \frac{1}{z} + b_0 + b_1 z + \ldots,$$
$$g'(z) = \frac{1}{z^2} + 0 + b_1 + 2b_2 z + \ldots,$$

so that

$$\int_{|w|=r} \bar{g}(w)g'(w)dw = i\int \bar{g}(w)g'(w)wd\theta$$
$$= i\int (\frac{1}{\bar{w}} + \bar{b}_0 + \bar{b}_1\bar{w} + \dots)(-\frac{1}{w} + b_1 w + 2b_2 w + \dots)d\theta$$
$$= 2\pi i(-\frac{1}{r^2} + |b_1|^2 r^2 + 2|b_2|r^4 + \dots)$$

Thus,

$$0 \le \text{area}(D_r) = \pi(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}).$$

Taking $r \to 1$ gives the result.

Solution 1.16 Let $F(z) = z\sqrt{f(z^2)/z^2}$. Then the quantity inside the square root is even and doesn't vanish in $\mathbb{D}$, so F is odd, univalent and

$$F(z) = z + \frac{a_2}{2}z + \dots.$$

Thus

$$g(z) = \frac{1}{F(z)} = \frac{1}{z} - \frac{a_2}{2}z + \dots,$$

is univalent and satisfies Theorem **??**, so $|a_2| \le 2$.

Solution 1.17 By pre-composing with a Möbius transformation and post-composing by a linear map, we may assume $z = 0$, $f(0) = 0$ and $f'(0) = 1$. Then the right hand inequality is just Schwarz's lemma applied to f^{-1}. To prove the left hand inequality, suppose f never equals w in $\mathbb{D}$. Then

$$g(z) = \frac{wf(z)}{w - f(z)}$$
$$= w(z + a_2 z^2 + \dots)\frac{1}{w}[(1 + \frac{1}{w}(z + a_2 z^2 + \dots) + \frac{1}{w^2}(z + a_2 z^2 + \dots)^2 + \dots)]$$
$$= z + (a_2 + \frac{1}{w})z^2 + \dots,$$

is univalent with $f(0) = 0$ and $f'(0) = 1$. Applying Corollary 1.16 to f and g gives

$$\frac{1}{|w|} \le |a_2| + |a_2 + \frac{1}{w}| \le 2 + 2 = 4.$$

Thus the omitted point w lies outside $D(0, 1/4)$, as desired.

Solution 1.18 Define

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

Then F is conformal, $F(0) = 0$ and $F'(0) = 1$, so Lemma **??** says that $|F''(0)| \leq 4$. A computation shows

$$F''(0) = \frac{f''(z)}{f'(z)}(1-|z|^2) + (-2\bar{z}),$$

and $\varphi' = (\log f')' = f''/f'$, so

$$|\varphi'|(1-|z|^2) \leq |F''(0)| + |2z| \leq 4+2 = 6.$$

Solution 1.19 Fix a point $w \in \mathbb{D}$ and write the Koebe transform of f,

$$F(z) = \frac{f(\tau(z)) - f(w)}{(1-|w|^2)f'(w)},$$

where

$$\tau(z) = \frac{z+w}{1-\bar{w}z}.$$

This is univalent, so by Corollary 1.16, $|a_2(w)| \leq 2$. Differentiation and setting $z = 0$ shows

$$F'(z) = \frac{f'(\tau(z))\tau'(z)}{(1-|w|^2)f'(w)},$$

$$F''(z) = \frac{f''(\tau(z))\tau'(z)^2 + f'(\tau(z))\tau''(z)}{(1-|w|^2)f'(w)},$$

$$\tau'(0) = 1-|w|^2, \tau''(0) = -2(1-|w|^2),$$

$$F''(0) = \frac{f''(w)}{f(w)}(1-|w|^2) - 2\bar{w}.$$

This implies that the coefficient of z^2 (as a function of w) in the power series of F is

$$a_2(w) = \frac{1}{2}((1-|w|^2)\frac{f''(w)}{f'(w)} - 2\bar{w}).$$

Using $|a_2| \leq 2$ and multiplying by $w/(1-|w|^2)$, we get

$$|\frac{wf''(w)}{f'(w)} - \frac{2|w|^2}{1-|w|^2}| \leq \frac{4|w|}{1-|w|^2}.$$

Thus

$$\frac{2|w|^2-4|w|}{1-|w|^2} \le \frac{wf''(w)}{f'(w)} \le \frac{4|w|+2|w|^2}{1-|w|^2}.$$

Now divide by $|w|$ and use partial fractions,

$$\frac{-1}{1-|w|} + \frac{-3}{1+|w|} \le \frac{1}{|w|}\frac{wf''(w)}{f'(w)} \le \frac{3}{1-|w|} + \frac{1}{1+|w|}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\log|f'(re^{i\theta})| &= \frac{\partial}{\partial r}\mathrm{Re}\log f'(z) \\
&= \mathrm{Re}\frac{z}{|z|}\frac{\partial}{\partial z}\log f'(z) \\
&= \frac{1}{|z|}\mathrm{Re}(\frac{zf''(z)}{f'(z)})
\end{aligned}$$

Since $w = re^{i\theta}$ and $f'(0) = 1$, we can integrate to get

$$\log(1-r) - 3\log(1+r) \le \log|f'(re^{i\theta})| \le -3\log(1-r) + \log(1+r).$$

Exponentiating gives the result.