ON THE BEHAVIOUR OF MEROMORPHIC FUNCTIONS IN THE NEIGHBOURHOOD OF AN ISOLATED SINGULARITY

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Introduction

1. The classical theorem of Picard states that in the neighbourhood of an isolated essential singularity a meromorphic function takes all values except at most two. This result has given rise to a very extensive further study. In Nevanlinna's value distribution theory, far-reaching results have been obtained on the number of a-points near the singularity and on the magnitude of certain mean values associated with these numbers ([3]). Further, the generalizations of Picard's Theorem, due to Julia [1] and Ostrowski [4], yield information on the location of a-points in the vicinity of the singularity.

In this paper we once more return to the question of the behaviour of a meromorphic $f(z)$ in the neighbourhood of an isolated essential singularity. Our previous investigations concerning *normal* meromorphic functions ([2], especially Theorem 9) have led us to the idea of characterizing the growth of $f(z)$ near the singular point with the help of the spherical derivative

$$\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Theorem 1 below shows that the behaviour of $f(z)$ near an isolated singularity can, in fact, be accurately described in terms of $\varrho(f(z))$. The slowest possible growth of $\varrho(f(z))$ can be expressed by means of a universal constant.

Theorem 1 distinguishes a particular class of meromorphic functions for which $\varrho(f(z))$ is of slow growth in the vicinity of the singular point. These functions are proved to be equivalent with functions which are normal in every simply connected subdomain of a neighbourhood of the singularity (Theorem 2). This characterization provides a convenient tool for the study of these functions.

Contact with previous results is afforded by Theorem 3 which states that the above slowly growing functions are equivalent with functions exceptional in the sense of Julia.

The paper concludes with certain remarks concerning the characteristic function of these functions.

§ 1. Growth of the spherical derivative

2. Let us suppose that $f(z)$ is meromorphic in a neighbourhood of $z = \infty$. In the following, this particular choice of the possible singular point does not offer any formal advantages, but in the literature the isolated singularity is usually placed at infinity.

It follows readily by simple computation that if $f(z)$ is meromorphic still at $z = \infty$, then

$$\varrho(f(z)) = O\left(\frac{1}{|z|^2}\right).$$

If, instead, $z = \infty$ is a singular point, the approach of $\varrho(f(z))$ towards zero is essentially slower. To this effect, we can prove

Theorem. 1. *Let $f(z)$ be single-valued and meromorphic in a neighbourhood of the essential singularity $z = \infty$. Then a universal positive constant k exists such that*

$$(1) \qquad \overline{\lim_{z \to \infty}} \; |z|\varrho(f(z)) \geqq k$$

for all $f(z)$, while there exist, for any positive ε, functions for which

$$(2) \qquad \overline{\lim_{z \to \infty}} \; |z|\,\varrho(f(z)) < k + \varepsilon.$$

Proof: Let $f(z)$ be meromorphic for $|z_0| \leqq |z| < \infty$, and let $f(a) = w$, $|a| \geqq |z_0|$. If $s(\alpha, \beta)$ denotes the distance of the points α and β on the Riemann sphere, we have for every z lying on the circle $|z| = |a|$,

$$(3) \qquad s(f(z), w) \leqq \int_a^z \varrho(f(z))|dz|,$$

the integral being extended along $|z| = |a|$.

Let us now make the antithesis that (1) is not true. Given an arbitrary $\eta > 0$, a neighbourhood $|z| > M$ of $z = \infty$ then exists in which $|z|\varrho(f(z)) < \eta$ for some $f(z)$. Hence, by (3), as soon as $|z| = |a| > M$, we have

$$(4) \qquad s(f(z), w) < \pi\eta.$$

We shall show that for small η this leads to a contradiction.

Considering Picard's Theorem, there is of course no loss of generality to suppose that $f(z)$ takes the value zero in every neighbourhood of $z = \infty$. Let us now choose $\eta < \frac{1}{4}$, and let $|z| = r > M$ be a circle containing a zero of $f(z)$. By (4), a circular ring $r \leqq |z| \leqq R$ then exists in which

$$(5) \qquad s(f(z), 0) \leqq 2\pi\eta,$$

i.e. $f(z) \neq \infty$, with equality in (5) at some point on $|z| = R$.

Denoting by S_r and S_R the sets of values w taken by $f(z)$ on $|z| = r$ and $|z| = R$, respectively, we conclude that these sets are bounded and mutually disjoint in the w-plane. Boundedness is clear, and that the sets cannot have any points in common follows from (4), if we apply it on $|z| = R$ for a value w for which $s(w,0) = 2\pi\eta$.

The desired contradiction is now obtained as follows: Since the value set of $f(z)$ in $r \leq |z| \leq R$ contains a curve joining S_r to S_R, it contains a point w_0 lying in the same part of the complement of S_r and S_R as the point $w = \infty$. In this part of the complement, let L be a Jordan curve from w_0 to infinity, and $w = b$ the first point on L such that $f(z) \neq b$ in $r < |z| < R$. Then $|f(z)-b|$ has a positive lower bound on the boundary of the ring, while it is arbitrarily small in the interior. This, however, violates the minimum principle. Hence, the antithesis is false, and we infer the existence of a positive constant $k (\geq \frac{1}{4})$ such that (1) is valid.

In order to prove the latter part of the Theorem, we have only to show that there exist functions $f(z)$ for which

$$(6) \qquad \varlimsup_{z \to \infty} |z|\varrho(f(z)) < \infty.$$

Such examples can readily be given. For instance, consider a function $f(z) = F(\log z)$, where $F(w)$ is meromorphic and double-periodic for $w \neq \infty$, with one period equal to $2\pi i$. This $f(z)$ is single-valued and meromorphic for $z \neq 0, \infty$, and

$$(7) \qquad \varrho(f(z)) = \frac{1}{|z|} \varrho(F(w)) \qquad\qquad (w = \log z).$$

Because $F(w)$ is double-periodic, $\varrho(F(w))$ is obviously bounded, and the validity of (6) follows. The Theorem is thus completely established.

We note that there also exist functions $f(z)$, meromorphic in the whole finite plane $z \neq \infty$, which satisfy the condition (6). (Cf. Theorem 3 below).

3. In the case that $f(z)$ is meromorphic in a neighbourhood of a *finite* point $z = a$, the situation is as follows:

If $f(z)$ is meromorphic at $z = a$, then $\varrho(f(z))$ is *bounded* in a vicinity of a. If $z = a$ is a singular point, then

$$\inf_{f} \left\{ \varlimsup_{z \to a} |z-a|\varrho(f(z)) \right\} = k,$$

where k is, of course, the same constant as above.

§ 2. Normal meromorphic functions

4. In this section we study meromorphic functions $f(z)$ whose growth around the isolated singularity $z = \infty$ is minimal in the sense that (2) holds for some finite positive ε.

To this end, we introduce *normal* meromorphic functions. By definition ([2]), a meromorphic $f(z)$ is normal in a simply connected domain G if and only if the family $\{f(S(z))\}$, where $z' = S(z)$ denotes a one-one mapping of G onto itself, is normal in the sense of Montel. In multiply connected domains, $f(z)$ is said to be normal if it is normal on the universal covering surface.

It is well known that a family $\mathfrak{F}$ of meromorphic functions is normal in a domain if and only if

$$(8) \qquad\qquad \sup_{f \in \mathfrak{F}} \varrho(f(z)) < \infty$$

in every compact part of the domain. Applying this condition to the family $\{f(S(z))\}$, we proved in [2] that $f(z)$ is normal in a (simply or multiply connected) domain if and only if

$$(9) \qquad\qquad \varrho(f(z))|dz| = O(d\sigma(z)),$$

where $d\sigma(z)$ denotes the element of length in the hyperbolic metric of the domain.

In multiply connected domains G, another form of normality can be introduced as follows: a meromorphic $f(z)$ is said to be *weakly normal* in G, if $f(z)$ is normal in every simply connected subdomain of G. By the principle of hyperbolic measure, it follows from the condition (9) that normality implies weak normality. The converse is generally not true.

We remark that weak normality can also be defined with respect to sets G which are not necessarily open domains themselves.

5. We proved in [2] (Theorem 9) that a meromorphic $f(z)$ cannot be normal in any neighbourhood of an isolated essential singularity. This follows immediately also from Theorem 1. For if $f(z)$ is normal in a vicinity of $z = \infty$, i.e. if (9) is fulfilled, it follows by an easy computation that

$$\varrho(f(z)) = O\left(\frac{1}{|z| \ \log \ \dfrac{1}{|z|}}\right).$$

Hence, $\varrho(f(z)) = o\left(\dfrac{1}{|z|}\right)$, implying that $f(z)$ is meromorphic still at $z = \infty$.

6. Contrary to the above, the slightly wider class consisting of functions *weakly* normal around the singularity $z = \infty$, is not void. As a matter of fact, this is exactly the class for whose functions the minimal condition $|z|\varrho(f(z)) = O(1)$ is valid.

Theorem 2. *Let $f(z)$ be meromorphic in the neighbourhood U:
$0 < |z_0| \leqq |z| < \infty$ of the essential singularity $z = \infty$. Then*

(10)
$$\overline{\lim_{z \to \infty}} |z| \varrho(f(z)) < \infty$$

if and only if $f(z)$ is weakly normal in U.

Proof: The proof becomes somewhat simpler, if we perform the inversion and study the following situation: $f(z)$ is meromorphic for $0 < |z| \leqq R (< \infty)$, $z = 0$ being an essential singularity.

The sufficiency part of the Theorem is clear. For if $f(z)$ is weakly normal in $0 < |z| \leqq R$, it is normal in every semicircle $(|z| < R) \cap (\vartheta < \arg z < \vartheta + \pi)$, $0 \leqq \vartheta \leqq \pi$. An elementary computation then shows that $|z| \varrho(f(z)) = O(1)$, and (10) follows by inversion.

In order to establish the necessity of the condition, we consider an arbitrary simply connected subdomain G of $0 < |z| \leqq R$. Let ζ be a point of G, and let $w = \varphi(z)$ give a one-one mapping of G onto $|w| < 1$ such that $\varphi(\zeta) = 0$. Then $d\sigma_G(\zeta) = |\varphi'(\zeta)| |d\zeta|$.

In order to estimate $|\varphi'(\zeta)|$, we apply Koebe's distortion theorem. If d_ζ denotes the shortest distance from the point ζ to the boundary, Koebe's theorem yields $d_\zeta |\varphi'(\zeta)| \geqq \frac{1}{4}$. Hence,

$$\frac{d\sigma_G(z)}{|dz|} \geqq \frac{1}{4 d_z} .$$

By hypothesis, $z = 0$ lies outside G. Consequently, $d_z \leqq |z|$, and it follows that

$$\frac{d\sigma_G(z)}{|dz|} \geqq \frac{1}{4 |z|} .$$

From this we conclude that if (10) holds, then a fortiori $\varrho(f(z)) |dz| = O(d\sigma_G(z))$. Consequently, $f(z)$ is normal in G, and the Theorem is completely proved.

7. From Theorem 2 we can immediately infer certain properties of functions satisfying the minimal condition (10).

Let us assume that a function having an isolated singularity at $z = \infty$ and satisfying (10) possesses an asymptotic value α at infinity along a Jordan curve. By Theorem 2, $f(z)$ is normal in a neighbourhood of $z = \infty$ slit along the asymptotic path. By Theorem 2 in [2], $f(z)$ then converges uniformly towards the asymptotic value α, no matter how $z \to \infty$. This being impossible, we conclude that if (10) holds, $f(z)$ cannot possess any asymptotic values.

If $f(z)$ omits a value, this is asymptotic by Iversen's Theorem. Hence, if (10) holds, $f(z)$ takes all complex values. Especially, for analytic functions $\varrho(f(z))$ is always of more rapid growth than $O(1/|z|)$.

8. The functions possessing the minimal growth (10) or, which is the same, being weakly normal in a neighbourhood of $z = \infty$, admit a further characterization whereby an interesting contact is obtained with certain well-known previous investigations.

Utilizing Montel's theory of normal families, Julia (see e.g. [1]) established the following version of Picard's Theorem: Let $f(z)$ be meromorphic in the vicinity of the singular point $z = \infty$, let $\Gamma: z = \sigma(t)$ be an arbitrary Jordan curve with endpoint at infinity, and $U_\varepsilon: |z - \sigma(t)| < \varepsilon |\sigma(t)|$, $\varepsilon > 0$, a neighbourhood of Γ. If $f(z)$ possesses an asymptotic value at infinity, then Γ can be rotated in such a position that in every corresponding neighbourhood U_ε, $\varepsilon > 0$, $f(z)$ takes all values except at most two.

Julia also considered the problem when a family $\{f(\sigma_\nu z)\}$, $\sigma_\nu \to \infty$, is normal. Ostrowski [4] proved that Julia's above-mentioned result holds for $f(z)$ if and only if $\{f(\sigma_\nu z)\}$ is not normal for at least one sequence σ_ν. In the opposite case, i.e. if $\{f(\sigma_\nu z)\}$ is normal for all sequences σ_ν, Ostrowski calls $f(z)$ *exceptional* in the sense of Julia and gives necessary and sufficient conditions of a fairly explicit kind under which $f(z)$ is exceptional.

These results are in close connection with our above considerations. For by the criterion (8), it follows from $\varrho(f(\zeta z)) = \dfrac{1}{|z|} |w| \varrho(f(w))$, $w = \zeta z$, that $\{f(\zeta z)\}$, $\zeta \to \infty$, is a normal family if and only if $|z| \varrho(f(z)) = O(1)$. Hence, the following equivalence prevails between Julia exceptional functions and our slowly growing (or weakly normal) functions.

Theorem 3. *Let $f(z)$ be single-valued and meromorphic in a neighbourhood of the essential singularity $z = \infty$. Then*

$$\varlimsup_{z \to \infty} |z| \varrho(f(z)) < \infty$$

if and only if $f(z)$ is exceptional in the sense of Julia.

9. We conclude the paper by comparing the growth of the spherical derivative $\varrho(f(z))$ to that of the characteristic function $T(r)$ of $f(z)$. For simplicity, we suppose that $f(z)$ is meromorphic for $z \neq \infty$, and recall that $T(r)$ can then be represented as the mean value

$$(11) \qquad T(r) = \frac{1}{\pi} \int_0^r \left(\iint_{|z| \leq t} \varrho(f(z))^2 dx dy \right) \frac{dt}{t} \qquad (z = x + iy)$$

It is well known that

(12) $$T(r) = O(\log r)$$

if and only if $f(z)$ is a rational function. However, the transition from rational to non-rational functions is continuous in the sense that there exist non-rational meromorphic (or entire) functions $f(z)$ for which the growth of $T(r)$ differs arbitrarily little from (12). Such examples can easily be constructed by means of Weierstrassian products.

As regards the growth of $\varrho(f(z))$, we saw above that the situation is different. The transition from rational to non-rational functions is *not* continuous: for rational functions $\varrho(f(z)) = O(1/|z|^2)$, while for non-rational $f(z)$ the minimal growth is $\varrho(f(z)) = O(1/|z|)$.

One can ask how rapidly $T(r)$ can grow if $\varrho(f(z))$ is of the minimal growth $O(1/|z|)$. From (11) we immediately get the upper estimate

$$T(r) = O(\log^2 r).$$

This bound can in fact be attained. For

$$f(z) = \prod_{0}^{\infty} \frac{2^n - z}{2^n + z}$$

is Julia's exceptional function ([4]), for which $n(r,0) \geqq A \log r \ (A > 0)$, and hence, $T(r) \geqq N(r,0) + O(1) \geqq \dfrac{A}{2} \log^2 r + O(1)$.

References

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